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STOCHASTIC CALCULUS FOR GAUSSIAN PROCESSES AND APPLICATION TO HITTING TIMES

PEDRO LEI AND DAVID NUALART*

Abstract. In this paper we establish a change-of-variable formula for a class of Gaussian processes with a covariance function satisfying minimal regularity and integrability conditions. The existence of the local time and a version of Tanaka's formula are derived. These results are applied to a general class of self-similar processes that includes the bifractional Brownian motion. On the other hand, we establish a comparison result on the Laplace transform of the hitting time for a fractional Brownian motion with Hurst parameter $H < \frac{1}{2}$.

1. Introduction

There has been a recent interest in establishing change-of-variable formulas for a general class of Gaussian process which are not semimartingales, using techniques of Malliavin calculus. The basic example of such process is the fractional Brownian motion, and, since the pioneering work by Decreusefond and Üstünel [8], different versions of the Itô formula have been established (see the recent monograph by Biagini, Hu, Øksendal and Zhang [4] and the references therein).

In [1] the authors have considered the case of a Gaussian Volterra process of the form $X_t = \int_0^t K(t,s)dW_s$, where W is a Wiener process and $K(t,s)$ is a square integrable kernel satisfying some regularity and integrability conditions, and they have proved a change-of-variable formula for a class of processes which includes the fractional Brownian motion with Hurst parameter $H > \frac{1}{4}$. A more intrinsic approach based on the covariance function (instead of the kernel K) has been developed by Cheridito and Nualart in [5] for the fractional Brownian motion. In this paper an extended divergence operator is introduced in order to establish an Itô formula in the case of an arbitrary Hurst parameter $H \in (0,1)$. In [13], Kruk, Russo and Tudor have developed a stochastic calculus for a continuous Gaussian process $X = \{X_t, t \in [0,T]\}$ with covariance function $R(s,t) = E(X_tX_s)$ which has a bounded planar variation. This corresponds to the case of the fractional Brownian motion with Hurst parameter $H \geq \frac{1}{2}$. In [12] Kruk and Russo have extended the stochastic calculus for the Skorohod integral to the case of Gaussian processes with a singular covariance, which includes the case of the fractional Brownian motion with Hurst parameter $H < \frac{1}{2}$. The approach of [12] based on

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the duality relationship of Malliavin calculus and the introduction of an extended domain for the divergence operator is related with the method used in the present paper, although there are clear differences in the notation and basic assumptions.

In [15], Mocioalca and Viens have constructed the Skorohod integral and developed a stochastic calculus for Gaussian processes having a covariance structure of the form $E[|B_t - B_s|^2] \sim \gamma^2(|t-s|)$, where γ satisfies some minimal regularity conditions. In particular, the authors have been able to consider processes with a logarithmic modulus of continuity, and even processes which are not continuous.

The purpose of this paper is to extend the methodology introduced Cheridito and Nualart in [5] to the case of a general Gaussian process whose covariance function R is absolutely continuous in one variable and the derivative satisfies an appropriate integrability condition, without assuming that R has planar bounded variation. The main result is a general Itô's formula formulated in terms of the extended divergence operator, proved in Section 3. As an application we establish the existence of a local time in L^2 and a version of Tanaka's formula in Section 4. In Section 5 the results of the previous sections are applied to the case of a general class of self-similar processes that includes the bifractional Brownian motion with parameters $H \in (0,1)$ and $K \in (0,1]$ and the extended bifractional Brownian motion with parameters $H \in (0,1)$ and $K \in (1,2)$ such that $HK \in (0,1)$. Finally, using the stochastic calculus developed in Section 3, we have been able, in Section 6, to generalize the results by Decreusefond and Nualart (see [7]) on the distribution of the hitting time, to the case of a fractional Brownian motion with Hurst parameter $H < \frac{1}{2}$. More precisely, we prove that if the Hurst parameter is less than $\frac{1}{2}$, then the hitting time τ_a, for $a > 0$, satisfies $E(\exp(-\alpha\tau_a^{2H})) \geq e^{-a\sqrt{2\alpha}}$ for any $\alpha > 0$.

2. Preliminaries

Let $X = \{X_t, t \in [0,T]\}$ be a continuous Gaussian process with zero mean and covariance function $R(s,t) = E(X_t X_s)$, defined on a complete probability space $(\Omega, \mathcal{F}, P)$. For the sake of simplicity we will assume that $X_0 = 0$. Consider the following condition on the covariance function:

(H1) For all $t \in [0,T]$, the map $s \mapsto R(s,t)$ is absolutely continuous on $[0,T]$, and for some $\alpha > 1$,

$$\sup_{0 \leq t \leq T} \int_0^T \left| \frac{\partial R}{\partial s}(s,t) \right|^\alpha ds < \infty.$$

Our aim is to develop a stochastic calculus for the Gaussian process X, assuming condition (H1). In this section we introduce some preliminaries.

Denote by $\mathcal{E}$ the space of step functions on $[0,T]$. We define in $\mathcal{E}$ the scalar product

$$\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle_{\mathcal{H}} = R(t,s).$$

Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to this scalar product. The mapping $\mathbf{1}_{[0,t]} \to X_t$ can be extended to a linear isometry from $\mathcal{H}$ into the Gaussian subspace of $L^2(\Omega)$ spanned by the random variables $\{X_t, t \in [0,T]\}$. This Gaussian subspace is usually called the first Wiener chaos of the

Gaussian process X. The image of an element $\varphi \in \mathcal{H}$ by this isometry will be a Gaussian random variable denoted by $X(\varphi)$. For example, if $X = B$ is a standard Brownian motion, then the Hilbert space $\mathcal{H}$ is isometric to $L^2([0,T])$, and $B(\varphi)$ is the Wiener integral $\int_0^T \varphi_t dB_t$. A natural question is whether the elements of the space $\mathcal{H}$ can be indentified with real valued functions on $[0,T]$, and in this case, $X(\varphi)$ will be interpreted as the stochastic integral of the function φ with respect to the process X. For instance, in the case of the fractional Brownian motion with Hurst parameter $H \in (0,1)$, this question has been discussed in detail by Pipiras and Taqqu in the references [18, 19].

We are interested in extending the inner product $\langle \varphi, \mathbf{1}_{[0,t]} \rangle_{\mathcal{H}}$ to elements φ that are not necessarily in the space $\mathcal{H}$. Suppose first that $\varphi \in \mathcal{E}$ has the form

$$\varphi = \sum_{i=1}^{n} a_i \mathbf{1}_{[0,t_i]},$$

where $0 \le t_i \le T$. Then the inner product $\langle \varphi, \mathbf{1}_{[0,t]} \rangle_{\mathcal{H}}$ can be expressed as follows

$$\langle \varphi, \mathbf{1}_{[0,t]} \rangle_{\mathcal{H}} = \sum_{i=1}^{n} a_i R(t_i, t) = \sum_{i=1}^{n} a_i \int_0^{t_i} \frac{\partial R}{\partial s}(s,t) ds = \int_0^T \varphi(s) \frac{\partial R}{\partial s}(s,t) ds. \quad (2.1)$$

If β is the conjugate of α, i.e. $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, applying Hölder's inequality, we obtain

$$\left| \langle \varphi, \mathbf{1}_{[0,t]} \rangle_{\mathcal{H}} \right| = \left| \int_0^T \varphi(s) \frac{\partial R}{\partial s}(s,t) ds \right| \le \|\varphi\|_\beta \sup_{0 \le t \le T} \left(\int_0^T |\frac{\partial R}{\partial s}(s,t)|^\alpha ds \right)^{\frac{1}{\alpha}}.$$

Therefore, if (H1) holds, we can extend the inner product $\langle \varphi, \mathbf{1}_{[0,t]} \rangle_{\mathcal{H}}$ to functions $\varphi \in L^\beta([0,T])$ by means of formula (2.1), and the mapping $\varphi \to \langle \varphi, \mathbf{1}_{[0,t]} \rangle_{\mathcal{H}}$ is continuous in $L^\beta([0,T])$. This leads to the following definition.

Definition 2.1. Given $\varphi \in L^\beta([0,T])$ and $\psi = \sum_{j=1}^m b_j \mathbf{1}_{[0,t_j]} \in \mathcal{E}$, we set

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \sum_{j=1}^{m} b_j \int_0^T \varphi(s) \frac{\partial R}{\partial s}(s, t_j) ds.$$

In particular, this implies that for any φ and ψ as in the above definition,

$$\langle \varphi \mathbf{1}_{[0,t]}, \psi \rangle_{\mathcal{H}} = \int_0^t \varphi(s) d\langle \mathbf{1}_{[0,s]}, \psi \rangle_{\mathcal{H}}. \quad (2.2)$$

3. Stochastic Calculus for the Skorohod Integral

Following the argument of Alós, Mazet and Nualart in [1], in this section we establish a version of Itô's formula. In order to do this we first discuss the extended divergence operator for a continuous Gaussian stochastic process $X = \{X_t, t \in [0,T]\}$ with mean zero and covariance function $R(s,t) = E(X_t X_s)$, defined in a complete probability space $(\Omega, \mathcal{F}, P)$, satisfying condition (H1), and such that $X_0 = 0$. The Gaussian family $\{X(\varphi), \varphi \in \mathcal{H}\}$ introduced in the Section 2 is an isonormal Gaussian process associated with the Hilbert space $\mathcal{H}$, and we can construct the Malliavin calculus with respect to this process (see [17] and the references therein for a more complete presentation of this theory).

