5-1-2004

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## Recommended Citation

Delzell, C. (2004). Impossibility of $C^{\infty}$ variation or formal power series variation in solutions to Hilbert's 17th problem. *Journal of Algebra, 275* (1), 233-249. https://doi.org/10.1016/j.jalgebra.2003.10.032

Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Algebra 275 (2004) 233–249
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

# Impossibility of $C^\infty$ variation or formal power series variation in solutions to Hilbert's 17th problem ☆

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Received 30 January 2003
Communicated by Leonard Lipshitz

**Abstract**

No matter how a positive semidefinite polynomial $f \in \mathbb{R}[X_1, \ldots, X_n]$ is represented (according to E. Artin's 1926 solution to Hilbert's 17th problem) in the form $f = \sum p_i r_i^2$ (with $0 \leqslant p_i \in \mathbb{R}$ and $r_i \in \mathbb{R}(X_1, \ldots, X_n)$), the $p_i$ and the coefficients of the $r_i$ cannot be chosen to depend in a $C^\infty$ (i.e., infinitely differentiable) manner upon the coefficients of $f$ (unless $\deg f \leqslant 2$); formal powers series variation is also impossible. This answers a question we had raised in 1990 [Contemp. Math., vol. 155, Amer. Math. Soc., 1994, pp. 107–117], where we had already shown that real analytic variation was impossible; and Gonzalez-Vega and Lombardi [Math. Z. 225 (3) (1997) 427–451] then showed that for every fixed, finite $r \in \mathbb{N}$, $C^r$ variation is possible, improving upon their and the author's result that continuous, piecewise-polynomial variation is possible.


*Keywords:* $C^\infty$ functions; Sums of squares; Basic closed semianalytic sets; Positive semidefinite polynomials; Hilbert's 17th problem; Formal power series; Weierstraß polynomials

## 1. Introduction

Suppose $n \in \mathbb{N} := \{0, 1, \ldots\}$, $X := (X_1, \ldots, X_n)$ are indeterminates, and $f \in \mathbb{R}[X]$ is psd (positive semidefinite), i.e., $\forall x := (x_1, \ldots, x_n) \in \mathbb{R}^n$, $f(x) \geqslant 0$. Hilbert's 17th problem [23] was to prove that we can always write such an $f$ in the form

☆ See http://at.yorku.ca/cgi-bin/amca/cacv-60 for an abstract of this paper (dated June 1999).
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1 Partially supported by NSF-DMS-ANT-9401509.

0021-8693/$ – see front matter © 2004 Published by Elsevier Inc.
doi:10.1016/j.jalgebra.2003.10.032

$$f = \sum_i r_i^2, \quad \text{for some } r_i \in \mathbb{R}(X).$$

E. Artin solved this problem in [1], and went on to prove that if $K \subseteq \mathbb{R}$ is a subfield, $f \in K[X]$, and $f$ is psd, then we can write

$$f = \sum p_i r_i^2, \tag{1.0.1}$$

for some $r_i \in K(X)$ and $p_i \in K$ such that $p_i \geqslant 0$.

*Parametrization of Hilbert's 17th problem*

Now let $d \in \mathbb{N}$, let $m := m_{nd} = \binom{n+d}{n}$, let $C := (C_1, \dots, C_m)$ be indeterminates, and let $f_{nd} := f_{nd}(C; X) \in \mathbb{Z}[C; X]$ be the general polynomial of degree $d$ in $X$ with coefficients $C$:

$$f_{nd} = \sum_{|\alpha| \leqslant d} C_{j(\alpha)} X^\alpha,$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \sum \alpha_i$, $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$, and $j$ is any fixed bijection:

$$j := j_{nd} : \{\alpha \mid |\alpha| \leqslant d\} \to \{1, \dots, m\}.$$

Writing $c = (c_1, \dots, c_m) \in \mathbb{R}^m$, let

$$P_{nd} = \{c \in \mathbb{R}^m \mid f_{nd}(c; X) \text{ is psd in } X\}.^2 \tag{1.0.2}$$

Let $B$ be any subring of the ring of all functions $\mathbb{R}^m \to \mathbb{R}$.

**Question 1.1.** For which subrings $B$ can we solve Hilbert's 17th problem so that the $p_i$ and the coefficients of the $r_i$ in (1.0.1) are functions in $B$? (Obviously, we seek $B$ as small as possible.)

Precisely, for which $B$ is the following true?

For all $n, d \in \mathbb{N}$, there exist $s \in \mathbb{N}$, $p_1, \dots, p_s \in B$, and $g_1, \dots, g_s, h_1, \dots, h_s \in B[X]$ such that

$$\forall c \in \mathbb{R}^m, \quad f_{nd}(c; X) = \sum_{i=1}^s p_i(c) \left( \frac{g_i(c; X)}{h_i(c; X)} \right)^2; \tag{1.1.1}$$

$$\forall c \in P_{nd}, \ \forall i, \quad p_i(c) \geqslant 0; \quad \text{and} \tag{1.1.2}$$

[2] $P_{nd}$ is a closed, convex, semialgebraic cone; its interior $P_{nd}^\circ$ consists of those $c \in P_{nd}$ such that the $X$-homogenization (in $\mathbb{R}[X_1, \dots, X_n, X_{n+1}]$) of $f(c; X)$ is positive at all $(x_1, \dots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} \setminus \{(0, \dots, 0)\}$; and $P_{nd} = \overline{P_{nd}^\circ}$. But we do not need these facts here.

$$\forall c \in P_{nd}, \quad \text{if} \quad f_{nd}(c; X) \neq 0 \in \mathbb{R}[X],^{3} \quad \text{then} \quad \forall i, \quad h_i(c; X) \neq 0. \tag{1.1.3}$$

**Note 1.2.** If we write $g_i = \sum_\alpha g_{i,\alpha} X^\alpha$ with $g_{i,\alpha} \in B$, for finitely many $\alpha \in \mathbb{N}^n$, then $g_i(c; X)$ means $\sum_\alpha g_{i,\alpha}(c) X^\alpha \in \mathbb{R}[X]$. Similarly for $h_i(c; X)$.

When seeking function rings $B$ that satisfy the conditions of Question 1.1, we prefer those $B$ such that for all $p \in B$ and for all subfields $K \subseteq \mathbb{R}$, $p$ takes values in $K$ at $K$-rational points, i.e., $p(K^m) \subseteq K$; such $B$ give a parametrized version of the full force of Artin's result (1.0.1), "uniformly" for all $K \subseteq \mathbb{R}$. On the other hand, if we show that some function ring $B$ does *not* satisfy the conditions of 1.1, then it is immaterial whether all functions in $B$ take values in $K$ at $K$-rational points.

