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FEYNMAN-KAC FORMULA FOR THE SOLUTION OF CAUCHY'S PROBLEM WITH TIME DEPENDENT LÉVY GENERATOR

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Abstract. We exploit an equivalence between the well-posedness of the homogeneous Cauchy problem for a time dependent Lévy generator L, and the well-posedness of the martingale problem for L, to obtain the Feynman-Kac representation of the solution of

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= L(t)\,u(t,x) + c(t,x)u(t,x), \quad t>0,\; x\in\mathbb{R}^d, \\ u(0,x) &= \varphi(x), \quad \varphi\in C_0^2\left(\mathbb{R}^d\right), \end{aligned}$$

where c is a bounded continuous function.

1. Introduction

Solutions of many partial differential equations can be represented as expectation functionals of stochastic processes known as Feynman-Kac formulas; see [8], [12] and [13] for pioneering work of these representations.

Feynman-Kac formulas are useful to investigate properties of partial differential equations in terms of appropriate stochastic models, as well as to study probabilistic properties of Markov processes by means of related partial differential equations; see e.g. [9] for the case of diffusion processes. Feynman-Kac formulas naturally arise in the potential theory for Schödinger equations [4], in systems of relativistic interacting particles with an electromagnetic field [11], and in mathematical finance [6], where they provide a bridge between the probabilistic and the PDE representations of pricing formulae. In recent years there has been a growing interest in the use of Lévy processes to model market behaviours (see e.g. [1] and references therein). This leads to consider Feynman-Kac formulas for Cauchy's problems with Lévy generators. Also, Feynman-Kac representations have been used recently to determine conditions under which positive solutions of semi-linear equations exhibit finite-time blow up, see [2] for the autonomous case and [14] and [17] for the nonautonomous one. A well-known reference on the interplay between the Cauchy problem for second-order differential operators, and the martingale problem for diffusion processes is [20]. In particular, in [20] it is proved that existence and uniqueness for the Cauchy problem associated to

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a diffusion operator is equivalent to existence and uniqueness of the martingale problem for the same operator.

The purpose of this paper is to prove that such an equivalence also holds for equations with Lévy generators and their corresponding martingale problems, and to provide a Feynman-Kac representation of the solution of the Cauchy problem.

Let $\varphi \in C_0^2\left(\mathbb{R}^d\right)$ be the space of C^2-functions $\varphi : \mathbb{R}^d \to \mathbb{R}$, vanishing at infinite. The Lévy generators we consider here are given by the expresion

$$\begin{aligned} L(t)\varphi(x) &= \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(t,x)\frac{\partial^2\varphi(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(t,x)\frac{\partial \varphi(x)}{\partial x_i} \\ &\quad + \int_{\mathbb{R}^d}\left(\varphi(x+y) - \varphi(x) - \frac{\langle y, \nabla\varphi(x)\rangle}{1+|y|^2}\right)\mu(t,x,dy), \end{aligned} \tag{1.1}$$

$t \geq 0$, $\varphi \in C_0^2\left(\mathbb{R}^d\right)$, where $\langle x,y\rangle = \sum_{i=1}^{d} x_i y_i$, $a : [0,\infty) \times \mathbb{R}^d \to S_d^+$. Here S_d^+ is the space of symmetric, non-negative definite, square real matrices of order d, $b : [0,\infty)\times\mathbb{R}^d \to \mathbb{R}^d$ and $\mu(t,x,\cdot)$ is a Lévy measure, that is, $\mu(t,x,\cdot)$ is a σ-finite measure on $\mathbb{R}^d$ that satisfies $\mu(t,x,\{0\}) = 0$ and $\int_{\mathbb{R}^d}|y|^2(1+|y|^2)^{-1}\mu(t,x,dy) < \infty$ for all $t \geq 0$ and $x \in \mathbb{R}^d$. These operators represent the infinitesimal generators of the most general stochastically continuous, $\mathbb{R}^d$-valued Markov processes with independent increments. We establish the equivalence (see Theorem 4.3 below) between the existence and uniqueness of solutions of the homogeneous Cauchy problem

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= L(t)\,u(t,x), \quad t > s \geq 0,\ x \in \mathbb{R}^d, \\ u(s,x) &= \varphi_s(x), \quad \varphi_s \in C_0^2\left(\mathbb{R}^d\right), \end{aligned} \tag{1.2}$$

and that of solutions of the martingale problem for $\{L(t)\}_{t\geq 0}$ on $C_0^2\left(\mathbb{R}^d\right)$. In order to achieve this, we use some ideas introduced in [20], several properties of the Howland semigroup (see e.g. [3]), and some results about the martingale problem given in [7]. By means of this equivalence, using the Howland evolution semigroup (whose definition is based on the classical idea of considering "time"to be a new variable in order to transform a nonautonomous Cauchy problem into an autonomous one) and Theorem 9.7, p. 298 from [5] (in the autonomous case), we are able to prove (see Theorem 5.1 below) that the solution of the Cauchy problem (5.1), given below, admits the representation