We denote by $\mathcal{S}$ the space of smooth and cylindrical random variables of the form

$$F = f(X(\varphi_1), \dots, X(\varphi_n)), \tag{3.1}$$

where $f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$ (f is an infinitely differentiable function which is bounded together with all its partial derivatives), and, for $1 \le i \le n$, $\varphi_i \in \mathcal{E}$. The derivative operator, denoted by D, is defined by

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(X(\varphi_1), \dots, X(\varphi_n))\varphi_i,$$

if $F \in \mathcal{S}$ is given by (3.1). In this sense, DF is an $\mathcal{H}$-valued random variable. For any real number $p \ge 1$ we introduce the seminorm

$$\|F\|_{1,p} = (E(|F|^p) + E(\|DF\|_{\mathcal{H}}^p))^{\frac{1}{p}},$$

and we denote by $\mathbb{D}^{1,p}$ the closure of $\mathcal{S}$ with respect to this seminorm. More generally, for any integer $k \ge 1$, we denote by D^k the kth derivative operator, and $\mathbb{D}^{k,p}$ the closure of $\mathcal{S}$ with respect to the seminorm

$$\|F\|_{k,p} = \left(E(|F|^p) + \sum_{j=1}^k E(\|D^j F\|_{\mathcal{H}^{\otimes j}}^p) \right)^{\frac{1}{p}}.$$

The divergence operator δ is introduced as the adjoint of the derivative operator. More precisely, an element $u \in L^2(\Omega; \mathcal{H})$ belongs to the domain of δ if there exists a constant c_u depending on u such that

$$|E(\langle u, DF \rangle_{\mathcal{H}})| \le c_u \|F\|_2,$$

for any smooth random variable $F \in \mathcal{S}$. For any $u \in \text{Dom}\delta$, $\delta(u) \in L^2(\Omega)$ is then defined by the duality relationship $E(F\delta(u)) = E(\langle u, DF \rangle_{\mathcal{H}})$, for any $F \in \mathbb{D}^{1,2}$ and in the above inequality we can take $c_u = \|\delta(u)\|_2$. The space $\mathbb{D}^{1,2}(\mathcal{H})$ is included in the domain of the divergence.

If the process X is a Brownian motion, then $\mathcal{H}$ is $L^2([0,T])$ and δ is an extension of the Itô stochastic integral. Motivated by this example, we would like to interpret $\delta(u)$ as a stochastic integral for u in the domain of the divergence operator. However, it may happen that the process X itself does not belong to $L^2(\Omega; \mathcal{H})$. For example, this is true if X is a fractional Brownian motion with Hurst parameter $H \le \frac{1}{4}$ (see [5]). For this reason, we need to introduce an extended domain of the divergence operator.

Definition 3.1. We say that a stochastic process $u \in L^1(\Omega; L^\beta([0,T]))$ belongs to the extended domain of the divergence $\text{Dom}^E\delta$ if

$$|E(\langle u, DF \rangle_{\mathcal{H}})| \le c_u \|F\|_2,$$

for any smooth random variable $F \in \mathcal{S}$, where c_u is some constant depending on u. In this case, $\delta(u) \in L^2(\Omega)$ is defined by the duality relationship

$$E(F\delta(u)) = E(\langle u, DF \rangle_{\mathcal{H}}),$$

for any $F \in \mathcal{S}$.

Note that the pairing $\langle u, DF\rangle_{\mathcal{H}}$ is well defined because of Definition 2.1.

In general, the domains $\mathrm{Dom}\delta$ and $\mathrm{Dom}^E\delta$ are not comparable because $u \in \mathrm{Dom}\delta$ takes values in the abstract Hilbert space $\mathcal{H}$ and $u \in \mathrm{Dom}^E\delta$ takes values in $L^\beta([0,T])$. In the particular case of the fractional Brownian motion with Hurst parameter $H < \frac{1}{2}$ we have (see [7])

$$\mathcal{H} = I_{T-}^{\frac{1}{2}-H}(L^2) \subset L^{\frac{1}{H}}([0,T]),$$

and assumption (H1) holds for any $\alpha < \frac{1}{2H-1}$. As a consequence, if β is the conjugate of α, then $\beta > \frac{1}{2H}$, so $\mathcal{H} \subset L^\beta([0,T])$ and $\mathrm{Dom}\delta \subset \mathrm{Dom}^E\delta$.

If u belongs to $\mathrm{Dom}^E\delta$, we will make use of the notation

$$\delta(u) = \int_0^T u_s \delta X_s,$$

and we will write $\int_0^t u_s\delta X_s$ for $\delta(u\mathbf{1}_{[0,t]})$, provided $u\mathbf{1}_{[0,t]} \in \mathrm{Dom}^E\delta$.

We are going to prove a change-of-variable formula for $F(t, X_t)$ involving the extended divergence operator. Let $F(t,x)$ be a function in $\mathcal{C}^{1,2}([0,T]\times\mathbb{R})$ (the partial derivatives $\frac{\partial F}{\partial x}$, $\frac{\partial^2 F}{\partial x^2}$ and $\frac{\partial F}{\partial t}$ exist and are continuous). Consider the following growth condition.

(H2) There exist positive constants c and $\lambda < \frac{1}{4}(\sup_{0\le t\le T} R(t,t))^{-1}$ such that

$$\sup_{0\le t\le T}\left(|F(t,x)| + |\frac{\partial F}{\partial x}(t,x)| + |\frac{\partial^2 F}{\partial x^2}(t,x)| + |\frac{\partial F}{\partial t}(t,x)|\right) \le c\exp(\lambda|x|^2). \quad (3.2)$$

Using the integrability properties of the supremum of a Gaussian process, condition (3.2) implies

$$E\left(\sup_{0\le t\le T}|F(t,X_t)|^2\right) \le c^2 E\exp(2\lambda \sup_{0\le t\le T}|X_t|^2) < \infty,$$

and the same property holds for the partial derivatives $\frac{\partial F}{\partial x}$, $\frac{\partial^2 F}{\partial x^2}$ and $\frac{\partial F}{\partial t}$. We need the following additional condition on the covariance function.

(H3) The function $R_t := R(t,t)$ has bounded variation on $[0,T]$.

Theorem 3.2. *Let F be a function in $\mathcal{C}^{1,2}([0,T]\times\mathbb{R})$ satisfying (H2). Suppose that $X = \{X_t, t\in[0,T]\}$ is a zero mean continuous Gaussian process with covariance function $R(t,s)$, such that $X(0) = 0$, satisfying (H1) and (H3). Then for each $t \in [0,T]$ the process $\{\frac{\partial F}{\partial x}(s,X_s)\mathbf{1}_{[0,t]}(s), 0 \le t \le T\}$ belongs to extended domain of the divergence $\mathrm{Dom}^E\delta$ and the following holds*

$$F(t,X_t) = F(0,0) + \int_0^t \frac{\partial F}{\partial s}(s,X_s)ds + \int_0^t \frac{\partial F}{\partial x}(s,X_s)\delta X_s + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(s,X_s)dR_s. \quad (3.3)$$

Proof. Suppose that G is a random variable of the form $G = I_n(h^{\otimes n})$, where I_n denotes the multiple stochastic integral of order n with respect to X and h is a step function in $[0,T]$. The set of all these random variables form a total subset of $L^2(\Omega)$. Taking into account Definition 3.1 of the extended divergence operator, it is enough to show that for any such G,

$$E(GF(t,X_t)) - E(GF(0,0)) - \int_0^t E(G\frac{\partial F}{\partial s}(s,X_s)ds - \frac{1}{2}\int_0^t E(G\frac{\partial^2 F}{\partial x^2}(s,X_s))dR_s$$
$$= E(\langle DG, \mathbf{1}_{[0,t]}(\cdot)\frac{\partial F}{\partial x}(\cdot, X.)\rangle_{\mathcal{H}}). \tag{3.4}$$

First we reduce the problem to the case where the function F is smooth in x. For this purpose we replace F by

$$F_k(t,x) = k\int_{-1}^1 F(t,x-y)\varepsilon(ky)dy,$$

where ε is a nonnegative smooth function supported by $[-1,1]$ such that $\int_{-1}^1 \varepsilon(y)dy = 1$. The functions F_k are infinitely differentiable in x and their derivatives satisfy the growth condition (3.2) with some constants c_k and λ_k.

Suppose first that G is a constant, that is, $n = 0$. The right-hand side of Equality (3.4) vanishes. On the other hand, we can write

$$E(GF_k(t,X_t)) = G\int_{\mathbb{R}} F_k(t,x)p(R_t,x)dx,$$

where $p(\sigma,y) = (2\pi\sigma)^{-1/2}\exp(-x^2/2\sigma)$. We know that $\frac{\partial p}{\partial\sigma} = \frac{1}{2}\frac{\partial^2 p}{\partial x^2}$. As a consequence, integrating by parts, we obtain

$$\begin{aligned}
&E(GF_k(t,X_t)) - GF(0,0) - G\int_0^t\int_{\mathbb{R}} \frac{\partial F_k}{\partial s}(s,x)p(R_s,x)dxds \\
&= \frac{1}{2}G\int_0^t \left(\int_{\mathbb{R}} F_k(s,x)\frac{\partial^2 p}{\partial x^2}(R_s,x)dx\right)dR_s \\
&= \frac{1}{2}G\int_0^t \left(\int_{\mathbb{R}} \frac{\partial^2 F_k}{\partial x^2}(s,x)p(R_s,x)dx\right)dR_s \\
&= \frac{1}{2}G\int_0^t E\left(\frac{\partial^2 F_k}{\partial x^2}(s,X_s)\right)dR_s,
\end{aligned}$$

which completes the proof of (3.4), when G is constant.