**Remark 1.3.** (1.1.1) is equivalent to:

$$\forall c \in \mathbb{R}^m, \quad h(c; X)^2 f_{nd}(c; X) = \sum_{i=1}^{s} p_i(c) g_i'(c; X)^2, \tag{1.1.1$'$}$$

where $h = h_1 \cdots h_s \in B[X]$ is a common denominator, and

$$g_i' = g_i \prod_{j \neq i} h_j.$$

With this alternative notation, (1.1.3) is then equivalent to:

$$\forall c \in P_{nd}, \quad \text{if} \quad f_{nd}(c; X) \neq 0, \quad \text{then} \quad h(c; X) \neq 0. \tag{1.1.3$'$}$$

The main result of this paper is that the ring $B$ in Question 1.1 may *not* be taken to be $C^\infty(\mathbb{R}^m)$, the ring of $C^\infty$ (i.e., infinitely differentiable) functions $p : \mathbb{R}^m \to \mathbb{R}$:

**Theorem 1.4.** *For $n \geqslant 1$ and $d \geqslant 4$, there exists no $C^\infty$ varying solution to the 17th problem. I.e., there exist no $s \in \mathbb{N}$, no $p_1, \ldots, p_s \in C^\infty(\mathbb{R}^m)$, and no $g_1, \ldots, g_s, h \in C^\infty(\mathbb{R}^m)[X]$ such that* (1.1.1$'$), (1.1.2), *and* (1.1.3$'$) *hold.*

This will follow immediately from Theorem 1.5 below, which deals with the following simpler situation.

From now on let $m = 2$, let $C := (C_1, C_2)$, let $c := (c_1, c_2) \in \mathbb{R}^2$, and let

$$f(C; X_1) = X_1^4 + C_1 X_1^2 + C_2 = \left(X_1^2 + \frac{1}{2}C_1\right)^2 + \left(C_2 - \frac{1}{4}C_1^2\right). \tag{1.4.1}$$

[3] Note that for all $c \in \mathbb{R}^m$, $f_{nd}(c; X) \neq 0 \in \mathbb{R}[X]$ if and only if $c \neq (0, \ldots, 0) \in \mathbb{R}^m$.

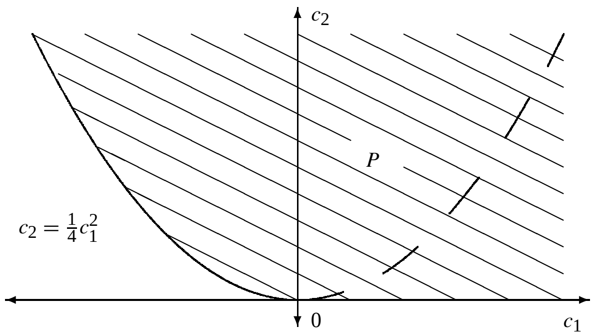
Let $P = \{c \in \mathbb{R}^2 \mid f(c; X_1)$ is psd in $X_1\}$; this is a 2-dimensional cross-section of the 5-dimensional set $P_{1,4}$ of (1.0.2). Then

$$P = \left\{c \in \mathbb{R}^2 \;\middle|\; [c_1 \geqslant 0 \wedge c_2 \geqslant 0] \text{ or } [c_1 \leqslant 0 \wedge c_2 \geqslant c_1^2/4]\right\}, \tag{1.4.2}$$

since, e.g.,

$$\min_{x \in \mathbb{R}} f(c; x) = \begin{cases} c_2 & \text{if } c_1 \geqslant 0, \\ c_2 - c_1^2/4 & \text{if } c_1 \leqslant 0, \end{cases} \tag{1.4.3}$$

by (1.4.1). (These minima are achieved for $x = 0$ or $x = \sqrt{-c_1/2} \in \mathbb{R}$, respectively.)




Let $U \subseteq \mathbb{R}^2$ be any open neighborhood of $(0,0)$, and let $C^\infty(U) = \{C^\infty$ functions $U \to \mathbb{R}\}$. We now restrict (1.1.1′), (1.1.2), and (1.1.3′) to $U$, and replace $f_{nd}$ by $f$, obtaining, respectively:

$$\forall c \in U, \quad h(c; X_1)^2 f(c; X_1) = \sum_{i=1}^{s} p_i(c) g_i(c; X_1)^2; \tag{1.1.1″}$$

$$\forall c \in P \cap U,\ \forall i, \quad p_i(c) \geqslant 0; \quad \text{and} \tag{1.1.2″}$$

$$\forall c \in P \cap U, \quad h(c; X_1) \neq 0.^4 \tag{1.1.3″}$$

**Theorem 1.5.** *There exist no open neighborhood $U \subseteq \mathbb{R}^2$ of $(0,0)$, no $s \in \mathbb{N}$, no $p_1, \ldots, p_s \in C^\infty(U)$, and no $g_1, \ldots, g_s, h \in C^\infty(U)[X_1]$ satisfying* (1.1.1″), (1.1.2″), *and* (1.1.3″). *I.e., there are no* germs *at* $(0,0)$ *of $C^\infty$ functions of $c_1, c_2$ that provide the weights and the coefficients of a representation of $X_1^4 + c_1 X_1^2 + c_2$ as a weighted sum of squares in $\mathbb{R}(X_1)$ (as in* (1.1.1″)), *where the weights are nonnegative for $(c_1, c_2)$ in (the germ at $(0,0)$ of) $P$.*

[4] In (1.1.3′) we required (for any given $c \in P_{nd}$) the hypothesis that $f_{nd}(c; X) \neq 0 \in \mathbb{R}[X]$; when $f_{nd}$ is replaced by $f$ in (1.1.3″), this hypothesis becomes $f(c; X_1) \neq 0 \in \mathbb{R}[X_1]$, which is satisfied for all $c \in \mathbb{R}^2$.

Since for any $n \geqslant 1$ and $d \geqslant 4$, $X_1^4 + c_1X_1^2 + c_2$ can be obtained from $f_{nd}$ merely by setting some of the (other) coefficients $c_i$ equal to 0 or 1, Theorem 1.4 now follows, upon taking $U = \mathbb{R}^2$ in 1.5.

We shall prove Theorem 1.5 in Section 6 below, after first proving (5.2) that formal power series variation is also impossible in Artin's theorem. Both of these results require some review of well-known facts about $C^\infty$ functions (Section 3) and formal power series (Section 4). First, however, we review earlier work on Question 1.1 in Section 2 below.

## 2. Review of earlier answers to Question 1.1

Artin's theorem itself (1.0.1) may be considered to be a trivial "parametrization," for it says that we may take $B$ in 1.1 to be the ring (here denoted by $\mathbb{R}_K^{(\mathbb{R}^m)}$) of *all* functions from $\mathbb{R}^m$ to $\mathbb{R}$ with values in $K$ at $K$-rational points;[5] in other words, Artin did not consider whether the variation of the sum-of-squares representation in (1.0.1) (or in (1.1.1)) could be continuous, or could have other interesting properties.

The first non-trivial parametrization of (1.0.1) was by Henkin [20], who found that the variation can be given by $\mathbb{Z}$*-piecewise-polynomial* functions, here denoted by $\mathrm{PWP}(\mathbb{R}^m)$; i.e., $P_{nd}$ (or $\mathbb{R}^m$) can be decomposed into $\mathbb{Z}$-semialgebraic "pieces" $S_1 \cup \cdots \cup S_k$, on each of which the $p_i$ and the coefficients of the $r_i$ in (1.0.1) are given by functions in $\mathbb{Z}[C]$. van den Dries gave another proof of this in [15].