$$u(t,x) = E_x\left[\varphi(X(t))\exp\left(\int_0^t c(t-s,X(s))ds\right)\right],$$

where $X \equiv \{X(t)\}_{t\geq 0}$ is a strong Markov process on $\mathbb{R}^d$ with respect to the filtration $\mathcal{G}_t^X = \mathcal{F}_{t+}^X$, which is right continuous and quasi-left continuous, and solves the martingale problem for $\{L(t)\}_{t\geq 0}$ on $C_0^2\left(\mathbb{R}^d\right)$. Here E_x denotes the expectation with respect to the process X starting at x.

2. Non-negativity of Solutions

Let us consider the Lévy generator defined in (1.1). It is known (see e.g. [18]) that the space $C_c^{\infty}\left(\mathbb{R}^d\right)$ of continuous functions $g : \mathbb{R}^d \to \mathbb{R}$, having compact support and possessing continuous derivatives of all orders, is a core for the common domain $\mathcal{D}$ of the family of linear operators $\{L(t)\}_{t\geq 0}$, and that $C_0^2\left(\mathbb{R}^d\right) \subset \mathcal{D}$.

Notice that $\{L\left(t\right)\}_{t\geq 0}$ satisfies the positive maximun principle, namely,

$$\text{If } \varphi\left(x\right) = \sup_{y\in\mathbb{R}^d} \varphi\left(y\right) \geq 0,\ \varphi \in \mathcal{D}, \text{ then } L\left(t\right)\varphi\left(x\right) \leq 0 \text{ for all } t \geq 0. \tag{2.1}$$

In fact, if $\varphi\left(x\right) = \sup_{y\in\mathbb{R}^d} \varphi\left(y\right) \geq 0$, then $\nabla\varphi\left(x\right) = 0$ and $\left(\frac{\partial^2\varphi(x)}{\partial x_i \partial x_j}\right)_{1\leq i,j\leq d}$ is a symmetric non-positive definite matrix. Since

$$\begin{aligned} L\left(t\right)\varphi\left(x\right) &= \frac{1}{2} trace\left(a\left(t,x\right)H_{\varphi}\left(x\right)\right) + \left\langle b\left(t,x\right), \nabla\varphi\left(x\right)\right\rangle \\ &\quad + \int_{\mathbb{R}^d} \left(\varphi\left(x+y\right) - \varphi\left(x\right) - \frac{\left\langle y, \nabla\varphi\left(x\right)\right\rangle}{1+\left|y\right|^2}\right)\mu\left(t,x,dy\right), \end{aligned}$$

where $b\left(t,x\right) = \left(b_i\left(t,x\right)\right)_{1\leq i\leq d}$, $a\left(t,x\right) = \left(a_{ij}\left(t,x\right)\right)_{1\leq i,j\leq d}$ and $H_{\varphi}\left(x\right)$ is the Hessian matrix, we have

$$L\left(t\right)\varphi\left(x\right) \leq 0,$$

because $trace\left(AB\right) \leq 0$ if $A \in S_d^+$, $B \in S_d^-$, and by assumption $\varphi\left(x+y\right) - \varphi\left(x\right) \leq 0$ for all $y \in \mathbb{R}^d$.

We note also that (2.1) implies that $L\left(t\right)\varphi \leq 0$ for any nonnegative constant function $\varphi \in \mathcal{D}$. In fact, here $L\left(t\right)\varphi = 0$ for all functions which do not depend on space.

Let us now turn to the differential equation (1.2).

We are going to assume that the homogeneous Cauchy problem (1.2) is well-posed, and that the evolution family $\{U\left(t,s\right)\}_{t\geq s\geq 0}$ that solves (1.2) is an evolution family of contractions, i.e., a family of operators on $C_0\left(\mathbb{R}^d\right)$ such that

(i) $U(t,s) = U(t,r)U(r,s)$ and $U(t,t) = I$ for all $t \geq r \geq s \geq 0$ (here I is the identity operator).

(ii) For each $\varphi \in C_0\left(\mathbb{R}^d\right)$, the function $(t,s) \mapsto U(t,s)\varphi$ is continuous for $t \geq s \geq 0$.

(iii) $\|U(t,s)\| \leq 1$ for all $t \geq s \geq 0$.