Suppose now that $n \geq 1$. In this case $E(G) = 0$. On the other hand, using the fact that the multiple stochastic integral I_n is the adjoint of the iterated derivative operator D^n we obtain

$$E(GF_k(t,X_t)) = E(I_n(h^{\otimes n})F_k(t,X_t)) = E\left(\langle h^{\otimes n}, \frac{\partial^n F_k}{\partial x^n}(t,X_t)\mathbf{1}_{[0,t]}^{\otimes n}\rangle_{\mathcal{H}^{\otimes n}}\right)$$
$$= E\left(\frac{\partial^n F_k}{\partial x^n}(t,X_t)\right)\langle h, \mathbf{1}_{[0,t]}\rangle_{\mathcal{H}}^n. \tag{3.5}$$

Note that $E(GF_k(t,X_t))$ is the product of two factors. Therefore, its differential will be expressed as the sum of two terms

$$d\left(E(GF_k(t,X_t))\right) = \langle h, \mathbf{1}_{[0,t]}\rangle_{\mathcal{H}}^n d\left(E(\frac{\partial^n F_k}{\partial x^n}(t,X_t))\right) + E(\frac{\partial^n F_k}{\partial x^n}(t,X_t))d\left(\langle h, \mathbf{1}_{[0,t]}\rangle_{\mathcal{H}}^n\right). \tag{3.6}$$

Using again the integration by parts formula and the fact that the Gaussian density satisfies the heat equation we obtain

$$\begin{aligned} d\left(E(\frac{\partial^n F_k}{\partial x^n}(t,X_t))\right) &= d\left(\int_{\mathbb{R}} \frac{\partial^n F_k}{\partial x^n}(t,x)p(R_t,x)dx\right) \\ &= \left(\int_{\mathbb{R}} \frac{\partial^{n+1} F_k}{\partial t \partial x^n}(t,x)p(R_t,x)dx\right)dt + \frac{1}{2}\left(\int_{\mathbb{R}} \frac{\partial^n F_k}{\partial x^n}(t,x)\frac{\partial^2 p}{\partial x^2}(R_t,x)dx\right)dR_t \\ &= \left(\int_{\mathbb{R}} \frac{\partial^{n+1} F_k}{\partial t \partial x^n}(t,x)p(R_t,x)dx\right)dt + \frac{1}{2}\left(\int_{\mathbb{R}} \frac{\partial^{n+2} F_k}{\partial x^{n+2}}(t,x)p(R_t,x)dx\right)dR_t \\ &= E(\frac{\partial^{n+1} F_k}{\partial t \partial x^n}(t,X_t))dt + \frac{1}{2}E(\frac{\partial^{n+2} F_k}{\partial x^{n+2}}(t,X_t))dR_t. \end{aligned} \tag{3.7}$$

Equation (3.5) applied to $\frac{\partial^2 F_k}{\partial x^2}$ and to $\frac{\partial F_k}{\partial t}$ yields

$$E(G\frac{\partial^2 F_k}{\partial x^2}(t,X_t)) = E(\frac{\partial^{n+2} F_k}{\partial x^{n+2}}(t,X_t))\langle h, \mathbf{1}_{[0,t]}\rangle_{\mathcal{H}}^n, \tag{3.8}$$

and

$$E(G\frac{\partial F_k}{\partial t}(t,X_t)) = E(\frac{\partial^{n+1} F_k}{\partial t \partial x^n}(t,X_t))\langle h, \mathbf{1}_{[0,t]}\rangle_{\mathcal{H}}^n, \tag{3.9}$$

respectively. Then, substituting (3.8), (3.9) and (3.7) into the first summand in the right-hand side of (3.6) we obtain

$$d(E(GF_k(t,X_t))) = E(G\frac{\partial F_k}{\partial t}(t,X_t))dt + \frac{1}{2}E(G\frac{\partial^2 F_k}{\partial x^2}(t,X_t))dR_t + E(\frac{\partial^n F_k}{\partial x^n}(t,X_t)))d\left(\langle h, \mathbf{1}_{[0,t]}\rangle_{\mathcal{H}}^n\right). \tag{3.10}$$

Therefore, to show (3.4), it only remains to check that

$$E(\langle DG, \mathbf{1}_{[0,t]}(\cdot)\frac{\partial F_k}{\partial x}(\cdot, X_\cdot)\rangle_{\mathcal{H}}) = n\int_0^t E(\frac{\partial^n F_k}{\partial x^n}(s,X_s))\langle h, \mathbf{1}_{[0,s]}\rangle_{\mathcal{H}}^{n-1} d\left(\langle h, \mathbf{1}_{[0,s]}\rangle_{\mathcal{H}}\right).$$

Using the fact that $DG = nI_{n-1}(h^{\otimes(n-1)})h$, we get

$$E(\langle DG, \mathbf{1}_{[0,t]}(\cdot)\frac{\partial F_k}{\partial x}(\cdot, X_\cdot)\rangle_{\mathcal{H}}) = n\langle h, \mathbf{1}_{[0,t]}(\cdot)E(I_{n-1}(h^{\otimes(n-1)})\frac{\partial F_k}{\partial x}(\cdot, X_\cdot))\rangle_{\mathcal{H}}.$$

Then, taking into account (2.2), we can write

$$d(E(\langle DG, \mathbf{1}_{[0,t]}(\cdot)\frac{\partial F_k}{\partial x}(\cdot, X_\cdot)\rangle_{\mathcal{H}})) = nE(I_{n-1}(h^{\otimes(n-1)})\frac{\partial F_k}{\partial x}(t,X_t))d\left(\langle h, \mathbf{1}_{[0,t]}\rangle_{\mathcal{H}}\right).$$

Finally, using again that I_{n-1} is the adjoint of the derivative operator yields

$$E(I_{n-1}(h^{\otimes(n-1)})\frac{\partial F_k}{\partial x}(t,X_t)) = E(\frac{\partial^n F_k}{\partial x^n}(t,X_t))\langle h, \mathbf{1}_{[0,t]}\rangle_{\mathcal{H}}^{n-1},$$

which allows us to complete the proof for the function F_k. Finally, it suffices to let k tend to infinity. □

4. Local Time

In this section, we will apply the Itô formula obtained in Section 3 to derive a version of Tanaka's formula involving the local time of the process X. In order to do this we first discuss the existence of the local time for a continuous Gaussian stochastic process $X = \{X_t, t \in [0,T]\}$ with mean zero defined on a complete probability space $(\Omega, \mathcal{F}, P)$, with covariance function $R(s,t)$. We impose the following additional condition which is stronger than (H3):

(H3a) The function $R_t = R(t,t)$ is increasing on $[0,T]$, and $R_t > 0$ for any $t > 0$.

The local time $L_t(x)$ of the process X (with respect to the measure induced by the variance function) is defined, if it exists, as the density of the occupation measure

$$m_t(B) = \int_0^t \mathbf{1}_B(X_s)dR_s, \quad B \in \mathcal{B}(\mathbb{R})$$

with respect to the Lebesgue measure. That is, for any bounded and measurable function g we have the occupation formula

$$\int_{\mathbb{R}} g(x)L_t(x)dx = \int_0^t g(X_s)dR_s.$$

Following the computations in [6] based on Wiener chaos expansions we can get sufficient conditions for the local time $L_t(x)$ to exists and to belong to $L^2(\Omega)$ for any fixed $t \in [0,T]$ and $x \in \mathbb{R}$. We denote by H_n the nth Hermite polynomial defined for $n \geq 1$ by

$$H_n(x) = \frac{(-1)^n}{n!} e^{-\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-\frac{x^2}{2}}),$$

and $H_0 = 1$. For $s, t \neq 0$ set $\rho(s,t) = \frac{R(s,t)}{\sqrt{R_s R_t}}$. For all $n \geq 1$ and $t \in [0,T]$ we define

$$\alpha_n(t) = \int_0^t \int_0^u \frac{|\rho(u,v)|^n}{\sqrt{n}} \frac{dR_v dR_u}{\sqrt{R_u R_v}},$$

and we introduce the following condition on the covariance function $R(s,t)$:

(H4) $\displaystyle\sum_{n=1}^{\infty} \alpha_n(T) < \infty.$

The following proposition is an extension of the result on the existence and Wiener chaos expansion of the local time for the fractional Brownian motion proved by Coutin, Nualart and Tudor in [6]. Recall that for all $\varepsilon > 0$ and $x \in \mathbb{R}$, $p(\varepsilon, x) = (2\pi\varepsilon)^{-1/2} \exp(-x^2/2\varepsilon)$.

Proposition 4.1. *Suppose that $X = \{X_t, t \in [0,T]\}$ is a zero mean continuous Gaussian process with covariance function $R(t,s)$, satisfying conditions (H3a) and (H4) and with $X(0) = 0$. Then, for each $a \in \mathbb{R}$, and $t \in [0,T]$, the random variables*

$$\int_0^t p(\varepsilon, X_s - a)dR_s$$

converge in $L^2(\Omega)$ to the local time $L_t(a)$, as ε tends to zero. Furthermore the local time $L_t(a)$ has the following Wiener chaos expansion

$$L_t(a) = \sum_{n=0}^{\infty} \int_0^t R_s^{-\frac{n}{2}} p(R_s, a) H_n(\frac{a}{\sqrt{R_s}}) I_n(\mathbf{1}_{[0,s]}^{\otimes n}) dR_s. \tag{4.1}$$

Proof. Applying Stroock's formula we can compute the Wiener chaos expansion of the random variable $p(\varepsilon, X_s - a)$ for any $s > 0$ as it has been done in [6], and we obtain

$$p(\varepsilon, X_s - a) = \sum_{n=0}^{\infty} \beta_{n,\varepsilon}(s) I_n(\mathbf{1}_{[0,s]}^{\otimes n}), \tag{4.2}$$

where

$$\beta_{n,\varepsilon}(s) = (R_s + \varepsilon)^{-\frac{n}{2}} p(R_s + \varepsilon, a) H_n(\frac{a}{\sqrt{R_s + \varepsilon}}). \tag{4.3}$$

From (4.2), integrating with respect to the measure dR_s, we deduce the Wiener chaos expansion

$$\int_0^t p(\varepsilon, X_s - a) dR_s = \sum_{n=0}^{\infty} \int_0^t \beta_{n,\varepsilon}(s) I_n(\mathbf{1}_{[0,s]}^{\otimes n}(\cdot)) dR_s. \tag{4.4}$$