Shortly thereafter, Kreisel [24] asked whether this variation could even be polynomial (i.e., with only one piece $S_1$), or at least continuous. We showed [7] that for $d \geqslant 4$, polynomial variation is impossible, i.e., that we may not take $B = \mathbb{R}[C]$.[6] We also showed [8] that for all $d \geqslant 0$, continuous (even continuous "$\mathbb{Z}$-semialgebraic") variation is possible; i.e., we may take $B = \mathrm{CSA}_{P_{nd}}(\mathbb{R}^m)$, the ring of $\mathbb{Z}$-semialgebraic functions $p : \mathbb{R}^m \to \mathbb{R}$ *whose restrictions to* $P_{nd}$ *are continuous*.[7] A ($\mathbb{Z}$-)semialgebraic function is defined to be one with a ($\mathbb{Z}$-)semialgebraic graph; such a function will usually not take values in $K$ at $K$-rational points $c \in P_{nd} \cap K^m$ (where $K \subseteq \mathbb{R}$ is as in (1.0.1)), unless $K$ is *real closed*

[5] Actually, the usual statement of Artin's theorem (1.0.1) gives only functions $p : P_{nd} \to \mathbb{R}$ (with $p(P_{nd} \cap K^m) \subseteq K$); but these functions may be extended to the rest of the desired domain $\mathbb{R}^m$ (i.e., for $c \in \mathbb{R}^m \setminus P_{nd}$) so as to continue to satisfy (1.1.1′), using the identity $f = ((f+1)/2)^2 + (-1)((f-1)/2)^2$ (which also appeared in [1]). Condition (1.1.2) is still (vacuously) satisfied with $p_2(c) = -1$ for $c \in \mathbb{R}^m \setminus P_{nd}$, since the $p_i$ need be nonnegative only for $c \in P_{nd}$; and (1.1.3′) is also obvious under this extension (where $h(c; X) = 2 \neq 0$ for $c \in \mathbb{R}^m \setminus P_{nd}$).

[6] For $d \leqslant 2$, however, we may even take $B = \mathbb{Z}[C]$ ($\subset \mathbb{R}[C] \subset \mathcal{O}(\mathbb{R}^m) \subset C^\infty(\mathbb{R}^m)$); see [6]. In fact, when $d = 2$, we may even take the common denominator $h$ in (1.1.1′) to be in $\mathbb{Z}[C]$ (and not merely in $\mathbb{Z}[C; X]$); thus, psd *quadratic* forms can be continuously represented as nonnegatively weighted sums of squares of $X$-linear *forms* (and not merely rational functions as in (1.1.1)). Admittedly, the continuity result for $h \in Z[C]$ was established only for $c \in P_{n,2}$, and not necessarily for all $c \in \mathbb{R}^{\binom{n}{2}}$. Upper and lower bounds for the number of such continuously varying squared linear forms needed for this are in [10].

[7] Actually, [8] constructed only functions $p : P_{nd} \to \mathbb{R}$ (continuous and $\mathbb{Z}$-semialgebraic). To extend such $p$ from $P_{nd}$ to all of $\mathbb{R}^m$ (to conform to the formulation of the problem in Question 1.1) we need only add the trick, mentioned in footnote 5 above, for discontinuously extending the sum-of-squares representation (1.1.1) to the rest of $\mathbb{R}^m$. But the only way we know to get functions defined and continuous even outside of $P_{nd}$ that answer Question 1.1 is to show that we may even take $B = \mathrm{SIPD}(\mathbb{R}^m)$ ($\subset \mathrm{CSA}_{P_{nd}}(\mathbb{R}^m)$), as in the next paragraph.

(e.g., when $K$ is $\mathbb{R}$, or the real algebraic numbers). This makes semialgebraic functions unsatisfactory for parametrizing Artin's theorem over non-real-closed fields $K$, such as $\mathbb{Q}$ (Hilbert actually considered the case $K = \mathbb{Q}$ in his *Grundlagen der Geometrie* [22], and even emphasized the need to allow *arbitrary* $K \subseteq \mathbb{R}$ when he formulated the 17th problem in his famous list of 23 problems [23]).

Thus the above result answered Kreisel's question about continuous variation in Artin's theorem only over *real closed* subfields $K \subseteq \mathbb{R}$;[8] the answer to his question over arbitrary $K \subseteq \mathbb{R}$ was not found until [11] (and re-discovered (using a different method) by González-Vega and Lombardi [17]; see also our joint article [13], and the treatments in [14] and [27]): the $p_i$ and the coefficients of the $r_i$ in (1.0.1) may be taken from the ring SIPD($\mathbb{R}^m$) of "sup-inf-polynomially definable" functions (i.e., functions of the form $\sup_i \inf_j p_{ij}(c)$, for finitely many polynomial functions $p_{ij} \in \mathbb{Z}[C]$). Such functions are not only continuous, but also piecewise-polynomial (with integer coefficients); thus for any $K \subset \mathbb{R}$, they take values in $K$ at $K$-rational points $c \in K^m$, as Hilbert would have wanted. Thus this result combines the best features of [20] and [8] above.

In the 1990s, Question 1.1 was studied for various function-rings $B$ *bigger* than $\mathbb{R}[C]$ (where the answer is no, as mentioned above), and/or *smaller* than SIPD($\mathbb{R}^m$) (where the answer is yes). First, in [12] we showed that in 1.1, $B$ can *not* be taken to be $\mathcal{O}(\mathbb{R}^m)$, the ring of real analytic functions on $\mathbb{R}^m$;[9] we then asked whether $B$ could be taken to be $C^\infty(\mathbb{R}^m)$. Meanwhile, González-Vega and Lombardi [18] showed that for each fixed $r \in \mathbb{N}$, $B$ may be taken to be $C^r(\mathbb{R}^m) \cap \text{SIPD}(\mathbb{R}^m)$ (where $C^r(\mathbb{R}^m)$ denotes the ring of functions $p : \mathbb{R}^m \to \mathbb{R}$ all of whose $r$th-order partial derivatives exist and are continuous), improving upon the earlier result about SIPD($\mathbb{R}^m$). They also considered a weakening of Question 1.1 obtained by replacing "$\forall c \in P_{nd}$" with "$\forall c \in P_{nd}^\circ$" in (1.1.3′) (recall footnote 2); i.e., they considered allowing the "denominator" $h$ in (1.1.1′) to vanish for $c$ on the boundary $\partial P_{nd}$ of $P_{nd}$, and outside $P_{nd}$. They then showed that $B$ may be taken to be either (a) the subring of functions that are continuous and semialgebraic on $\mathbb{R}^m$, Nash on $P_{nd}^\circ$, and zero outside $P_{nd}^\circ$; or (b) the subring of functions that are $C^\infty$ on $\mathbb{R}^m$, analytic on $P_{nd}^\circ$, and zero outside $P_{nd}^\circ$.

Our question about $C^\infty(\mathbb{R}^m)$ (with the original version of (1.1.3′)), however, remained unanswered until now; Theorem 1.4 above states that for $d \geqslant 4$, $B$ may *not* be taken to be $C^\infty(\mathbb{R}^m)$ in 1.1.

[8] Actually, a Boolean combination of $\mathbb{Z}$-polynomial inequalities defining a $\mathbb{Z}$-semialgebraic set such as $P_{nd}$ over $\mathbb{R}$ also defines a corresponding set $P_{nd,R} \subseteq R^m$, where $R$ is *any* real closed field. Likewise, the $\mathbb{Z}$-polynomial inequalities defining a $\mathbb{Z}$-semialgebraic function such as any $p \in \text{CSA}(P_{nd})$ also define a corresponding function $p_R : P_{nd,R} \to R$. And for any subfield $K \subseteq R$, $p_R$ will take values in $K$ at $K$-rational points of $P_{nd,R}$ provided $K$, too, is real closed. Consequently, Question 1.1 has sometimes been formulated with $\mathbb{R}$ replaced by an arbitrary real closed field $R$; then the continuous semialgebraic variation in Artin's theorem constructed in [8] is seen to work uniformly over all real closed fields $R$. Similarly for the SIPD variation discussed later in the above sentence.