Proposition 2.1. *Assume that u is a classical solution of (1.2) such that $u\left(t,\cdot\right) \in C_0^2\left(\mathbb{R}^d\right)$ for all $t \geq s \geq 0$, and that $0 \leq \varphi_s \equiv \varphi \in C_0^2\left(\mathbb{R}^d\right)$. Then $u\left(t,x\right) \geq 0$, $t \geq s \geq 0$, $x \in \mathbb{R}^d$.*

Proof. Suppose that for some $r > s$,

$$\inf_{x\in\mathbb{R}^d} u\left(r,x\right) = -c < 0.$$

Fix $T > r$ and let $\delta > 0$ be such that $\delta < \frac{c}{T-s}$. Define on $[s,T] \times \mathbb{R}^d$ the function

$$v_{\delta}\left(t,x\right) = u\left(t,x\right) + \delta\left(t-s\right),$$

which coincides with u when $t = s$. Clearly, this function has a negative infimun on $[s,T] \times \mathbb{R}^d$ and tends to the positive constant $\delta\left(t-s\right)$ as $|x| \to \infty$ and $t \in$

$(s,T]$ is fixed. Consequently, v_δ has a global (negative) minimun at some point $(t_0,x_0)\in(s,T]\times\mathbb{R}^d$. This implies that

$$\frac{\partial v_\delta}{\partial t}(t_0,x_0)\le 0,$$

and by the positive maximun principle (2.1),

$$L(t)\,v_\delta(t_0,x_0)\ge 0.$$

Therefore

$$\left(\frac{\partial v_\delta}{\partial t}-L(t)\,v_\delta\right)(t_0,x_0)\le 0.$$

On the other hand, since u solves (1.2),

$$\left(\frac{\partial v_\delta}{\partial t}-L(t)\,v_\delta\right)(t_0,x_0)=\left(\frac{\partial u}{\partial t}-L(t)\,u\right)(t_0,x_0)+\delta-(t-s)\,(L(t)\,\delta)\,(x_0)\ge\delta.$$

This contradiction proves the proposition. □

Corollary 2.2. *The differential equation (1.2) can have at most one solution u such that $u(t,\cdot)\in C_0^2\left(\mathbb{R}^d\right)$, $t\ge s\ge 0$.*

3. Markov Process Associated to the Evolution Family

From Proposition 2.1 we deduce that $U(t,s)$ is a nonnegative contraction on $C_0\left(\mathbb{R}^d\right)$ for $t\ge s\ge 0$, i.e., $\{U(t,s)\}_{t\ge s\ge 0}$ is an evolution family of contractions such that $U(t,s)\varphi\ge 0$ for each $\varphi\ge 0$. This follows from the fact that $C_0^2\left(\mathbb{R}^d\right)$ is dense in $C_0\left(\mathbb{R}^d\right)$, and that, by definition of $\{U(t,s)\}_{t\ge s\ge 0}$, for any $\varphi_s\in C_0^2\left(\mathbb{R}^d\right)$ the function

$$u(t,x)=U(t,s)\,\varphi_s(x)$$

is a solution of (1.2) such that $u(t,\cdot)\in C_0^2\left(\mathbb{R}^d\right)$. Thus, by Riesz representation theorem for nonnegative functionals on $C_0\left(\mathbb{R}^d\right)$ (see e.g. [21], p. 5), we have that for each $t\ge s\ge 0$ and $x\in\mathbb{R}^d$, there exists a measure $P(s,x,t,\cdot)$ on the Borel σ-field $\mathcal{B}\left(\mathbb{R}^d\right)$ on $\mathbb{R}^d$, such that $P\left(s,x,t,\mathbb{R}^d\right)\le 1$ and

$$U(t,s)\,\varphi(x)=\int_{\mathbb{R}^d}\varphi(y)\,P(s,x,t,dy)\,,\quad \varphi\in C_0\left(\mathbb{R}^d\right).$$

Since $\{U(t,s)\}_{t\ge s\ge 0}$ is an evolution family, in order to prove that $P(s,x,t,\Gamma)$, $t\ge s\ge 0$, $x\in\mathbb{R}^d$, $\Gamma\in\mathcal{B}\left(\mathbb{R}^d\right)$, is a transition probability function it suffices to show that $P(s,x,t,\cdot)$ is a probability measure for all $t\ge s\ge 0$, $x\in\mathbb{R}^d$. To this end, from the evolution family $\{U(t,s)\}_{t\ge s\ge 0}$ on $C_0\left(\mathbb{R}^d\right)$, we define the family of operators $\{T(t)\}_{t\ge 0}$ on $C_0\left([0,\infty)\times\mathbb{R}^d\right)$ by

$$(T(t)\,f)\,(r,x)=\begin{cases} U(r,r-t)\,f(r-t,x)\,, & r>t\ge 0,\ x\in\mathbb{R}^d,\\ U(r,0)\,f(0,x)\,, & 0\le r\le t.\end{cases}\tag{3.1}$$

Notice that $\{T(t)\}_{t\ge 0}$ is a positivity-preserving, strongly continuous semigroup of contractions on $C_0\left([0,\infty)\times\mathbb{R}^d\right)$, which is called the Howland semigroup of