We need to show this expression converges in $L^2(\Omega)$ to the right-hand side of Equation (4.1), denoted by $\Lambda_t(a)$, as ε tends to zero. For every n and s we have $\lim_{\varepsilon \to 0} \beta_{n,\varepsilon}(s) = \beta_n(s)$, where

$$\beta_n(s) = R_s^{-\frac{n}{2}} p(R_s, a) H_n(\frac{a}{\sqrt{R_s}}).$$

We claim that

$$|\beta_{n,\varepsilon}(s)| \leq c \frac{2^{n/2}}{n!} \Gamma\left(\frac{n+1}{2}\right) R_s^{-\frac{n+1}{2}}. \tag{4.5}$$

In fact, from the properties of Hermite polynomials it follows that

$$H_n(y) e^{-y^2/2} = (-1)^{[\frac{n}{2}]} 2^{n/2} \frac{2}{n! \sqrt{\pi}} \int_0^{\infty} s^n e^{-s^2} g(ys\sqrt{2}) ds,$$

where $g(r) = \cos r$ for n even, and $g(r) = \sin r$ for n odd. Thus, $|g|$ is dominated by 1, and this implies

$$|H_n(y) e^{-y^2/2}| \leq c \frac{2^{n/2}}{n!} \Gamma\left(\frac{n+1}{2}\right).$$

Substituting this estimate into (4.3) yields (4.5). The estimate (4.5) implies that, for any $n \geq 1$, the integral $\int_0^t \beta_n(s) I_n(\mathbf{1}_{[0,s]}^{\otimes n}) dR_s$ is well defined as a random variable in $L^2(\Omega)$, and it is the limit in $L^2(\Omega)$ of $\int_0^t \beta_{n,\varepsilon}(s) I_n(\mathbf{1}_{[0,s]}^{\otimes n}) dR_s$ as ε tends to zero. In fact, (4.5) implies that

$$\begin{aligned}\left\|\int_0^t \beta_n(s) I_n(\mathbf{1}_{[0,s]}^{\otimes n}) dR_s\right\|_2 &\le \int_0^t |\beta_n(s)| \left\|I_n(\mathbf{1}_{[0,s]}^{\otimes n})\right\|_2 dR_s \\ &\le c\sqrt{n!}2^{n/2}\Gamma\left(\frac{n+1}{2}\right)\int_0^t R_s^{-\frac{n+1}{2}} R_s^{\frac{n}{2}} dR_s \\ &\le= c\sqrt{n!}2^{n/2}\Gamma\left(\frac{n+1}{2}\right)\sqrt{R_t}.\end{aligned}$$

For $n = 0$, $\beta_{n,\varepsilon}(s) = p(R_s + \varepsilon, a)$, and clearly $\int_0^t p(R_s + \varepsilon, a) dR_s$ converges to $\int_0^t p(R_s, a) dR_s$ as ε tends to zero. In the same way, using dominated convergence, we can prove that

$$\lim_{\varepsilon\to 0}\left\|\int_0^t (\beta_{n,\varepsilon}(s) - \beta_n(s))\, I_n(\mathbf{1}_{[0,s]}^{\otimes n}) dR_s\right\|_2 = 0.$$

Set

$$\alpha_{n,\varepsilon} = E\left(\int_0^t \beta_{n,\varepsilon}(s) I_n(\mathbf{1}_{[0,s]}^{\otimes n}) dR_s\right)^2.$$

To show the convergence in $L^2(\Omega)$ of the series (4.4) to the right-hand side of (4.1) it suffices to prove that $\sup_\varepsilon \sum_{n=1}^\infty \alpha_{n,\varepsilon} < \infty$. Using (4.5) and Stirling formula we have

$$\begin{aligned}\alpha_{n,\varepsilon} &= \int_0^t\int_0^t E(I_n(\mathbf{1}_{[0,u]}^{\otimes n}) I_n(\mathbf{1}_{[0,v]}^{\otimes n}))\beta_{n,\varepsilon}(u)\beta_{n,\varepsilon}(v) dR_v dR_u \\ &= 2n!\int_0^t\int_0^u R(u,v)^n \beta_{n,\varepsilon}(u)\beta_{n,\varepsilon}(v) dR_v dR_u \\ &\le c\frac{2^n}{n!}\Gamma\left(\frac{n+1}{2}\right)^2 \int_0^t\int_0^u |R(u,v)|^n (R_u R_v)^{-\frac{n+1}{2}} dR_v dR_u \\ &\le c\int_0^t\int_0^u \frac{|\rho(u,v)|^n}{\sqrt{n}}\frac{dR_v dR_u}{\sqrt{R_v R_u}} \\ &= \alpha_n(t).\end{aligned}$$

Therefore, taking into account hypothesis (H4), we conclude that

$$\sup_\varepsilon \sum_{n=1}^\infty \alpha_{n,\varepsilon} < \sum_{n=1}^\infty \alpha_n(T) < \infty,$$

and this proves the convergence in $L^2(\Omega)$ of the series (4.4) to a limit denoted by $\Lambda_t(a)$.

Finally, we have to show that $\Lambda_t(a)$ is the local time $L_t(a)$. The above estimates are uniform in $a \in \mathbb{R}$. Therefore, we can deduce that the convergence of $\int_0^t p(\varepsilon, X_s - a) dR_s$ to $\Lambda_t(a)$ holds in $L^2(\Omega \times \mathbb{R}, P \times \mu)$, for any finite meausre μ. As a consequence, for any continuous function g with compact support we have that

$$\int_{\mathbb{R}}\left(\int_0^t p(\varepsilon, X_s - a) dR_s\right) g(a) da$$

converges in $L^2(\Omega)$, as ε tends to zero, to $\int_{\mathbb{R}} \Lambda_t(a)g(a)da$. Clearly, this sequence also converges to $\int_0^t g(X_s)dR_s$. Hence,

$$\int_{\mathbb{R}} \Lambda_t(a)g(a)da = \int_0^t g(X_s)dR_s,$$

which imples that $\Lambda_t(a)$ is a version of the local time $L_t(a)$ □

Corollary 4.2. *Condition (H4) holds if*

$$\int_0^T \int_0^T \frac{1-\ln(1-|\rho(u,v)|)}{\sqrt{R_vR_u}\cdot\sqrt{1-|\rho(u,v)|}}dR_v dR_u < \infty. \tag{4.6}$$

Proof. We can write

$$\sum_{n=1}^{\infty} \alpha_n(T) = \frac{1}{2}\int_0^T \int_0^T \varphi(|\rho(u,v)|)\frac{dR_v dR_u}{\sqrt{R_vR_u}},$$

where $\varphi(x) = \sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$. If we define $g(x) = \varphi(x)\sqrt{1-x}$ for every $x \in [0,1)$, then

$$g(x) = \sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}\sqrt{1-x} = \sum_{n(1-x)<1} \frac{x^n}{\sqrt{n}}\sqrt{1-x} + \sum_{n(1-x)\geq 1} \frac{x^n}{\sqrt{n}}\sqrt{1-x}$$
$$\leq \sum_{n(1-x)<1} \frac{x^n}{n} + \sum_{n=0}^{\infty} x^n(1-x) \leq 1-\ln(1-x),$$

and the result follows. □

Notice that the Wiener chaos expansion (4.1) can also we written as

$$L_t(a) = \sum_{n=0}^{\infty} I_n\left(\int_{s_1\vee\cdots\vee s_n}^t R_s^{-\frac{n}{2}} p(R_s,a)H_n(\frac{a}{\sqrt{R_s}})dR_s\right).$$

In the particular case $a = 0$, the Wiener chaos expansion of $L_t(0)$ can be written as

$$L_t(0) = \sum_{k=0}^{\infty} \int_0^t R_s^{-k-\frac{1}{2}} \frac{(-1)^k}{\sqrt{2\pi}2^k k!} I_{2k}(\mathbf{1}_{[0,s]}^{\otimes 2k})dR_s.$$

Using arguments of Fourier analysis, in [3] it is proved that if the covariance function $R(s,t)$ satisfies

$$\int_0^T \int_0^T (R_u + R_v - 2R(u,v))^{-\frac{1}{2}} dR_u dR_v < \infty, \tag{4.7}$$

then for any $t \in [0,T]$ the local time L_t of X exists and is square integrable, i.e. $E\left(\int_{\mathbb{R}} L_t^2(x)dx\right) < \infty$. We can write

$$\begin{aligned} R_u + R_v - 2R(u,v) &= R_u + R_v - 2\rho(u,v)\sqrt{R_uR_v} \\ &= (\sqrt{R_u} - \sqrt{R_v})^2 + 2\sqrt{R_uR_v}(1-\rho(u,v)) \\ &\geq 2\sqrt{R_uR_v}(1-\rho(u,v)). \end{aligned}$$

Therefore, condition (4.7) is implied by

$$\int_0^T \int_0^T \frac{1}{\sqrt[4]{R_u R_v} \cdot \sqrt{1-\rho(u,v)}} dR_u dR_v < \infty, \tag{4.8}$$

which can be compared with the above assumption (4.6). Notice that both conditions have different consequences. In fact, (4.8) implies $E\left(\int_{\mathbb{R}} L_t^2(x)dx\right) < \infty$; whereas (4.6) implies only that for each x, $E(L_t^2(x)) < \infty$.

We can now establish the following version of Tanaka formula.