[9] This contrasts with the situation for Euler's theorem [16] that every positive rational number $r$ is the sum of four squares of rationals: Heilbronn [19] showed that those four rational numbers can be chosen to vary analytically in $r$, answering another question of Kreisel; a similar result [9] holds for Siegel's generalization [31] of Euler's theorem, that in *every* number field $K$, every totally positive element is the sum of four squares in $K$.

**Summary of answers to Question 1.1**

$B$ can be any of the following rings (except those crossed out):

$\mathbb{R}_K^{(\mathbb{R}^m)}$ (= *all* functions $p:\mathbb{R}^m \to \mathbb{R}$ with $p(K^m) \subseteq K$; recall footnote 5); Artin [1] (1926)

PWP($\mathbb{R}^m$) (piecewise-$\mathbb{Z}$-polynomial functions); Henkin [20] (1960)

$\mathrm{CSA}_{P_{nd}}(\mathbb{R}^m)$ ($\mathbb{Z}$-semialgebraic functions continuous on $P_{nd}$; recall footnote 7) Delzell [8] (1984)

SIPD($\mathbb{R}^m$) (= $\{\sup_i \inf_j p_{ij} \mid p_{ij} \in \mathbb{Z}[C]\}$) Delzell [11] (1989, 1993); González-Vega, Lombardi [17], [13] (1993)

$C^r(\mathbb{R}^m) \cap \mathrm{SIPD}(\mathbb{R}^m)$, for any fixed $r \in \mathbb{N}$; González-Vega, Lombardi [18] (1997) (idea: $\sup\{0, p_i(c)\}^{2r+1} \in C^{2r}$)

~~$C^\infty(\mathbb{R}^m)$~~ (or ~~$\mathbb{R}[[C]]$~~) (for $d \geqslant 4$; recall footnote 6); Theorems 1.4 & 5.2 in this paper

~~$\mathcal{O}(\mathbb{R}^m)$~~ (for $d \geqslant 4$; recall footnote 6) Delzell [12] (1994)

~~$\mathbb{R}[C]$~~ (for $d \geqslant 4$; recall footnote 6) Delzell [7] (1982)

At the moment we have no further candidates for function-rings $B$ to consider in Question 1.1; so perhaps this line of investigation into the possible kinds of variation in Artin's theorem is finally complete.

*Review of continuity results in other sum-of-squares representations*

The real Nullstellensatz and Positivstellensatz (both due originally to Krivine [25] (1964), and re-discovered by Dubois, Prestel, Risler, and Stengle; or see, e.g., [27]) also admit continuous versions in certain cases: [11,13,17,30]. While psd quartic polynomials in $\mathbb{R}[X_1, X_2]$ are sums of squares of quadratic polynomials in $\mathbb{R}[X_1, X_2]$,[10] such (denominator-free) sum-of-squares representations must vary discontinuously [6]. On the other hand, for every $d \geqslant 0$, a continuously varying representation of psd $f \in \mathbb{R}[X_1]$ of degree $\leqslant d$ as sums of squares in $\mathbb{R}[X_1]$ was explicitly constructed by M. Ziegler in

[10] Hilbert [21]; modern expositions have been given by Choi and Lam [4], Swan (1993, unpublished), and Rajwade [28].

1988 (unpublished); see Cornelsen's master's thesis [5] for an exposition. That thesis also presents Prestel's continuously varying representation of (most of) those $f \in \mathbb{R}[X_1]$ that are sums of $2m$th powers in $\mathbb{R}(X_1)$ (for any $m \geqslant 1$) as sums of $2m$th powers. Finally, Cornelsen's thesis also presents T. Backmeister's (unpublished) proof of continuous variation in the weak isotropy of torsion quadratic forms over $\mathbb{R}(X_1, \ldots, X_n)$ (weak isotropy is presented in [27, §3.5]). Finally, Reznick [29] considered certain *sub*sets of $P_{nd}$, in which even $\mathbb{Q}$-*linear* variation is possible in Artin's theorem.

## 3. Taylor series of $C^\infty$ functions

For $m \geqslant 1$, let $U \subseteq \mathbb{R}^m$ be any open neighborhood of $\mathbf{0} := (0, \ldots, 0)$. $C^\infty(U)$ denotes the ring of functions $p : U \to \mathbb{R}$ whose partial derivatives of all orders exist on $U$. We write $\tau_C(p)$, or simply $\overline{p}$, for the Taylor (or Maclaurin) series of $p$ at $\mathbf{0}$ in the indeterminates $C$, viz.,

$$\sum_{\beta \in \mathbb{N}^m} \frac{1}{\beta_1! \cdots \beta_m!} \frac{\partial^{|\beta|} p}{\partial c_1^{\beta_1} \cdots \partial c_m^{\beta_m}}(\mathbf{0}) C^\beta.$$

$\tau_C$ is an $\mathbb{R}$-algebra homomorphism[11] $C^\infty(U) \to \mathbb{R}[[C]]$ (= the ring of formal, i.e., not necessarily convergent, power series in $C := (C_1, \ldots, C_m)$). Borel's lemma[12] states that $\tau_C$ is surjective; we shall not use this fact. We say that $p$ is *flat* at $\mathbf{0}$ if $p$ belongs to the (prime) ideal $\ker \tau_C$; i.e., if $\overline{p} = 0$. $\tau_C$ extends to an $\mathbb{R}$-algebra homomorphism $C^\infty(U)[X] \to \mathbb{R}[[C]][X]$ by

$$\tau_C\left(\sum_\alpha p_\alpha X^\alpha\right) = \sum_\alpha \overline{p_\alpha} X^\alpha \quad \big(p_\alpha \in C^\infty(U)\big).$$

For $m = 1$ we consider also *half*-open sets $U \subseteq \mathbb{R}^1$ of the form $U = [0, \delta)$, $\delta > 0$. Then $C^\infty([0, \delta))$ denotes the ring of functions $p : [0, \delta) \to \mathbb{R}$ all of whose derivatives exist on $[0, \delta)$, where, at $c_1 = 0$, we refer only to the *right*-hand derivatives of $p$. For $p \in C^\infty([0, \delta))$ we still have the "right-hand" Taylor series $\overline{p} \in \mathbb{R}[[C_1]]$ of $p$ at $c_1 = 0$.

## 4. An ordering on $\mathbb{R}((T))$, and its real closure

Let $T$ be a single indeterminate, and write a typical element $a := a(T) := \sum_{i=k}^\infty a_i T^i \in \mathbb{R}[[T]] \setminus \{0\}$, with $a_i \in \mathbb{R}$, $k \in \mathbb{N}$, and $a_k \neq 0$. We extend the unique field ordering $>$ on $\mathbb{R}$

[11] Here, the fact that $\tau_C(pq) = \tau_C(p)\tau_C(q)$, for $p, q \in C^\infty(U)$, is the Leibniz product-rule for higher-order partial derivatives.