$\{U(t,s)\}_{t\geq s\geq 0}$. Let us denote by A the infinitesimal generator of $\{T(t)\}_{t\geq 0}$, and define the operator $\hat{A}$ by

$$\left(\hat{A}f\right)(r,x) = -\frac{\partial f(r,x)}{\partial r} + L(r)\, f(r,x)\,, \quad r \geq 0,\ x \in \mathbb{R}^d, \tag{3.2}$$

whose domain is the space of functions f which are differentiable at t, such that $f(t,\cdot) \in C_0^2\left(\mathbb{R}^d\right)$ and $\hat{A}f \in C_0\left([0,\infty)\times\mathbb{R}^d\right)$. Let us denote by D the linear span of all functions $f \in C_0\left([0,\infty)\times\mathbb{R}^d\right)$ of the form

$$f(t,x) \equiv f_{\alpha,\varphi}(t,x) = \alpha(t)\,\varphi(x)\,, \quad \alpha \in C_c^1([0,\infty)) \quad \text{and} \quad \varphi \in C_0^2\left(\mathbb{R}^d\right), \tag{3.3}$$

where $C_c^1([0,\infty))$ is the space of continuous functions on $[0,\infty)$ having compact support and continuous first derivative. Then

$$(T(t)\, f_{\alpha,\varphi})(r,x) = \begin{cases} \alpha(r-t)\, U(r,r-t)\,\varphi(x)\,, & r > t \geq 0,\ x \in \mathbb{R}^d, \\ \alpha(0)\, U(r,0)\,\varphi(x)\,, & 0 \leq r \leq t, \quad x \in \mathbb{R}^d. \end{cases}$$

Thus,

$$\begin{aligned} (Af_{\alpha,\varphi})(r,x) &= \frac{d}{dt}(T(t)\, f_{\alpha,\varphi})(r,x)\mid_{t=0} \\ &= -\alpha'(r)\,\varphi(x) + \alpha(r)\, L(r)\,\varphi(x) \\ &= -\frac{\partial f_{\alpha,\varphi}(r,x)}{\partial r} + L(r)\, f_{\alpha,\varphi}(r,x) \\ &= \left(\hat{A}f_{\alpha,\varphi}\right)(r,x)\,. \end{aligned} \tag{3.4}$$

Since D is dense in $C_0\left([0,\infty)\times\mathbb{R}^d\right)$, this proves that the operator A is the closure of $\hat{A}\mid_D$.

Notice that the infinitesimal generator A of the Howland semigroup $\{T(t)\}_{t\geq 0}$ is conservative. Hence $\{T(t)\}_{t\geq 0}$ is a Feller semigroup, i.e. $\{T(t)\}_{t\geq 0}$ is a positivity-preserving, strongly continuous semigroup of contractions on $C_0\left([0,\infty)\times\mathbb{R}^d\right)$, whose infinitesimal generator is conservative. Therefore (see [7], Theorem 2.7, p. 169), there exists a time-homogeneous Markov process $\{Z(t)\}_{t\geq 0}$ with state space $[0,\infty)\times\mathbb{R}^d$ and sample paths in the Skorohod space $D_{[0,\infty)\times\mathbb{R}^d}[0,\infty)$, such that

$$(T(t)\, f)(r,x) = \int_{[0,\infty)\times\mathbb{R}^d} f(v,y)\,\widetilde{P}(t,(r,x),d(v,y))\,, \quad f \in C_0\left([0,\infty)\times\mathbb{R}^d\right),$$

where $\widetilde{P}\left(t,(r,x),\widetilde{\Gamma}\right)$, $t \geq 0$, $(r,x) \in [0,\infty)\times\mathbb{R}^d$, $\widetilde{\Gamma} \in \mathcal{B}\left([0,\infty)\times\mathbb{R}^d\right)$, is a transition probability function for $\{Z(t)\}_{t\geq 0}$. Recalling that

$$U(t,s)\,\varphi(x) = \int_{\mathbb{R}^d} \varphi(y)\, P(s,x,t,dy)\,, \quad \varphi \in C_0\left(\mathbb{R}^d\right),$$

we obtain, by definition of $\{T(t)\}_{t\geq 0}$, that for any $r > t \geq 0$,

$$\begin{aligned}
&\int_{[0,\infty)\times\mathbb{R}^d} f(\upsilon, y)\,\widetilde{P}(t,(r,x),d\upsilon dy) \\
&= \int_{\mathbb{R}^d} f(r-t,y)\,P(r-t,x,r,dy) \\
&= \int_{[0,\infty)\times\mathbb{R}^d} f(\upsilon,y)\,P(r-t,x,r,dy)\,\delta_{r-t}(d\upsilon), \quad f \in C_0\left([0,\infty)\times\mathbb{R}^d\right),
\end{aligned}$$

where δ_l denotes the measure with unit mass at l, and for any $0 \leq r \leq t$,

$$\begin{aligned}
&\int_{[0,\infty)\times\mathbb{R}^d} f(\upsilon, y)\,\widetilde{P}(t,(r,x),d\upsilon dy) \\
&= \int_{\mathbb{R}^d} f(0,y)\,P(0,x,r,dy) \\
&= \int_{[0,\infty)\times\mathbb{R}^d} f(\upsilon,y)\,P(0,x,r,dy)\,\delta_0(d\upsilon), \quad f \in C_0\left([0,\infty)\times\mathbb{R}^d\right).
\end{aligned}$$