Theorem 4.3. *Suppose that $X = \{X_t, 0 \le t \le T\}$ is a zero-mean continuous Gaussian process,with $X_0 = 0$, and such that the covariance function $R(s,t)$ satisfies conditions (H1), (H3a) and (H4). Let $y \in \mathbb{R}$. Then, for any $0 < t \le T$, the process $\{\mathbf{1}_{(y,\infty)}(X_s)\mathbf{1}_{[0,t]}(s), 0 \le s \le T\}$ belongs to $\operatorname{Dom}^E \delta$ and the following holds*

$$\delta\left[\mathbf{1}_{(y,\infty)}(X_\cdot)\mathbf{1}_{[0,t]}(\cdot)\right] = (X_t - y)^+ - (-y)^+ - \frac{1}{2}L_t(y).$$

Proof. Let $\varepsilon > 0$ and for all $x \in \mathbb{R}$ set

$$f_\varepsilon(x) = \int_{-\infty}^x \int_{-\infty}^v p(\varepsilon, z-y) dz dv.$$

Theorem 3.2 implies that

$$f_\varepsilon(X_t) = f_\varepsilon(0) + \int_0^t f_\varepsilon'(X_s)\delta X_s + \frac{1}{2}\int_0^t f_\varepsilon''(X_s) dR_s.$$

Then we have that $f_\varepsilon'(X_s)\mathbf{1}_{[0,t]}(s)$ converges to $\mathbf{1}_{(y,\infty)}(X_s)\mathbf{1}_{[0,t]}(s)$ in $L^2(\Omega\times\mathbb{R})$ and $f_\varepsilon(X_t)$ converges to $(X_t - y)^+$ in $L^2(\Omega)$. Finally, by Proposition 4.1, $\int_0^t f_\varepsilon''(X_s)dR_s$ converges to $L_t(y)$ in $L^2(\Omega)$. This completes the proof. □

5. Example: Self-Similar Processes

In this section, we are going to apply the results of the previous sections to the case of a self-similar centered Gaussian process X. Suppose that $X = \{X_t, t \ge 0\}$ is a stochastic process defined on a complete probability space $(\Omega, \mathcal{F}, P)$. We say that X is self-similar with exponent $H \in (0,1)$ if for any $a > 0$, the processes $\{X(at), t \ge 0\}$ and $\{a^H X(t), t \ge 0\}$ have the same distribution. It is well-known that *fractional Brownian motion* is the only H-self-similar centered Gaussian process with stationary increments. Suppose that $X = \{X_t, t \ge 0\}$ is a continuous Gaussian centered self-similar process with exponent H. Let $R(s,t)$ be the covariance function of X. To simplify the presentation we assume $E(X_1^2) = 1$. The process X satisfies the condition (H3a) because

$$R_t = R(t,t) = t^{2H} R(1,1) = t^{2H}.$$

The function R is homogeneous of order $2H$, that is, for $a > 0$ and $s,t \ge 0$, we have

$$R(as, at) = E(X_{as}X_{at}) = E(a^H X_s a^H X_t) = a^{2H} R(s,t).$$

For any $x \ge 0$, we define

$$\varphi(x) = R(1,x).$$

Notice that for any $x > 0$,

$$\varphi(x) = R(1, x) = x^{2H} R(\frac{1}{x}, 1) = x^{2H} \varphi\left(\frac{1}{x}\right).$$

On the other hand, applying Cauchy-Schwarz inequality we get that function φ satisfies $|\varphi(x)| \leq x^H$ for all $x \in [0, 1]$. The next proposition provides simple sufficient conditions on the function φ for the process X to satisfy the assumptions (H1) and (H4).

Proposition 5.1. *Suppose that $X = \{X_t, t \geq 0\}$ is a zero mean continuous self-similar Gaussian process with exponent of self-similarity H and covariance function $R(s, t)$. Let $\varphi(x) = R(1, x)$. Then*

(i) *(H1) holds on any interval $[0, T]$ for $\alpha > 1$ if $\alpha(2H - 1) + 1 > 0$ and φ is absolutely continuous and satisfies*

$$\int_0^1 |\varphi'(x)|^\alpha dx < \infty. \tag{5.1}$$

(ii) *(H4) holds on any interval $[0, T]$ if for some $\varepsilon > 0$ and for all $x \in [0, 1]$*

$$|\varphi(x)| \leq x^{H+\varepsilon}. \tag{5.2}$$

Proof. We first prove (i). We write

$$\int_0^T |\frac{\partial R}{\partial s}(s, t)|^\alpha ds = \int_0^t |\frac{\partial R}{\partial s}(s, t)|^\alpha ds + \int_t^T |\frac{\partial R}{\partial s}(s, t)|^\alpha ds.$$

For $s \leq t$, $R(s, t) = t^{2H} \varphi(\frac{s}{t})$ and $\frac{\partial R}{\partial s}(s, t) = t^{2H-1} \varphi'(\frac{s}{t})$. Applying (5.1) and the change of variables by $x = \frac{s}{t}$, we have

$$\begin{aligned}\int_0^t |\frac{\partial R}{\partial s}(s, t)|^\alpha ds &= \int_0^t t^{\alpha(2H-1)} |\varphi'\left(\frac{s}{t}\right)|^\alpha ds \\ &= t^{\alpha(2H-1)+1} \int_0^1 |\varphi'(x)|^\alpha dx. \end{aligned}\tag{5.3}$$

For $s > t$, $R(s, t) = s^{2H} \varphi(\frac{t}{s})$ and

$$\frac{\partial R}{\partial s}(s, t) = 2H s^{2H-1} \varphi(\frac{t}{s}) - s^{2H-2} t \varphi'(\frac{t}{s}).$$

Then,

$$\int_t^T |\frac{\partial R}{\partial s}(s, t)|^\alpha ds \leq C \left(\int_t^T s^{2H-1} |\varphi\left(\frac{t}{s}\right)|^\alpha ds + \int_t^T s^{2H-2} t |\varphi'\left(\frac{t}{s}\right)|^\alpha ds \right).$$

With the change of variables $x = \frac{t}{s}$ we can write

$$\begin{aligned}\int_t^T s^{2H-1} |\varphi\left(\frac{t}{s}\right)|^\alpha ds &\leq \|\varphi\|_\infty^\alpha t^{(2H-1)\alpha+1} \int_{\frac{t}{T}}^1 x^{\alpha(1-2H)-2} dx, \\ &= \frac{\|\varphi\|_\infty^\alpha}{\alpha(1-2H)-1} [t^{(2H-1)\alpha+1} - T^{(2H-1)\alpha+1}] \end{aligned}\tag{5.4}$$

and

$$\int_t^T s^{2H-2}t|\varphi'\left(\frac{t}{s}\right)|^\alpha ds \le t^{(2H-1)\alpha+1}\int_{\frac{t}{T}}^1 |\varphi'(x)|^\alpha x^{(2-2H)\alpha-2}dx$$
$$\le t^{(2H-1)\alpha+1}\left(\frac{t}{T}\right)^{(2-2H)\alpha-2}\int_{\frac{t}{T}}^1 |\varphi'(x)|^\alpha dx$$
$$\le \frac{t^{\alpha-1}}{T^{(2-2H)\alpha-2}}\int_{\frac{t}{T}}^1 |\varphi'(x)|^\alpha dx. \tag{5.5}$$

Now, (H1) follows from (5.3), (5.4) and (5.5).

In order to show (ii) we need to show that

$$\sum_{n=1}^\infty \alpha_n(T) = \sum_{n=1}^\infty \int_0^T \int_0^u \frac{1}{\sqrt{n}} \frac{|R(u,v)|^n}{(R_uR_v)^{\frac{n+1}{2}}} dR_v dR_u < \infty.$$

For any $0 < v < u$, we have $R(u,v) = u^{2H}\varphi(\frac{v}{u})$, and the change of variable $x = \frac{v}{u}$ yields

$$\alpha_n(T) = \frac{(2H)^2}{\sqrt{n}}\int_0^T\int_0^u |R(u,v)|^n(uv)^{H(1-n)-1}dvdu$$
$$= \frac{(2H)^2}{\sqrt{n}}\int_0^T\int_0^1 |R(1,x)|^n u^{2H-1}x^{H(1-n)-1}dvdu$$
$$= \frac{2HT^{2H}}{\sqrt{n}}\int_0^1 |\varphi(x)|^n x^{H(1-n)-1}dx$$
$$\le \frac{2HT^{2H}}{\sqrt{n}}\int_0^1 x^{n\varepsilon+H-1}dx$$
$$= \frac{2HT^{2H}}{\sqrt{n}}\frac{1}{n\varepsilon+H}.$$

Therefore, we have

$$\sum_{n=1}^\infty \alpha_n(T) \le \frac{2HT^{2H}}{\varepsilon}\sum_{n=1}^\infty n^{-\frac{3}{2}} < \infty. \tag{5.6}$$

This completes the proof of (ii). □

Example 5.2. The bifractional Brownian motion is a centered Gaussian process $X = \{B_t^{H,K}, t \ge 0\}$, with covariance

$$R(t,s) = R^{H,K}(t,s) = 2^{-K}((t^{2H}+s^{2H})^K - |t-s|^{2HK}), \tag{5.7}$$

where $H \in (0,1)$ and $K \in (0,1]$. We refer to Houdré and Villa [10] for the definition and basic properties of this process. Russo and Tudor [20] have studied several properties of the bifractional Brownian motion and analyzed the case $HK = \frac{1}{2}$. Tudor and Xiao [21] have derived small ball estimates and have proved a version of the Chung's law of the iterated logarithm for the bifractional Brownian motion. In [14], the authors have shown a decomposition of the bifractional Brownian motion with parameters H and K into the sum of a fractional Brownian motion with Hurst parameter HK plus a stochastic process with absolutely continuous trajectories. The stochastic calculus with respect to the bifractional Brownian motion has been recently developed in the references [13] and [12]. A Tanaka formula for the bifractional Brownian motion in the case $HK \le \frac{1}{2}$ by Es-Sebaiy and Tudor in [9]. A multidimensional Itô's formula for the bifractional Brownian motion has been established in [2].