[12] Émile Borel proved this for $m = 1$ in [2, p. 44]. I thank Prof. Armand Borel for this reference. I do not know whether É. Borel ever stated this result for $m > 1$. At any rate, proofs (for all $m \geqslant 1$) can be found, e.g., in [26, p. 30].

to a ring-order on $\mathbb{R}[[T]]$ by defining $a > 0 \Leftrightarrow a_k > 0$. We further extend $>$ (uniquely) to the field of fractions

$$\mathbb{R}((T)) := \left\{ \sum_{i=k}^{\infty} a_i T^i \;\middle|\; k \in \mathbb{Z}, a_i \in \mathbb{R} \right\},$$

and thence to the real closure

$$\mathcal{R} := \bigcup_{e=1}^{\infty} \mathbb{R}\bigl(\bigl(T^{1/e}\bigr)\bigr) \quad \bigl(= \text{the field of (formal) Puiseux series over } \mathbb{R}\bigr).$$

We write

$$\mathcal{V} := \bigcup_{e=1}^{\infty} \mathbb{R}\bigl[\bigl[T^{1/e}\bigr]\bigr] \quad \text{(a valuation ring)}, \quad \text{with maximal ideal}$$

$$\mathcal{M} := \bigcup_{e=1}^{\infty} T^{1/e} \cdot \mathbb{R}\bigl[\bigl[T^{1/e}\bigr]\bigr] = \bigl\{ a \in \mathcal{R} \bigm| a(\mathbf{0}) \text{ is defined and } a(\mathbf{0}) = 0 \bigr\}.$$

**Lemma 4.1.** *Suppose $S$ is an indeterminate, $\varepsilon > 0$, $q \in C^{\infty}([0, \varepsilon))$, $e \in \mathbb{N}^{+} := \{1, 2, \ldots\}$, $\bar{q} := \tau_S(q) \in \mathbb{R}[[S]]$, and $0 < \bar{q}(T^{1/e})$ $(\in \mathcal{V})$. Then there exists a $\delta \in (0, \varepsilon^e)$ such that for all $t \in (0, \delta)$, $q(t^{1/e}) > 0$.*

**Proof.** Write $\bar{q} = \sum_{i=k}^{\infty} q_i S^i$, with $k \in \mathbb{N}$, $q_i \in \mathbb{R}$, $q_k \neq 0$; then in fact $q_k > 0$. Introducing the variable $s = t^{1/e}$, we get

$$\lim_{t \to 0^+} \frac{q(t^{1/e})}{t^{k/e}} = \lim_{s \to 0^+} \frac{q'(s)}{ks^{k-1}} = \cdots = \lim_{s \to 0^+} \frac{q^{(k)}(s)}{k!} = q_k > 0. \qquad \square$$

**Lemma 4.2.** *The order $>$ on $\mathcal{R}$ restricts to a dense order on $\mathcal{M}$. In fact, for each $e \in \mathbb{N}^{+}$, $T^{1/e}\mathbb{R}[T^{1/e}]$ is (order-)dense in $T^{1/e}\mathbb{R}[[T^{1/e}]]$.*

**Sketch of Proof.** Given $a, b \in T^{1/e}\mathbb{R}[[T^{1/e}]]$ such that $a < b$, we are to find $\gamma_1, \gamma_2, \gamma_3 \in T^{1/e}\mathbb{R}[T^{1/e}]$ such that $\gamma_1 < a < \gamma_2 < b < \gamma_3$. For this, take truncations $\tilde{a}, \tilde{b} \in T^{1/e}\mathbb{R}[T^{1/e}]$ of $a$ and $b$ sufficiently long so that $\tilde{a} < \tilde{b}$, and then let $\gamma_1 = \tilde{a} - T$, $\gamma_2 = (\tilde{a} + \tilde{b})/2$, and $\gamma_3 = \tilde{b} + T$. $\square$

## 5. Impossibility of formal power series variation in solutions to Hilbert's 17th problem

Write

$$\mathbb{R}^2_{(0,0)} = \bigl\{ (T, \gamma), (-T, \gamma) \bigm| \gamma \in \mathcal{M} \bigr\} \cup \bigl\{ (0, T), (0, -T) \bigr\}.$$

Thus the elements $(\alpha_1, \alpha_2) \in \mathbb{R}^2_{(0,0)}$ are "standard" parametrizations of the algebroid curve germs of type $[0, \delta)$ at $(0, 0)$, in which one of the coordinates (the first one, whenever possible) is chosen to be $T$ or $-T$.

Recalling the ordering $>$ on $\mathcal{M} \subseteq \mathcal{R}$ (Section 4), it then makes sense to speak of those $(\alpha_1, \alpha_2) \in \mathbb{R}^2_{(0,0)}$ (or even those $(\alpha_1, \alpha_2) \in \mathcal{M} \times \mathcal{M}$) that satisfy some Boolean combination of inequalities $p(\alpha_1, \alpha_2) \geqslant 0$, for various $p \in \mathbb{R}[[C_1, C_2]]$. Specifically, we write

$$P_{(0,0)} = \big\{(T, \gamma) \bigm| \gamma \in \mathcal{M}, \gamma \geqslant 0\big\} \cup \big\{(-T, \gamma) \bigm| \gamma \in \mathcal{M}, \gamma \geqslant T^2/4\big\} \cup \big\{(0, T)\big\}. \tag{5.0.1}$$

Recalling (1.4.2), $P_{(0,0)}$ is therefore the set of algebroid curve germs in $\mathbb{R}^2_{(0,0)}$ that "stay in $P$." The next lemma is the algebroid-curve-germ analog of (1.4.2).

**Lemma 5.1.** *With $f$ as in* (1.4.1),

$$\begin{aligned} P_{(0,0)} &= \big\{(\alpha_1, \alpha_2) \in \mathbb{R}^2_{(0,0)} \bigm| \forall \xi \in \mathcal{M}, f(\alpha_1, \alpha_2; \xi) \geqslant 0\big\} \\ &= \big\{(\alpha_1, \alpha_2) \in \mathbb{R}^2_{(0,0)} \bigm| \forall \xi \in \mathcal{R}, f(\alpha_1, \alpha_2; \xi) \geqslant 0\big\}. \end{aligned}$$

**Proof.** As in (1.4.3), for any $(\alpha_1, \alpha_2) \in \mathbb{R}^2_{(0,0)}$,

$$\min_{\xi \in \mathcal{M}} f(\alpha_1, \alpha_2; \xi) = \min_{\xi \in \mathcal{R}} f(\alpha_1, \alpha_2; \xi) = \begin{cases} \alpha_2 & \text{if } \alpha_1 \geqslant 0, \\ \alpha_2 - \alpha_1^2/4 & \text{if } \alpha_1 \leqslant 0. \end{cases}$$

(These minima are achieved for $\xi = 0$ or $\xi = \sqrt{-\alpha_1/2} \in \mathcal{M} \subset \mathcal{R}$, respectively.[13]) Thus the nonnegativity of this minimum in $\mathcal{R}$ is equivalent to the condition that $(\alpha_1, \alpha_2) \in P_{(0,0)}$, by (5.0.1). □

**Theorem 5.2.** *With $f$ as in* (1.4.1), *there exist no $s \in \mathbb{N}$, no $p_1, \ldots, p_s \in \mathbb{R}[[C_1, C_2]]$, and no $g_1, \ldots, g_s, h \in \mathbb{R}[[C_1, C_2]][X_1]$ such that*

$$h(C_1, C_2; X_1)^2 f(C_1, C_2; X_1) = \sum_{i=1}^{s} p_i(C_1, C_2) g_i(C_1, C_2; X_1)^2, \tag{5.2.1}$$

$$\forall (\alpha_1, \alpha_2) \in P_{(0,0)},\ \forall i \in \{1, \ldots, s\}, \quad p_i(\alpha_1, \alpha_2) \geqslant 0 \in \mathcal{M}, \quad \textit{and} \tag{5.2.2}$$

$$h(0, 0; X_1) \neq 0 \in \mathbb{R}[X_1]. \tag{5.2.3}$$

The proof will use the following three lemmas.