Therefore

$$\widetilde{P}(t,(r,x),C\times\Gamma) = \begin{cases} P(r-t,x,r,\Gamma)\,\delta_{r-t}(C), & \text{if } r > t \geq 0, \\ P(0,x,r,\Gamma)\,\delta_0(C), & \text{if } 0 \leq r \leq t, \end{cases} \tag{3.5}$$

where $C \in \mathcal{B}([0,\infty))$ and $\Gamma \in \mathcal{B}\left(\mathbb{R}^d\right)$.

Since $\widetilde{P}(t,(r,x),\cdot)$ is a probability measure on $\left([0,\infty)\times\mathbb{R}^d, \mathcal{B}\left([0,\infty)\times\mathbb{R}^d\right)\right)$, it follows from (3.5) that $P(s,x,t,\cdot)$ is a probability measure on $\left(\mathbb{R}^d,\mathcal{B}\left(\mathbb{R}^d\right)\right)$. Thus, if there exists an evolution family of contractions $\{U(t,s)\}_{t\geq s\geq 0}$ on $C_0\left(\mathbb{R}^d\right)$ that solves the homogeneous Cauchy problem (1.2), then there also exists a transition probability function

$$P(s,x,t,\Gamma), \; t \geq s \geq 0, \; x \in \mathbb{R}^d, \; \Gamma \in \mathcal{B}\left(\mathbb{R}^d\right), \tag{3.6}$$

such that

$$U(t,s)\,\varphi(x) = \int_{\mathbb{R}^d} \varphi(y)\,P(s,x,t,dy), \; \varphi \in C_0\left(\mathbb{R}^d\right).$$

4. The Cauchy Problem and the Martingale Problem

Let $\{X(t)\}_{t\geq 0}$ be an $\mathbb{R}^d$-valued Markov process with sample paths in $D_{\mathbb{R}^d}[0,\infty)$, and with the transition probability function given by (3.6) (see [10], Theorem 3, p. 79).

Lemma 4.1. *For any $\varphi \in C_0^2\left(\mathbb{R}^d\right)$ and $s \geq 0$,*

$$M(t) = \varphi(X(t)) - \int_s^t L(\upsilon)\,\varphi(X(\upsilon))\,d\upsilon, \quad t \geq s,$$

is a martingale after time s with respect to the filtration $\mathcal{F}_t^X = \sigma(X(s),\; s \leq t)$.

Proof. Let $\varphi \in C_0^2\left(\mathbb{R}^d\right)$ and $t > r \geq s$. Then, almost surely,

$$\begin{aligned}
&E\left[M(t) \mid \mathcal{F}_r^X\right] \\
&= \int_{\mathbb{R}^d} \varphi(y)\, P(r, X(r), t, dy) - \int_r^t \int_{\mathbb{R}^d} L(\upsilon)\, \varphi(y)\, P(r, X(r), \upsilon, dy)\, d\upsilon \\
&\quad - \int_s^r L(\upsilon)\, \varphi(X(\upsilon))\, d\upsilon \\
&= U(t, r)\, \varphi(X(r)) - \int_r^t U(\upsilon, r)\, L(\upsilon)\, \varphi(X(r))\, d\upsilon - \int_s^r L(\upsilon)\, \varphi(X(\upsilon))\, d\upsilon \\
&= U(t, r)\, \varphi(X(r)) - \int_r^t \frac{\partial U(\upsilon, r)\, \varphi(X(r))}{\partial \upsilon} d\upsilon - \int_s^r L(\upsilon)\, \varphi(X(\upsilon))\, d\upsilon \\
&= U(t, r)\, \varphi(X(r)) - \left[U(t, r)\, \varphi(X(r)) - \varphi(X(r))\right] - \int_s^r L(\upsilon)\, \varphi(X(\upsilon))\, d\upsilon \\
&= \varphi(X(r)) - \int_s^r L(\upsilon)\, \varphi(X(\upsilon))\, d\upsilon = M(r).
\end{aligned}$$

□

Let $\{Y(t)\}_{t \geq 0}$ be the time-homogeneous Markov process on $[0, \infty)$ with transition probabilities

$$P(t, r, \Gamma) = \delta_{r-t}(\Gamma),\ \Gamma \in \mathcal{B}([0, \infty)),\ t, r \geq 0.$$

Then the Markov semigroup $\{V(t)\}_{t \geq 0}$ associated to $\{Y(t)\}_{t \geq 0}$ is given by

$$V(t)\, f(r) = f(r - t),\ r \geq t \geq 0,$$

and its generator Q satisfies

$$Qf = -f'$$

on the space $D(Q) = C_c^1([0, \infty))$.