Note that, if $K = 1$ then $B^{H,1}$ is a fractional Brownian motion with Hurst parameter $H \in (0,1)$, and we denote this process by B^H. Bifractional Brownian motion is a self-similar Gaussian process with non-stationary increment if K is not equal to 1.

Set

$$\varphi(x) = 2^{-K}((1+x^{2H})^K - (1-x)^{2HK}).$$

Then

$$\varphi'(x) = 2^{1-K}HK[x^{2HK-1}(1+x^{2H})^K + (1-x)^{2HK-1}],$$

which implies that (i) in Proposition 5.1 holds α such that $\alpha(2HK-1) > -1$. Notice that

$$\varphi(x) \le \frac{1}{2^K}[1 + x^{2H} - (1-x)^{2H}]^K. \tag{5.8}$$

Then, if $2H \le 1$

$$1 + x^{2H} - (1-x)^{2H} \le 2x^{2H}, \tag{5.9}$$

and when $2H > 1$,

$$1 + x^{2H} - (1-x)^{2H} \le x + x^{2H} \le 2x. \tag{5.10}$$

From the inequalities (5.8), (5.9) and (5.10) we obtain

$$\frac{\varphi(x)}{x^{HK}} = \frac{(1+x^{2H})^K - (1-x)^{2HK}}{2^K x^{HK}} \le x^{\min(H,1-H)K}. \tag{5.11}$$

Then condition (ii) in Proposition 5.1 holds with $\varepsilon = \min(H, 1-H)K$. As a consequence, the results in Sections 3, 4 and 5 hold for the bifractional Brownian motion.

Bardina and Es-Sebaiy considered in [2] an extension of bifractional Brownian motion with parameters $H \in (0,1)$, $K \in (1,2)$ and $HK \in (0,1)$ with covariance function (5.7). By the same arguments as above, Proposition 5.1 holds in this case with $\varepsilon = \min(H, 1-H)K$ in condition (ii). Thus, the results in Sections 3, 4 and 5 hold for this extension of the bifractional Brownian motion.

6. Hitting Times

Suppose that $X = \{X_t, t \ge 0\}$ is a zero mean continuous Gaussian process with covariance function $R(t,s)$, satisfying (H1) and (H3) on any interval $[0,T]$. We also assume that $X(0) = 0$. Moreover, we assume the following conditions:

(H5) $\limsup_{t\to\infty} X_t = +\infty$ almost surely.
(H6) For any $0 \le s < t$, we have $E(|X_t - X_s|^2) > 0$.
(H7) For any continuous function f ,

$$r \mapsto \int_0^t f(s)\frac{\partial R}{\partial s}(s,r)ds$$

is continuous on $[0,\infty)$.

For any $a > 0$, we denote by τ_a the hitting time defined by

$$\tau_a = \inf\{t \geq 0, X_t = a\} = \inf\{t \geq 0, X_t \geq a\}. \tag{6.1}$$

The map $a \mapsto \tau_a$ is left continuous and increasing with right limits.

We are interested in the distribution of the random variable τ_a. The explicit form of this distribution is known only in some special cases such as the standard Brownian motion. In this case the Laplace transform of the hitting time τ_a is given by

$$E(e^{-\alpha\tau_a}) = e^{-a\sqrt{2\alpha}},$$

for all $\alpha > 0$. This can be proved, for instance, using the exponential martingale

$$M_t = e^{\lambda X_t - \frac{1}{2}\lambda^2 t},$$

and Doob's optional stopping theorem. In the general case, the exponential process

$$M_t = \exp(\lambda X_t - \frac{1}{2}\lambda^2 R_t). \tag{6.2}$$

is no longer martingale. However, if we apply (3.2) for the divergence integral, we have

$$M_t = 1 + \lambda\delta(M\mathbf{1}_{[0,t]}) = 1 + \lambda\delta_t(M). \tag{6.3}$$

Substituting t by τ_a and taking the expectation in Equation (6.3), Decreusefond and Nualart have established in [7] an inequality of the form $E(e^{-\alpha R_{\tau_a}}) \leq e^{-a\sqrt{2\alpha}}$, assuming that the partial derivative of the covariance $\frac{\partial R}{\partial s}(t,s)$ is nonnegative and continuous. This includes the case of the fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. The purpose of this section is derive the converse inequality in the singular case where the partial derivative of the covariance is not continuous, assuming $\frac{\partial R}{\partial s}(t,s) \leq 0$ for $s < t$ (which includes the case of the fractional Brownian motion with Hurst parameter $H < \frac{1}{2}$), completing the analysis initiated in [7].

As in the case of the Brownian motion we would like to substitute t by τ_a in both sides of Equation (6.3) and then take the mathematical expectation in both sides of the equality. It is convenient to introduce also an integral in a, and, following the approach developed in [7], we claim that the following result holds.

Proposition 6.1. *Suppose X satisfies (H1), (H3), (H5), (H6) and (H7), then*

$$\int_0^\infty E(M_{\tau_a})\psi(a)da = c - \lim_{\delta\to 0} \lambda E \int_0^{S_T} d\tau_y \int_0^1 \psi(y)d\eta \int_0^\infty p_\delta((\tau_{y^+} \wedge T)\eta + \tau_y(1-\eta) - s)M_s \frac{\partial R}{\partial s}(\tau_y, s)ds, \tag{6.4}$$

where p is an infinitely differentiable function with support on $[-1,1]$ such that $\int_{-1}^1 p(x)dx = 1$, $\psi(x)$ be a nonnegative smooth function with compact support contained in $(0,\infty)$ such that $\int_0^\infty \psi(a)da = c$ and we use the notation $p_\varepsilon(x) = \frac{1}{\varepsilon}p(\frac{x}{\varepsilon})$.

Before proving this proposition we need several technical lemmas. The first lemma is an integration by parts formula, and it is a consequence of the definition of the extended divergence operator given in Definition 2.1.

Lemma 6.2. *For any $t > 0$ and any random variable of the form $F = f(X_{t_1}, \dots, X_{t_n})$, where f is an inifinitely differentiable function which is bounded together with all its partial derivatives, we have*

$$E(F\delta_t(M)) = E\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(X_{t_1}, \dots, X_{t_n}) \int_0^t M_s \frac{\partial R}{\partial s}(t_i, s) ds\right), \tag{6.5}$$

where $\delta_t(M)$ is given in Equation (6.3).

Proof. Using the Definition 2.1 of the extended divergence operator and Equation (2.1) we can write

$$\begin{aligned} E(F\delta_t(M)) &= E(\langle DF, M\mathbf{1}_{[0,t]}\rangle_{\mathcal{H}}) \\ &= E\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(X_{t_1}, \dots, X_{t_n}) \langle \mathbf{1}_{[0,t_i]}, M\mathbf{1}_{[0,t]}\rangle_{\mathcal{H}}\right) \\ &= E\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(X_{t_1}, \dots, X_{t_n}) \int_0^t M_s \frac{\partial R}{\partial s}(t_i, s) ds\right), \end{aligned}$$

which completes the proof of the lemma. □

For any $a > 0$, we know that $P(\tau_a < \infty) = 1$ by condition (H5). Set

$$S_t = \sup_{s \in [0,t]} X_s.$$

We know that for all $t > 0$, S_t belongs to $\mathbb{D}^{1,2}$ and $DS_t = \mathbf{1}_{[0,\tau_{S_t}]}$ (see [7] and [11]). Following the approach developed in [7], we introduce a regularization of the hitting time τ_a, and we establish its differentiability in the sense of Malliavin calculus.

Lemma 6.3. *Suppose that φ is a nonnegative smooth function with compact support in $(0, \infty)$ and define for any $T > 0$,*

$$Y = \int_0^\infty \varphi(a)(\tau_a \wedge T) da.$$

The random variable Y belongs to the space $\mathbb{D}^{1,2}$, and

$$D_r Y = -\int_0^{S_T} \varphi(y) \mathbf{1}_{[0,\tau_y]}(r) d\tau_y.$$

Proof. First, it is clear that Y is bounded because φ has compact support. On the other hand, for any $r > 0$, we can write

$$\{\tau_a > r\} = \{S_r < a\}.$$

Apply Fubini's theorem, we have

$$\begin{aligned} Y &= \int_0^\infty \varphi(a) \left(\int_0^{\tau_a \wedge T} d\theta \right) da = \int_0^\infty \int_0^\infty \varphi(a) \mathbf{1}_{\{\theta < \tau_a \wedge T\}} d\theta da \\ &= \int_0^\infty \int_0^\infty \varphi(a) \mathbf{1}_{\{\theta < \tau_a\}} \mathbf{1}_{\{\theta < T\}} da d\theta = \int_0^T \int_0^\infty \varphi(a) \mathbf{1}_{\{S_\theta < a\}} da d\theta \\ &= \int_0^T \int_{S_\theta}^\infty \varphi(a) da d\theta. \end{aligned}$$

The function $\psi(x) = \int_x^\infty \varphi(a)da$ is continuously differentiable with a bounded derivative, so $\psi(S_\theta) \in \mathbb{D}^{1,2}$ for any $\theta \in [0, T]$ because $S_\theta \in \mathbb{D}^{1,2}$ (see, for instance, [17]). Finally, we can show that $Y = \int_0^T \psi(S_\theta)d\theta$ belongs to $\mathbb{D}^{1,2}$ approximatig the integral by Riemann sums. Hence, taking the Malliavin derivative of Y, we obtain

$$\begin{aligned} D_r Y &= -\int_0^T \varphi(S_\theta) D_r S_\theta d\theta \\ &= -\int_0^T \varphi(S_\theta) \mathbf{1}_{[0, \tau_{S_\theta}]}(r) d\theta = -\int_0^{S_T} \varphi(y) \mathbf{1}_{[0,\tau_y]}(r) d\tau_y, \end{aligned}$$

where the last equality holds by changing variable $S_\theta = y$, which is equivalent to $\theta = \tau_y$. □

The following lemma provides an explicit formula for the expectation $E(p(Y)\delta_t(M))$, where p is a smooth function with compact support.