[13] And $\sqrt{-\alpha_1/2} = \sqrt{1/2} \cdot T^{1/2}$ if $\alpha_1 < 0$, since we are assuming that $(\alpha_1, \alpha_2) \in \mathbb{R}^2_{(0,0)}$.

**Lemma 5.3.** *Suppose* $0 \neq p \in \mathbb{R}[[C_1, C_2]]$. *Then*

$$p(C_1, C_2) = C_1^b \cdot u(C_1, C_2) \prod_{j=1}^{J} q_j(C_1, C_2)^{e_j}, \tag{5.3.1}$$

*where* $b, J \in \mathbb{N}$; $e_j \in \mathbb{N}^+$; $u \in \mathbb{R}[[C_1, C_2]]^\times$ (*i.e.*, $u(0,0) \neq 0$); *and the* $q_j$ *are distinct irreducible "*$C_2$*-Weierstrass polynomials"* $\in \mathbb{R}[[C_1]][C_2]$ (*i.e.*, $q_j = C_2^{s_j} + \sum_{i=0}^{s_j-1} y_{ji}(C_1)C_2^i$, *some* $s_j \in \mathbb{N}^+$, $y_{ji} \in C_1 \cdot \mathbb{R}[[C_1]]$). *These data* (*except for the order of the* $q_j$ *and* $e_j$) *are uniquely determined by* $p$.

**Proof.** First choose $b$ maximal such that $C_1^b \mid p$ in (the UFD) $\mathbb{R}[[C_1, C_2]]$. Then $p' := p(C_1, C_2)/C_1^b \in \mathbb{R}[[C_1, C_2]]$ is "regular with respect to $C_2$," i.e., $p'(0, C_2) \neq 0 \in \mathbb{R}[[C_2]]$. Therefore we may apply the Weierstrass preparation theorem to $p'$ to get $p' = up''$, for uniquely determined $u \in \mathbb{R}[[C_1, C_2]]^\times$ and $C_2$-Weierstrass polynomial $p'' \in \mathbb{R}[[C_1]][C_2]$.

Now write $p'' = \prod q_j^{e_j}$ for (up to order) uniquely determined $C_2$-monic, pairwise non-associate, irreducible $q_j \in \mathbb{R}((C_1))[C_2]$ (= a UFD), and $e_j \in \mathbb{N}^+$. Actually, we have $q_j \in \mathbb{R}[[C_1]][C_2]$, by Gauss' lemma; and every $C_2$-monic factor $v \in \mathbb{R}[[C_1]][C_2]$ of a $C_2$-Weierstrass polynomial is again a $C_2$-Weierstrass polynomial. □

**Lemma 5.4.** *Suppose* $0 \neq p \in \mathbb{R}[[C_1, C_2]]$. *Then*

$$p(T, C_2) = T^b u(T, C_2) \prod_{k=1}^{K^+} \left(C_2 - \zeta_k^+\right)^{e_k^+} \prod_{l=1}^{L^+} \left[\left(C_2 - \eta_l^+\right)^2 + {\nu_l^+}^2\right]^{f_l^+}, \tag{5.4.1$^+$}$$

$$p(-T, C_2) = (-T)^b u(-T, C_2) \prod_{k=1}^{K^-} \left(C_2 - \zeta_k^-\right)^{e_k^-} \prod_{l=1}^{L^-} \left[\left(C_2 - \eta_l^-\right)^2 + {\nu_l^-}^2\right]^{f_l^-}, \tag{5.4.1$^-$}$$

*where* $b, K^\pm, L^\pm \in \mathbb{N}$; $e_k^\pm, f_l^\pm \in \mathbb{N}^+$; $u \in \mathbb{R}[[C_1, C_2]]^\times$; *the* $\zeta_k^+$ *are distinct elements of* $\mathcal{M}$, *as are the* $\zeta_k^-$; *and the ordered pairs* $(\eta_l^+, \nu_l^+)$ *are distinct elements of* $\mathcal{M} \times (\mathcal{M} \setminus \{0\})$, *as are the* $(\eta_l^-, \nu_l^-)$. *The above data are unique up to order. Finally, for each* $C_2$*-linear or irreducible* $C_2$*-quadratic factor* $F$ *displayed in* (5.4.1$^\pm$), *there exists a unique* $j \leqslant J$ *such that* $F \mid q_j(\pm T, C_2)$ *in* $\mathcal{R}[C_2]$; *moreover, for this* $j$, $e_j$ *equals the multiplicity* ($= e_k^\pm$ *or* $f_l^\pm$) *of* $F$ *in* (5.4.1$^\pm$). (*Here,* $J, q_j, e_j$ *are as in* (5.3.1).)

**Proof.** (5.4.1$^\pm$) comes from (5.3.1) upon replacing $C_1$ by $\pm T$, and then factoring each $q_j(\pm T, C_2)$ into irreducible factors $\in \mathcal{R}[C_2]$; these new factors will have $C_2$-degree $\leqslant 2$, since $\mathcal{R}$ is real closed.

The last statement of 5.4 (about multiplicities) follows from the separability of the extension $\mathcal{R}/\mathbb{R}((T))$. □

**Lemma 5.5$^\pm$.** *Given the notation in* 5.4, *we may* (*and shall*) *re-index the* $e_k^\pm$ *and* $\zeta_k^\pm$ *so that for some* $K'^+ \in \{0, 1, \ldots, K^+\}$ *and some* $K'^- \in \{0, 1, \ldots, K^-\}$,

$$e^{\pm}_1, \dots, e^{\pm}_{K'^{\pm}} \quad \textit{are odd},$$
$$e^{\pm}_{K'^{\pm}+1}, \dots, e^{\pm}_{K^{\pm}} \quad \textit{are even}, \quad \textit{and}$$
$$\zeta^{\pm}_1 < \cdots < \zeta^{\pm}_{K'^{\pm}}. \tag{5.5.1$^{\pm}$}$$

*Then*

$$p(\pm T, C_2) = (\pm T)^b u(\pm T, C_2)\big[\alpha^{\pm}(C_2)^2 + \beta^{\pm}(C_2)^2\big] \prod_{k=1}^{K'^{\pm}} \big(C_2 - \zeta^{\pm}_k\big), \tag{5.5.2$^{\pm}$}$$

*where* $\alpha^{\pm}, \beta^{\pm} \in \mathcal{V}[C_2]$.