Lemma 4.2. *Let $\{X(t)\}_{t \geq 0}$ and $\{Y(t)\}_{t \geq 0}$ be as above. Then*

$$\{(Y(t), X(t))\}_{t \geq 0}$$

is a Markov process with values in $[0, \infty) \times \mathbb{R}^d$, wich has the same distribution as the Markov process $\{Z(t)\}_{t \geq 0}$ whose semigroup is given by (3.1).

Proof. We consider the space of functions D defined in (3.3). Since the Markov process $\{Y(t)\}_{t \geq 0}$ is a solution of the martingale problem for Q and, by Lemma 4.1, the Markov process $\{X(t)\}_{t \geq 0}$ is a solution of the martingale problem for $\{L(t)\}_{t \geq 0}$ on $C_0^2\left(\mathbb{R}^d\right)$, we have (see [7], Theorem 10.1, p. 253) that the Markov process $\{(Y(t), X(t))\}_{t \geq 0}$ is a solution of the martingale problem for $\hat{A}\mid_D$, where $\hat{A}$ is the operator defined in (3.2). This implies (see [7], Theorem 4.1, p. 182) that the processes $\{Z(t)\}_{t \geq 0}$ and $\{(Y(t), X(t))\}_{t \geq 0}$ have the same finite-dimensional distributions. But, since these processes have sample paths in $D_{[0,\infty)\times\mathbb{R}^d}[0, \infty)$, it follows (see [7], Corollary 4.3, p. 186) that they have the same distribution on $D_{[0,\infty)\times\mathbb{R}^d}[0, \infty)$. □

Theorem 4.3. *There exists an evolution family of contractions* $\{U(t,s)\}_{t\geq s\geq 0}$ *on* $C_0(\mathbb{R}^d)$*, which is unique and solves the homogeneous Cauchy problem (1.2), if and only if, there exists a Markov process* $\{X(t)\}_{t\geq 0}$ *on* $\mathbb{R}^d$*, unique in distribution, that solves the martingale problem for* $\{L(t)\}_{t\geq 0}$ *on* $C_0^2(\mathbb{R}^d)$*.*

Proof. The necessity follows from Lemma 4.1. Assume that $\{X(t)\}_{t\geq 0}$ is a Markov process on $\mathbb{R}^d$, unique in distribution, that solves the martingale problem for $\{L(t)\}_{t\geq 0}$ on $C_0^2(\mathbb{R}^d)$. Let $P(s,x,t,\Gamma)$ be a transition function for $\{X(t)\}_{t\geq 0}$, and let

$$U(t,s)f(x)=\int_{\mathbb{R}^d}P(s,x,t,dy)f(y),\quad t\geq s\geq 0,\ f\in C_0\left(\mathbb{R}^d\right).$$

Then $\{U(t,s)\}_{t\geq s\geq 0}$ is a positivity-preserving evolution family of contractions on $C_0(\mathbb{R}^d)$. Let $\{T(t)\}_{t\geq 0}$ be the semigroup on $C_0\left([0,\infty)\times\mathbb{R}^d\right)$ defined from the evolution family $\{U(t,s)\}_{t\geq s\geq 0}$ on $C_0(\mathbb{R}^d)$ by (3.1). Then, for $\varphi\in C_0^2(\mathbb{R}^d)$ and $\alpha\in C_c^1([s,\infty))$ satisfying $\alpha(s)=1$,

$$u(t,x)\equiv U(t,s)\varphi(x)=\left(T(t-s)\alpha(\cdot)\varphi(\cdot)\right)(t,x),\ t\geq s.$$