Lemma 6.4. *Suppose X satisfies (H1), (H3), (H5), (H6) and (H7). Then, for any infinitely differentiable function p with compact support,*

$$E(p(Y)\delta_t(M)) = -E\left(\int_0^t M_s p'(Y) \int_0^{S_T} \varphi(y) \frac{\partial R}{\partial s}(\tau_y, s) d\tau_y ds \right), \qquad (6.6)$$

Proof. Consider the random variable

$$\begin{aligned} Y &= \int_0^\infty \varphi(a)(\tau_a \wedge T) da = \int_0^T \int_{S_\theta}^\infty \varphi(a) da d\theta \\ &= \int_0^T \xi(S_\theta) d\theta, \end{aligned}$$

where $\xi(x) = \int_x^\infty \varphi(a)da$. Let $\{D_N, N \geq 1\}$ be an increasing sequence of finite subsets of $[0, T]$ such $\cup_{N=1}^\infty D_N$ is dense in $[0, T]$. Set $D_N = \{\sigma_i, 0 = \sigma_0 < \sigma_1 < \cdots < \sigma_N = T\}$ and $D_N^\theta = D_N \cap [0, \theta]$, and

$$S_\theta^N = \max\{X_t, t \in D_N^\theta\} = \max\{X_{\sigma_0}, \ldots, X_{\sigma(\theta)}\},$$

where $\sigma(\theta) = \sup D_N^\theta$. We also write $S_k^N = S_{\sigma_k}^N$. Define

$$\begin{aligned} Y_N &= \int_0^T \xi(S_\theta^N)d\theta = \sum_{k=1}^N (\sigma_k - \sigma_{k-1})\xi(\max\{X_{\sigma_0}, \dots, X_{\sigma_{k-1}}\}) \\ &= \sum_{k=1}^N (\sigma_k - \sigma_{k-1})\xi(S_{k-1}^N). \end{aligned}$$

Then, taking into account that $X_{\sigma_0} = X_0 = 0$, $p(Y_N)$ is a Lipschitz function F of the $N-1$ variables $\{X_{\sigma_1}, \dots, X_{\sigma_{N-1}}\}$, namely,

$$p(Y_N) = F(X_{\sigma_1}, \dots, X_{\sigma_{N-1}}) = p\left(\sum_{k=2}^N (\sigma_k - \sigma_{k-1})\xi(S_{k-1}^N)\right),$$

and, for all $1 \le i \le N-1$ the derivative of F respect to x_i is

$$\frac{\partial F}{\partial x_i} = -p'(Y_N) \sum_{k=i+1}^N (\sigma_k - \sigma_{k-1})\varphi(S_{k-1}^N)\mathbf{1}_{\{S_{k-1}^N = X_{\sigma_i}\}}.$$

By (6.5), we have

$$\begin{aligned} &E(p(Y_N)\delta_t(M)) \\ &= E\left(-p'(Y_N)\sum_{i=1}^{N-1}\sum_{k=i+1}^N (\sigma_k - \sigma_{k-1})\varphi(S_{k-1}^N)\mathbf{1}_{\{S_{k-1}^N = X_{\sigma_i}\}} \int_0^t M_s \frac{\partial R}{\partial s}(\sigma_i, s)ds\right) \\ &= -E\left(p'(Y_N)\sum_{k=2}^N (\sigma_k - \sigma_{k-1})\varphi(S_{k-1}^N)\int_0^t M_s(\sum_{i=1}^{k-1}\frac{\partial R}{\partial s}(\sigma_i, s)\mathbf{1}_{\{S_{k-1}^N = X_{\sigma_i}\}})ds\right) \\ &= -E\left(p'(Y_N)\int_{\sigma_1}^T \varphi(S_\theta^N)\int_0^t M_s\frac{\partial R}{\partial s}(\sigma^{\theta,N}, s)dsd\theta\right) + R_N, \end{aligned}$$

where

$$\sigma^{\theta,N} = \sum_{k=1}^N\sum_{i=0}^{k-1}\sigma_i\mathbf{1}_{(\sigma_{k-1},\sigma_k]}(\theta)\mathbf{1}_{\{\max(X_{\sigma_0},\dots,X_{\sigma_{k-1}}) = X_{\sigma_i}\}},$$

and the reminder term R_N is given by

$$R_N = -\varphi(0)\sum_{k=2}^N (\sigma_k - \sigma_{k-1})E\left(p'(Y_N)\mathbf{1}_{\{\max(X_{\sigma_0},\dots,X_{\sigma_{k-1}}=0\}}\int_0^t M_s\frac{\partial R}{\partial s}(0,s)ds\right).$$

As N tends to infinity, R_N converges to

$$-\varphi(0)\int_0^T E\left(p'(Y_N)\mathbf{1}_{\{S_\theta=0\}}\int_0^t M_s\frac{\partial R}{\partial s}(0,s)ds\right)d\theta = 0,$$

because S_θ has an absolutely continuous distribution for any $\theta > 0$. On the other hand, we claim that for all θ, $\sigma^{\theta,N}$ converges to τ_{S_θ} almost suterly as N goes to infinite. This is a consequence of the fact that X is continuous and the maximum is almost surely attained in a unique point by condition (H6). In addition, $p'(Y_N)$ converges to $p'(Y)$ and $\varphi(S_\theta^N)$ converges to $\varphi(S_\theta)$ almost surely. Therefore, by condition (H7), $\int_0^t M_s \frac{\partial R}{\partial s}(s, \tau_{S_\theta^N})ds$ converges pointwise to $\int_0^t M_s \frac{\partial R}{\partial s}(s, \tau_{S_\theta})ds$. On the other hand, by condition (H1),

$$\left| \int_0^t M_s \frac{\partial R}{\partial s}(\tau_{S_\theta}^n, s)ds \right| \leq \left(\int_0^T M_s^\beta ds \right)^{\frac{1}{\beta}} \sup_{0\leq t\leq T} \left(\int_0^T \left| \frac{\partial R}{\partial s}(s,t) \right|^\alpha ds \right)^{\frac{1}{\alpha}},$$

so by the dominated convergence theorem, we obtain

$$E(p(Y)\delta_t(M)) = -E\left(p'(Y) \int_0^T \varphi(S_\theta) \int_0^t M_s \frac{\partial R}{\partial s}(\tau_{S_\theta}, s)dsd\theta \right).$$

Finally, the change of variable $S_\theta = y$ yields

$$E(p(Y)\delta_t(M)) = -E\left(\int_0^t M_s p'(Y) \int_0^{S_T} \varphi(y) \frac{\partial R}{\partial s}(\tau_y, s)d\tau_y ds \right),$$

which completes the proof of the lemma. □

Proof of Proposition 6.1. Define

$$Y_{\varepsilon,a} = \int_0^\infty \varphi_\varepsilon(x-a)(\tau_x \wedge T)dx = \frac{1}{\varepsilon}\int_{a-\varepsilon}^a (\tau_x \wedge T)dx = \int_0^1 (\tau_{a-\varepsilon\xi} \wedge T)d\xi,$$

where $\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\mathbf{1}_{[-1,0]}(\frac{x}{\varepsilon})$, and by convention $\tau_x = 0$ if $x < 0$. Lemma 6.4 can be extended to the function $x \mapsto \varphi_\varepsilon(x-a)$ and to the random variable $Y_{\varepsilon,a}$ for any fixed a. Therefore, from (6.3) and Lemma 6.4 we optain

$$\begin{aligned} &\int_0^\infty E(p_\delta(Y_{\varepsilon,a} - t)M_t)dt \\ &= 1 + \lambda \int_0^\infty E(p_\delta(Y_{\varepsilon,a} - t)\delta(M\mathbf{1}_{[0,t]}))dt \\ &= 1 - \lambda \int_0^\infty E\left(\int_0^t M_s p_\delta'(Y_{\varepsilon,a} - t) \int_0^{S_T} \varphi_\varepsilon(y-a) \frac{\partial R}{\partial s}(\tau_y, s)d\tau_y ds \right) dt \\ &= 1 - \lambda \int_0^\infty E\left(p_\delta(Y_{\varepsilon,a} - s)M_s \int_0^{S_T} \varphi_\varepsilon(y-a)\frac{\partial R}{\partial s}(\tau_y, s)d\tau_y \right) ds, \end{aligned} \tag{6.7}$$

where the last inequality holds by integration by parts. Multiplying by $\psi(a)$ and integrating with respect to the variable a yields

$$\begin{aligned} &\int_{\mathbb{R}} \psi(a) \int_0^\infty E(p_\delta(Y_{\varepsilon,a} - t)M_t)dtda \\ &= c - \lambda E\left(\int_{\mathbb{R}} \int_0^{S_T} d\tau_y \left(\int_0^\infty p_\delta(Y_{\varepsilon,a} - s)M_s \frac{\partial R}{\partial s}(\tau_y, s)ds \right) \varphi_\varepsilon(y-a)\psi(a)da \right) \\ &= c - \lambda E\left(\int_0^{S_T} d\tau_y \frac{1}{\varepsilon} \int_y^{y+\varepsilon} \psi(a)da \left(\int_0^\infty p_\delta(Y_{\varepsilon,a} - s)M_s \frac{\partial R}{\partial s}(\tau_y, s)ds \right) \right) \\ &= c - \lambda E\left(\int_0^{S_T} d\tau_y \left(\int_0^\infty \left(\int_0^1 d\eta \psi(y+\varepsilon\eta)p_\delta(Y_{\varepsilon,y-\varepsilon\eta} - s) \right) M_s \frac{\partial R}{\partial s}(\tau_y, s)ds \right) \right), \end{aligned}$$

where the last equation holds by the change of variable $a = y + \varepsilon\eta$. Next, consider

$$Y_{\varepsilon,y+\varepsilon\eta} = \int_0^1 (\tau_{y+\varepsilon\eta-\varepsilon\xi} \wedge T)d\xi = \int_0^\eta (\tau_{y+\varepsilon\eta-\varepsilon\xi} \wedge T)d\xi + \int_\eta^1 (\tau_{y+\varepsilon\eta-\varepsilon\xi} \wedge T)d\xi.$$