**Proof.** (5.5.2$^{\pm}$) follows from (5.4.1$^{\pm}$) via the two-square identity:

$$\big(a^2 + b^2\big)\big(c^2 + d^2\big) = (ac + bd)^2 + (ad - bc)^2. \qquad \square$$

**Remark 5.6.** In 5.5$^{\pm}$ we may actually choose $\alpha^{\pm}, \beta^{\pm} \in \mathcal{M}[C_2]$ provided $L^{\pm} > 0$.

**Proof of 5.2.** Suppose $s, p_i, g_i, h$ satisfy (5.2.1–3); we seek a contradiction. We may assume each $p_i \neq 0$. Fix $i \leqslant s$, and apply 5.3 and 5.5$^{\pm}$ with $p_i$ in place of $p$, obtaining the odd $C_2$-roots $\zeta^{\pm}_{i,1} < \cdots < \zeta^{\pm}_{i,K'^{\pm}_i} \in \mathcal{M}$ of $p_i(\pm T, C_2) \in \mathbb{R}[[T, C_2]]$, for some $K'^{\pm}_i \in \mathbb{N}$.

**Claim 5.7.** $\forall k^- \in \{1, \dots, K'^{-}_i\}$, $\zeta^{-}_{i,k^-} \neq T^2/4$.

**Proof.** Otherwise, there exists a unique $j \leqslant J$ such that $q_j(-T, T^2/4) = 0$, and for this $j$, $e_j$ $(= e^{-}_{k^-})$ is odd (where $J, q_j, e_j, e^{-}_k$ are as in 5.3 and 5.4). Then $q_j$ must be the irreducible $C_2$-Weierstrass polynomial $C_2 - C_1^2/4$. Then there exists a unique $k^+ \in \{1, \dots, K'^{+}_i\}$ such that $\zeta^{+}_{i,k^+} = T^2/4$, and for this $k^+$, $e^{+}_{k^+}$ $(= e_j)$ is odd (again by 5.4). Pick $\gamma^{+}_1, \gamma^{+}_2 \in \mathcal{M}$ such that

$$0 \leqslant \gamma^{+}_1 \quad \text{and} \tag{5.7.1}$$
$$\zeta^{+}_{i,k'} < \gamma^{+}_1 < \frac{1}{4}T^2 = \zeta^{+}_{i,k^+} < \gamma^{+}_2 < \zeta^{+}_{i,k''} \tag{5.7.2}$$

for all $k' \in \{1, \dots, k^+ - 1\}$ and for all $k'' \in \{k^+ + 1, \dots, K'^{+}_i\}$; this is possible by 4.2 and (5.5.1$^{\pm}$). Then $p_i(T, \gamma^{+}_1)$ and $p_i(T, \gamma^{+}_2)$ have opposite signs, by (5.7.2) and (5.5.2$^{+}$); and $(T, \gamma^{+}_1)$ and $(T, \gamma^{+}_2)$ both belong to $P_{(0,0)}$, by (5.7.1–2) and (5.0.1). This violates (5.2.2), proving 5.7. $\square$

**Claim 5.8.** $\forall k^- \in \{1, \dots, K'^{-}_i\}$, $\zeta^{-}_{i,k^-} < T^2/4$.

**Proof.** Otherwise, $\zeta^-_{i,k^-} > T^2/4$ for some $k^-$, by 5.7; let $k^-$ be minimal with respect to this property. Pick $\gamma_1^-, \gamma_2^- \in \mathcal{M}$ such that

$$\zeta^-_{i,k'} < \frac{1}{4}T^2 \leqslant \gamma_1^- < \zeta^-_{i,k^-} < \gamma_2^- < \zeta^-_{i,k''} \tag{5.8.1}$$

for all $k' \in \{1, \ldots, k^- - 1\}$ and for all $k'' \in \{k^- + 1, \ldots, K_i'^-\}$ (4.2, (5.5.1$^-$)). Then $p_i(-T, \gamma_1^-)$ and $p_i(-T, \gamma_2^-)$ have opposite signs, by (5.8.1) and (5.5.2$^-$); and $(-T, \gamma_1^-)$ and $(-T, \gamma_2^-)$ both belong to $P_{(0,0)}$, by (5.8.1) and (5.0.1). This violates (5.2.2), proving 5.8. □

Returning to the proof of 5.2 itself, we "unfix" $i \in \{1, \ldots, s\}$ and choose $\gamma \in \mathcal{M}$ such that

$$\zeta^-_{i,K_i'^-} < \gamma < \frac{1}{4}T^2 \tag{5.9.1}$$

for all $i$ such that $K_i'^- > 0$; this is possible by 4.2 and 5.8. Pick $\delta \in \mathcal{M}$ such that $\delta \geqslant T^2/4$; thus for all $i$, $p_i(-T, \delta) \geqslant 0$, by (5.0.1), and (5.2.2). Since no $p_i(-T, \cdot)$ changes sign between $\gamma$ and $\delta$ (5.9.1), $p_i(-T, \gamma) \geqslant 0$, for all $i$. Therefore for all $\xi \in \mathcal{R}$,

$$\sum_i p_i(-T, \gamma) g_i(-T, \gamma; \xi)^2 \geqslant 0. \tag{5.9.2}$$

As for the left-hand side of (5.2.1), note that $h(-T, \gamma; X_1) \neq 0 \in \mathcal{V}[X_1]$, by (5.2.3) (since $\gamma(0) = 0$). Therefore $h(-T, \gamma; X_1)$ has only finitely many $X_1$-roots $\xi \in \mathcal{R}$. Therefore for all $l \in \mathbb{N}^+$ sufficiently large,

$$h\left(-T, \gamma; \sqrt{\frac{1}{2}}T^{1/2} + T^l\right) \neq 0 \in \mathcal{V}.$$

It remains to examine the sign of $f$ under these substitutions. Recalling (1.4.1),

$$\begin{aligned} f\left(-T, \gamma; \sqrt{\frac{1}{2}}T^{1/2} + T^l\right) &= \left[\left(\sqrt{\frac{1}{2}}T^{1/2} + T^l\right)^2 - \frac{1}{2}T\right]^2 + \left(\gamma - \frac{1}{4}T^2\right) \\ &= \left(\frac{1}{2}T + \sqrt{2}T^{l+\frac{1}{2}} + T^{2l} - \frac{1}{2}T\right)^2 + \left(\gamma - \frac{1}{4}T^2\right) \\ &= \left(2T^{2l+1} + 2\sqrt{2}T^{3l+\frac{1}{2}} + T^{4l}\right) + \left(\gamma - \frac{1}{4}T^2\right), \end{aligned}$$

an element of $\mathcal{M}$ that is negative for $l$ sufficiently large, by (5.9.1).