Hence, due to (3.2), (3.4) and the strong continuity of the semigoup $\{T(t)\}_{t\geq 0}$,

$$\begin{aligned}
&\frac{\partial u(t,x)}{\partial t}\\
=&\lim_{h\to 0}\frac{(T(t-s+h)\alpha(\cdot)\varphi(\cdot))(t+h,x)-(T(t-s)\alpha(\cdot)\varphi(\cdot))(t,x)}{h}\\
=&\lim_{h\to 0}\frac{(T(t-s+h)\alpha(\cdot)\varphi(\cdot))(t+h,x)-(T(t-s+h)\alpha(\cdot)\varphi(\cdot))(t,x)}{h}\\
&+\lim_{h\to 0}\frac{(T(t-s+h)\alpha(\cdot)\varphi(\cdot))(t,x)-(T(t-s)\alpha(\cdot)\varphi(\cdot))(t,x)}{h}\\
=&\lim_{h\to 0}T(h)\left(\frac{(T(t-s)\alpha(\cdot)\varphi(\cdot))(t+h,x)-(T(t-s)\alpha(\cdot)\varphi(\cdot))(t,x)}{h}\right)\\
&+\hat{A}\left(T(t-s)\alpha(\cdot)\varphi(\cdot)\right)(t,x)\\
=&L(t)\left(T(t-s)\alpha(\cdot)\varphi(\cdot)\right)(t,x)=L(t)U(t,s)\varphi(x).
\end{aligned}$$

This shows clearly that the positivity-preserving evolution family of contractions $\{U(t,s)\}_{t\geq s\geq 0}$ solves the homogeneous Cauchy problem (1.2). Uniqueness follows from Corollary 2.2. □

Conditions on the functions $a:[0,\infty)\times\mathbb{R}^d\to S_d^+$, $b:[0,\infty)\times\mathbb{R}^d\to\mathbb{R}^d$ and the Lévy measure $\mu(t,x,\Gamma)$ that ensure existence of a unique Markov process $\{X(t)\}_{t\geq 0}$ on $\mathbb{R}^d$ that solves the martingale problem for $\{L(t)\}_{t\geq 0}$ on $C_c^\infty(\mathbb{R}^d)$, are given in [15] and [19]. In particular, in [19] it is proved that if $a:[0,\infty)\times\mathbb{R}^d\to S_d^+$ and $b:[0,\infty)\times\mathbb{R}^d\to\mathbb{R}^d$ are bounded and continuous, and $\mu(t,x,\cdot)$ is a Lévy measure such that $\int_\Gamma|y|^2\left(1+|y|^2\right)^{-1}\mu(t,x,dy)$ is bounded and continuous in (t,x) for every $\Gamma\in\mathcal{B}\left(\mathbb{R}^d\setminus\{0\}\right)$, then there exists a unique (in law) strong Markov process $\{X(t)\}_{t\geq 0}$ on $\mathbb{R}^d$ that solves the martingale problem for $\{L(t)\}_{t\geq 0}$ on $C_c^\infty(\mathbb{R}^d)$.

5. The Feynman-Kac Formula

Assume now that there exists a Markov process $X = \{X(t)\}_{t\geq 0}$ on $\mathbb{R}^d$ with respect to the filtration $\mathcal{G}_t^X = \mathcal{F}_{t^+}^X$, right continuous and quasi-left continuous, that solves the martingale problem for $\{L(t)\}_{t\geq 0}$ on $C_0^2(\mathbb{R}^d)$. Then, by Theorem 4.3, there exists a unique evolution family of contractions $\{U(t,s)\}_{t\geq s\geq 0}$ on $C_0(\mathbb{R}^d)$, that solves the homogeneous Cauchy problem (1.2).

We consider the Cauchy problem

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= L(t)\, u(t,x) + c(t,x)u(t,x)\,, \quad t>0,\ x\in\mathbb{R}^d, \\ u(0,x) &= \varphi(x)\,, \quad \varphi\in C_0^2(\mathbb{R}^d)\,, \end{aligned} \tag{5.1}$$

where c is a given bounded continuous function. Any classical solution $u(t,x)$ of (5.1) satisfies the integral equation

$$u(t,x) = U(t,0)\,\varphi(x) + \int_0^t U(t,r)(cu)(r,x)dr. \tag{5.2}$$

Any solution of the integral equation (5.2) is called a **mild solution** of the Cauchy problem (5.1).

Let us consider now the Cauchy problem

$$\begin{aligned} \frac{\partial u(t,(r,x))}{\partial t} &= Au(t,(r,x)) + c(r,x)u(t,(r,x))\,, \\ u(0,(r,x)) &= \widetilde{\varphi}(r,x), \end{aligned} \tag{5.3}$$

where $\widetilde{\varphi}\in C_0((0,\infty)\times\mathbb{R}^d)$, is such that $\widetilde{\varphi}(0,x)=\varphi(x)$, $x\in\mathbb{R}^d$. As before, the classical solution of (5.3) satisfies the integral equation

$$u(t,(s,x)) = T(t)\widetilde{\varphi}(s,x) + \int_0^t T(t-r)(cu(r,\cdot)(s,x)dr. \tag{5.4}$$

Let $u(t,(s,x))$ be a solution of the integral equation (5.4). Using the definition of $T(t)$ given in (3.1) and the assumption that $\widetilde{\varphi}(0,x)=\varphi(x)$, $x\in\mathbb{R}^d$, we obtain

$$\begin{aligned} u(t,(t,x)) &= T(t)\widetilde{\varphi}(t,x) + \int_0^t T(t-r)(cu(r,\cdot))(t,x)dr \\ &= U(t,0)\varphi(x) + \int_0^t U(t,r)(cu(r,\cdot))(r,x)dr. \end{aligned}$$

Hence, $u(t,x)\equiv u(t,(t,x))$, $t\geq 0$, satisfies the integral equation (5.2), i.e. $u(t,x)$ is the mild solution of the Cauchy problem (5.1).