Taking the limit as ε goes to zero, and using the fact that τ is left continuous and with right limit, we obtain

$$\lim_{\varepsilon\to 0} \int_0^\eta (\tau_{y+\varepsilon\eta-\varepsilon\xi} \wedge T)d\xi = \int_0^\eta (\tau_{y^+} \wedge T)d\xi = (\tau_{y^+} \wedge T)\eta,$$
$$\lim_{\varepsilon\to 0} \int_\eta^1 (\tau_{y-\varepsilon\eta+\varepsilon\xi} \wedge T)d\xi = \int_\eta^1 (\tau_y \wedge T)d\xi = \tau_y(1-\eta).$$

This implies that

$$\lim_{\varepsilon\to 0} \int_0^1 \psi(y+\varepsilon\eta)p_\delta(Y_{\varepsilon,y+\varepsilon\eta} - s)d\eta = \int_0^1 \psi(y)p_\delta((\tau_{y^+} \wedge T)\eta + \tau_y(1-\eta) - s)d\eta.$$

This allows us to compute the limit of the right-hand side of Equation (6.7) as ε tends to zero, using the dominated convergence theorem. In fact,

$$\int_0^1 \psi(y+\varepsilon\eta)p_\delta(Y_{\varepsilon,y+\varepsilon\eta} - s)d\eta \le K,$$

where K is a constant, and assuming $\operatorname{supp}(p_\delta) \subseteq [0, T+\delta]$, we have using condition (H1),

$$\begin{aligned}
&E\left(\int_0^{S_T} d\tau_y \int_0^{T+\delta} \left| M_s \frac{\partial R}{\partial s}(\tau_y, s)\right| ds\right) \\
&\le E\left(\int_0^{S_T} d\tau_y \left(\int_0^{T+\delta} |M_s|^\beta ds\right)^{\frac{1}{\beta}} \left(\int_0^{T+\delta} |\frac{\partial R}{\partial s}(\tau_y, s)|^\alpha ds\right)^{\frac{1}{\alpha}}\right) \\
&\le TE\left(\left(\int_0^{T+\delta} |M_s|^\beta\right)^{\frac{1}{\beta}} ds\right) \sup_{z\in[0,T+\delta]} \left(\int_0^{T+\delta} |\frac{\partial R}{\partial s}(z, s)|^\alpha ds\right)^{\frac{1}{\alpha}} < \infty.
\end{aligned}$$

On the other hand, we know that $\lim_{\varepsilon\to 0} Y_{\varepsilon,a} = \tau_a \wedge T = \tau_a$ since $\tau_a \le T$. Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}} \psi(a) \int_0^\infty E(p_\delta(\tau_a - t)M_t)dtda \\
&= c - \lambda E \int_0^{S_T} d\tau_y \int_0^1 \psi(y)d\eta \int_0^\infty p_\delta((\tau_{y^+} \wedge T)\eta + \tau_y(1-\eta) - s)M_s \frac{\partial R}{\partial s}(\tau_y, s)ds.
\end{aligned} \tag{6.8}$$

Finally, for the left hand side of (6.8) we have

$$\lim_{\delta\to 0} \int_{\mathbb{R}} \psi(a) \int_0^\infty E(p_\delta(\tau_a - t)M_t)dtda = \int_0^\infty E(M_{\tau_a})\psi(a)da,$$

which implies the desired result. □

Proposition 6.1 implies the following inequalities which are the main result of this section.

Theorem 6.5. *Assume that X satisfies (H1), (H3), (H5), (H6) and (H7).*

(i) *If $\frac{\partial R}{\partial s}(t,s) \geq 0$ for all $s > t$, then for all $\alpha, a > 0$, we have*

$$E(\exp(-\alpha R_{\tau_a}) \leq e^{-a\sqrt{2\alpha}}. \tag{6.9}$$

(ii) *If $\frac{\partial R}{\partial s}(t,s) \leq 0$ for all $s > t$, then for all $\alpha, a > 0$, we have*

$$E(\exp(-\alpha R_{\tau_a})) \geq e^{-a\sqrt{2\alpha}}. \tag{6.10}$$

Proof. If we assume $\frac{\partial R}{\partial s}(t,s) \geq 0$, Proposition 6.1 implies

$$\int_0^\infty E(M_{\tau_a})\psi(a)da \leq c.$$

Therefore, $E(M_{\tau_a}) \leq 1$, namely,

$$E(\exp(\lambda a - \frac{1}{2}\lambda^2 R_{\tau_a})) \leq 1,$$

for any $\lambda > 0$, which implies (6.9).

To show (ii), we choose p_δ such that $p_\delta(x-y) = 0$ if $x > y$. Then, in the integral with respect to ds appearing in the right-hand side of (6.8) we can assume that $s > (\tau_{y^+} \wedge T)\eta + \tau_y(1-\eta) \geq \tau_y$, which implies $\frac{\partial R}{\partial s}(\tau_y, s) \leq 0$. Then,

$$\int_0^\infty E(M_{\tau_a})\psi(a)da \geq c,$$

which allows us to conclude the proof as in the case (i). □

Theorem 6.5 tells that the Laplace transform of the random variable R_{τ_a} can be compared with the Laplace transform of the hitting time of the ordinary Brownian motion at the level a, under some monotonicity conditions on the covariance function. This implies some consequences on the moments of R_{τ_a}. In the case (i), the inequality (6.9) implies for any $r > 0$,

$$\begin{aligned} E(R_{\tau_a}^{-r}) &= \frac{1}{\Gamma(r)} \int_0^\infty E(e^{-\alpha R_{\tau_a}})\alpha^{r-1}d\alpha \\ &\leq \frac{1}{\Gamma(r)} \int_0^\infty e^{-a\sqrt{2\alpha}}\alpha^{r-1}d\alpha = \frac{2^r\Gamma(r+\frac{1}{2})}{\sqrt{\pi}}a^{-2r}. \end{aligned} \tag{6.11}$$

On the other hand, for $0 < r < 1$,

$$\begin{aligned} E(R_{\tau_a}^{r}) &= \frac{r}{\Gamma(1-r)} \int_0^\infty (1 - E(e^{-\alpha R_{\tau_a}}))\alpha^{-r-1}d\alpha \\ &\geq \frac{r}{\Gamma(1-r)} \int_0^\infty (1 - e^{-a\sqrt{2\alpha}})\alpha^{-r-1}d\alpha. \end{aligned} \tag{6.12}$$

As a consequence, $E(R_{\tau_a}^r) = +\infty$ for $r \in (\frac{1}{2}, 1)$.

In the case (ii), the inequality (6.10) implies for any $r > 0$

$$\begin{aligned} E(R_{\tau_a}^{-r}) &= \frac{1}{\Gamma(r)} \int_0^\infty E(e^{-\alpha R_{\tau_a}})\alpha^{r-1}d\alpha \\ &\geq \frac{1}{\Gamma(r)} \int_0^\infty e^{-a\sqrt{2\alpha}}\alpha^{r-1}d\alpha = \frac{2^r\Gamma(r+\frac{1}{2})}{\sqrt{\pi}}a^{-2r}. \end{aligned} \tag{6.13}$$

On the other hand, for $0 < r < 1$,

$$E(R^r_{\tau_a}) = \frac{r}{\Gamma(1-r)} \int_0^\infty (1 - E(e^{-\alpha R_{\tau_a}}))\alpha^{-r-1} d\alpha$$
$$\leq \frac{r}{\Gamma(1-r)} \int_0^\infty (1 - e^{-a\sqrt{2\alpha}})\alpha^{-r-1} d\alpha, \qquad (6.14)$$

and, hence, $E(R^r_{\tau_a}) < \infty$ for $r \in (0, \frac{1}{2})$.

Example 6.6. Consider the case of a fractional Brownian motion Hurst parameter $H > \frac{1}{2}$. Recall that

$$R_H(t,s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

Conditions (H5), (H6) and (H7) are satisfied. We can write

$$\frac{\partial R_H}{\partial s}(t,s) = H(s^{2H-1} + \mathrm{sign}(t-s)|t-s|^{2H-1})$$

for all $s, t \in [0, T]$.

If $H > \frac{1}{2}$, then $\frac{\partial R_H}{\partial s}(t,s) \geq 0$ for all s, t, and by (6.9) in Theorem 6.5, $E(\exp(-\alpha\tau_a^{2H})) \leq e^{-a\sqrt{2\alpha}}$. This implies that $E(\tau_a^p) = +\infty$ for any $H < p$ and τ_a has finite negative moments of all order.

If $H < \frac{1}{2}$, then $\frac{\partial R_H}{\partial s}(t,s) \leq 0$ for $s > t$, and by (6.10) in Theorem 6.5, $E(\exp(-\alpha\tau_a^{2H})) \geq e^{-a\sqrt{2\alpha}}$. This implies that $E(\tau_a^p) < +\infty$ for any $p < H$.

In ([16]), Molchan proved that for the fractional Brownian motion with Hurst parameter $H \in (0,1)$,

$$P(\tau_a > t) = t^{H-1+o(1)},$$

as t tends to infinity. As a consequence, $E(\tau_a^p) > \infty$ if $p < 1-H$ and $E(\tau_a^p) = \infty$ if $p > 1-H$, which is stronger than the integrability results mentioned above.

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