Thus the left-hand side of (5.2.1) is negative under these substitutions, violating (5.9.2), and proving 5.2. □

## 6. Proof of Theorem 1.5

Recalling Theorem 1.5, suppose $U \subseteq \mathbb{R}^2$ is an open neighborhood of $(0,0)$, $s \in \mathbb{N}$, $p_1, \ldots, p_s \in C^\infty(U)$, and $g_1, \ldots, g_s, h \in C^\infty(U)[X_1]$; and suppose that all of these satisfy (1.1.1″), (1.1.2″), and (1.1.3″). We seek a contradiction. Taking Taylor series at $(0,0)$ in (1.1.1″), we get

$$\bar{h}(C_1, C_2; X_1)^2 f(C_1, C_2; X_1) = \sum \overline{p_i}(C_1, C_2)\overline{g_i}(C_1, C_2; X_1)^2$$

in $\mathbb{R}[[C_1, C_2]][X_1]$; this is of the form (5.2.1). Note that $\bar{h}(0,0; X_1) \neq 0 \in \mathbb{R}[X_1]$ (5.2.3), by (1.1.3″). The desired contradiction will then follow from 5.2 once we verify (5.2.2) for $\overline{p_i}$; i.e., once we verify that

$$\forall(\alpha_1, \alpha_2) \in P_{(0,0)},\ \forall i, \quad \overline{p_i}(\alpha_1, \alpha_2) \geqslant 0 \in \mathcal{V}.$$

So suppose $\overline{p_i}(\alpha_1, \alpha_2) < 0$ for some $i$ and some $(\alpha_1, \alpha_2) \in P_{(0,0)}$; we seek a contradiction. Pick $e \in \mathbb{N}^+$ such that $\alpha_2 \in \mathbb{R}[[T^{1/e}]]$.

**Case 1.** $\alpha_1 = -T$. Then $\alpha_2 \geqslant T^2/4$ (5.0.1).

**Subcase 1(a).** $\alpha_2 = T^2/4$. Then $\exists \delta > 0$ such that $\forall t \in (0, \delta)$, $p_i(-t, t^2/4) < 0$, by 4.1, violating (1.1.2″).

**Subcase 1(b).** $\alpha_2 > T^2/4$. There exists an open interval $I \subseteq \mathcal{M}$ containing $\alpha_2$ such that $\forall \beta \in I$, $\overline{p_i}(-T, \beta(T)) < 0$, by (5.5.2$^-$). Shrinking $I$ if necessary, we may arrange that $\forall \beta \in I$, $\beta \geqslant T^2/4$ (i.e., $(-T, \beta) \in P_{(0,0)}$), since $\alpha_2 > T^2/4$. Pick $\beta \in I \cap \mathbb{R}[T^{1/e}]$ (4.2). Let $\gamma \in \mathbb{R}[S]$ ($S$ a new indeterminate) be such that $\beta(T) = \gamma(T^{1/e})$. For $s \in \mathbb{R}$, define $q(s) = p_i(-s^e, \gamma(s))$; then $q$ is $C^\infty$ in $s$. Moreover, $\overline{q}(S) = \overline{p_i}(-S^e, \gamma(S))$, by the chain-rule for higher-order partial derivatives. So

$$\bar{q}\big(T^{1/e}\big) = \overline{p_i}\big(-T, \gamma\big(T^{1/e}\big)\big) = \overline{p_i}\big(-T, \beta(T)\big) < 0.$$

Apply 4.1 to $q$; we get $\delta > 0$ such that

$$\forall t \in (0, \delta), \quad 0 > q\big(t^{1/e}\big) = p_i\big(-t, \beta(t)\big). \tag{6.0.1}$$

Apply 4.1 similarly to $\gamma(s) - s^{2e}/4 \in C^\infty(\mathbb{R})$: $\beta(t) > t^2/4$ for $t \in (0, \delta')$, some $\delta' \in (0, \delta)$; thus $(-t, \beta(t)) \in P$. This, together with (6.0.1), violates (1.1.2″).

**Cases 2 and 3.** $\alpha_1 = T$ and $\alpha_1 = 0$, respectively. These cases are easier than Case 1.

The three cases, taken together, prove 1.5. □

## 7. Postscript on work of Broglia and Pernazza

Theorem 1.4 above implies our earlier result [12] that analytic variation is impossible in Artin's theorem (i.e., that $B$ in Question 1.1 cannot be taken to be $\mathcal{O}(\mathbb{R}^m)$). The latter result had originally been deduced (easily) from the fact that the closed semianalytic set $P \subset \mathbb{R}^2$ in (1.4.2) is not "basic" closed semianalytic (at the origin $(0,0)$); i.e., for every open neighborhood $U \subseteq \mathbb{R}^2$ of $(0,0)$, and for every $s \in \mathbb{N}$, and for every choice of $p_1, \ldots, p_s \in \mathcal{O}(U)$:

$$P \cap U \neq \{c \in U \mid p_1(c) \geqslant 0, \ldots, p_s(c) \geqslant 0\}.$$[14]

We had at one time hoped that Theorem 1.4 could be deduced in a similar (easy) way from the following statement. Suppose $U \subseteq \mathbb{R}^2$ is an open neighborhood of $(0,0)$, $s \in \mathbb{N}$, $p_1, \ldots, p_s \in C^\infty(U)$, and

$$P \cap U = \{c \in U \mid p_1(c) \geqslant 0, \ldots, p_s(c) \geqslant 0\}; \tag{7.0.1}$$

then for at least one $i \leqslant s$, $p_i$ is flat at $(0,0)$ (recall Section 3). I asked Francesca Acquistapace and Fabrizio Broglia whether this statement is true; they (together with Ludovico Pernazza) succeeded in proving a more general version of this statement (but for open rather than closed semianalytic sets, with the obvious analog for the definition of "basic" open semianalytic):

**Theorem 7.1** [3]. *Let $S \subseteq \mathbb{R}^m$ be an open semianalytic set whose germ $S_{\mathbf{0}}$ at $\mathbf{0} := (0, \ldots, 0)$ is not basic open semianalytic, and suppose that for some open neighborhood $U \subseteq \mathbb{R}^m$ of $\mathbf{0}$, there are $\phi_i \in C^\infty(U)$ such that*

$$S \cap U = \{c \in U \mid \phi_1(c) > 0, \ldots, \phi_s(c) > 0\}.$$

*Then for at least one $i \leqslant s$, $\phi_i$ is flat at $\mathbf{0}$.*

Unfortunately, the fact for every choice of $p_i \in C^\infty(U)$ satisfying (7.0.1), $p_i$ must be flat at $(0,0)$ for some $i$, does not seem to lead easily to the results of this paper. For example, if some of the $p_i$ in (1.1.1″) are flat, then some of the summands on the right-hand side of (1.1.1″) will be flat; but this does not seem to lead to the desired contradiction, since the left-hand side could also be flat (depending on $h$).

[14] This easy deduction goes as follows. We take as known the fact that $P$ is not basic. Then $P_{nd}$ is not basic (for $d \geqslant 4$). Now if $B$ in (1.1) could be taken to be $\mathcal{O}(\mathbb{R}^m)$, we would conclude that $P = \{c \in \mathbb{R}^m \mid p_1(c) \geqslant 0, \ldots, p_s(c) \geqslant 0\}$, where the $p_i \in \mathcal{O}(\mathbb{R}^m)$ are given in (1.1.1–2) ($\subseteq$ by (1.1.2), and $\supseteq$ by (1.1.1)). I.e., $P$ is basic, after all—contradiction.

## Acknowledgments

The author thanks Francesca Acquistapace, William Adkins, Fabrizio Broglia, David Howard-Jones, Henri Lombardi, Ludovico Pernazza, Alexander Prestel, and Markus Tressl for helpful conversations and correspondence. The referee's suggestions were also helpful.

Part of the work on the problem solved in this paper was done during a 3-month stay by the author at the Model Theory of Fields program, held in the Spring of 1998 at the Mathematical Sciences Research Institute in Berkeley, CA. The author thanks MSRI, as well as Lou van den Dries, David Marker, Anand Pillay, Carol Wood, and the other members of the program committee.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