Theorem 5.1. *Let $X=\{X(t)\}_{t\geq 0}$ be a strong Markov process on $\mathbb{R}^d$ with respect to the filtration $\mathcal{G}_t^X=\mathcal{F}_{t^+}^X$, right continuous and quasi-left continuous, that solves the martingale problem for $\{L(t)\}_{t\geq 0}$ on $C_0^2(\mathbb{R}^d)$. Then the classical solution $u(t,x)$ of (5.1) on $[0,T)\times\mathbb{R}^d$ admits the representation*

$$u(t,x) = E_x\left[\varphi(X(t))\exp\left(\int_0^t c(t-s,X(s))ds\right)\right]. \tag{5.5}$$

Proof. Let $\beta > \sup_{(r,x)\in[0,\infty)\times\mathbb{R}^d} c(r,x)$, and consider the function

$$V(r,x) = \beta - c(r,x), \qquad (r,x) \in [0,\infty) \times \mathbb{R}^d.$$

It follows from [5], Theorem 9.7, p. 298, that the classical solution $v(t,(r,x))$ of the Cauchy problem

$$\begin{aligned} \frac{\partial v\left(t,(r,x)\right)}{\partial t} &= Av\left(t,(r,x)\right) - V(r,x)v(t,\ (r,x)), \qquad (5.6) \\ v\left(0,(r,x)\right) &= \widetilde{\varphi}(r,x), \end{aligned}$$

with $t \in [0,T]$,and $(r,x) \in [0,\infty) \times \mathbb{R}^d$, admits the representation

$$\begin{aligned} v(t,(r,x)) &= \widetilde{E}_{(r,x)}\left[\widetilde{\varphi}(Z(t)) \exp\left(-\int_0^t V(Z(s))ds\right)\right] \qquad (5.7) \\ &= e^{-\beta t}\widetilde{E}_{(r,x)}\left[\widetilde{\varphi}(Z(t)) \exp\left(\int_0^t c(Z(s))ds\right)\right], \end{aligned}$$

where $\widetilde{E}_{(r,x)}$ denotes the expectation with respect to the process $\{Z\,(t)\}_{t\geq 0}$ starting at (r,x).

On the other hand, if $u(t,(r,x))$ is a classical solution of (5.3) for $t \in [0,T]$, $(r,x) \in [0,\infty) \times \mathbb{R}^d$, then clearly

$$e^{-\beta t}u(t,(r,x)), \quad t \in [0,T],\ (r,x) \in [0,\infty) \times \mathbb{R}^d,$$

is a classical solution of (5.6). Thus, using (5.7) and uniqueness of solutions of problem (5.3), we obtain that $u(t,(r,x))$ admits the representation

$$u(t,(r,x)) = \widetilde{E}_{(r,x)}\left[\widetilde{\varphi}(Z(t)) \exp\left(\int_0^t c(Z(s))ds\right)\right],$$

see [16], Theorem 1.2, p. 184. Thus, due to Lemma 4.2 and the definition of $\widetilde{\varphi}\,(\cdot,\cdot)$,

$$\begin{aligned} u(t,x) &\equiv u\left(t,(t,x)\right) \\ &= \widetilde{E}_{(t,x)}\left[\widetilde{\varphi}\left(Z(t)\right) \exp\left(\int_0^t c(Z(s))ds\right)\right] \\ &= E\left[\widetilde{\varphi}(Z(t)) \exp\left(\int_0^t c(Z(s))ds\,|Z(0) = (t,x)\right)\right] \\ &= E\left[\widetilde{\varphi}(Y(t),X(t)) \exp\left(\int_0^t c(Y(s),X(s))ds\,|(Y(0),X(0)) = (t,x)\right)\right] \\ &= E\left[\varphi(X(t)) \exp\left(\int_0^t c(t-s,X(s))ds\,|X(0) = x\right)\right] \\ &= E_x\left[\varphi(X(t)) \exp\left(\int_0^t c(t-s,X(s))ds\right)\right], \end{aligned}$$

and, since $u(t,x) \equiv u(t,(t,x))$ satisfies the integral equation (5.2), by uniqueness of the mild solution to (5.1) (see e.g. [16]), it follows that the solution $u(t,x)$ of (5.1) on $[0,T] \times \mathbb{R}^d$ admits the representation (5.5). □

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