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MULTIPLE Q-ADAPTED INTEGRALS AND ITÔ FORMULA OF NONCOMMUTATIVE STOCHASTIC CALCULUS IN FOCK SPACE

V. P. BELAVKIN AND M. F. BROWN.

Dedicated to Professor K. R. Parthasarathy on the occasion of his 75th birthday

Abstract. We study the continuity property of multiple Q-adapted quantum stochastic integrals with respect to noncommuting integrands given by the non-adapted multiple integral kernels in Fock scale. The noncommutative algebra of relatively (exponentially) bounded nonadapted quantum stochastic processes is studied in the kernel form as introduced in [10]. The differential Q-adapted formula generalizing Itô product formula for adapted integrals is presented in both strong and weak sense as a particular case of the quantum stochastic nonadapted Itô formula.

1. Introduction

Non-commutative generalization of adapted Itô stochastic calculus, developed by Hudson and Parthasarathy (HP) in [18], gave an adequate mathematical tool for studying unitary and endomorphic cocycles of open quantum dynamical systems singularly interacting with a boson quantum-stochastic field. The adapted HP quantum stochastic calculus and its kernel variant [23], [21] also made it possible to solve the old quantum measurement problem by describing such systems by quantum stochastic Langevin equation [15] with continuously observed output field [6], and constructing a quantum filtration theory [11] which explained a continuous spontaneous collapse under such observation in terms of now famous quantum stochastic Master equations, first derived in [4],[8]. The Belavkin filtering equations gave examples of quantum stochastic non-unitary, even nonlinear, nonstationary, adapted evolution equations in Hilbert and operator spaces normalized only in the mean as the exponential martingales. Their solutions require a proper definition of chronologically ordered unbounded quantum stochastic exponentials of noncommuting operators and maps which cannot be studied within the original HP-calculus approach and its extensions [1],[2],[25] to the bounded QS (quantum stochastic) semi-martingales. Moreover, the perturbation theory for such evolutions, usually studied by applying Duhamel principle in non-stochastic case, requires the development of quantum non-adapted calculus, since the stochastic Duhamel formula cannot be written in terms of the adapted stochastic integral

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even in the classical case. In order to solve these problems, a more general Fock scale approach to quantum stochastic calculus and integration was developed in [3],[10],[12] which does not require the usual boundedness and adaptedness of the stochastic integrands and the resulting quantum integral semimartingales. Based on the canonical Pseudo-Poisson representation of quantum Itô algebra discovered by Belavkin in [3], the Fock scale analysis gives a simple constructive expression of the quantum functional Itô formula and is essentially basis free, allowing a unified algebraic treatment of any, even infinite number of quantum noise modes.

In this paper we give a brief overview of nonadapted quantum stochastic calculus closely following the nonadapted quantum Itô formula part of the paper [12], but consider the case of Q-adapted quantum stochastic integrals introduced in [13] in the natural Fock scale of Hilbert spaces as a special nonadapted case of [12]. To this end we shall explore the Belavkin notation for indefinite $\star$-algebraic structure of the kernel calculus as the general property of a natural pseudo-Euclidean representation for Schürmann's tripples associated with infinitely divisible states, obtained by Belavkin for the general nonstationary case in [12]. As a particular case of non-adapted Itô formula we establish a non-commutative Q-adapted generalization of the adapted Itô formula that is the principal formula of the classical stochastic calculus. In the $\mathrm{Q} = \mathrm{I}$ case this formula coincides with the well-known Hudson-Parthasarathy formula [18] for the product of a pair of non-commuting quantum processes and gives its functional extension. In the commutative case this gives a Q-adapted generalization of the Itô formula for classical stochastic processes as the case of the general nonadapted classical Itô formula, first obtained in the case of Wiener integrals in a weak form by classical stochastic methods by Nualart [24]. We also note that Fock scales are also used in the stochastic analysis of classical white noise by Hida, Kuo, Pothoff, Streit, starting from [17],[19], and independently by Berezanskii and Kondrat'ev, starting from [14]. However, while the classical stochastic analysis is mostly concerned with the study of the generalized stochastic functionals in nuclear Fock scales, the quantum noise analysis is concerned with the analysis of generalized operators (kernels) in nonnuclear Fock scale which was first introduced by Belavkin [3],[10],[12] and recently used also by Ji and Obata in their abstract Fock space approach to nondifferential quantum stochastic analysis.

Here we shall consider mostly differetial problems of quantum stochastic analysis, adopting Guichardet representation of Fock space as L^2-space over the finite subsets from a nonatomic measure space, regarding these subsets as almost totally ordered chains following the notation from [10],[12]. In this notation the integral quantum stochastic calculus is similar in spirit to the kernel calculus of Maassen-Lindsay-Meyer [21], [23], with the difference that all the main objects are constructed not in terms of Maassen-Meyer kernels but in terms of the operators kernels represented in the Fock state space. In this unifying approach we employ a much more general notion of multiple stochastic integral, non-adapted in general, but focusing on Q-adapted processes, which reduces to the notion of the kernel representation of an operator only in the case of a scalar (non-random) operator integrand. The possibility of defining a non-adapted single integral in terms of the kernel calculus in the case of quantum single mode noise was shown independently by Belavkin [3] and Lindsay [20], but the general repeated and the multiple non-adapted integrals were first introduced and studied within the quantum stochastic analysis in Fock scales in [10],[12].

2. Rigged Guichardet-Fock Space

Let $(\mathbb{X}, \lambda)$ be an essentially ordered space, that is, a measurable space $\mathbb{X}$ with a σ-finite measure $\lambda : \mathfrak{F}_{\mathbb{X}} \ni \triangle \mapsto \lambda(\triangle) \geq 0$ and an ordering relation $x \leq x'$ with the property that any n-tuple $(x_1, \ldots, x_n) \in \mathbb{X}^n$ can be identified up to a permutation with a chain $\vartheta = \{x_1 < \cdots < x_n\}$ modulo the product measure $\prod_{i=1}^n \mathrm{d}x_i$ of $\mathrm{d}x := \lambda(\mathrm{d}x)$. In particular we consider the order induced by the linear order in $\mathbb{R}_+$ by a measurable map $t : \mathbb{X} \to \mathbb{R}_+$ relatively to which λ is absolutely continuous with respect to the Lebesque measure $\mathrm{d}t$ on $\mathbb{R}_+$, then $x < x' \Leftrightarrow t(x) < t(x')$, where $t(x)$ is identified with the time of the point x.

We shall identify the finite chains ϑ with increasingly indexed n-tuples $\boldsymbol{s} \equiv (x_1, \ldots, x_n)$ with $x_i \in \mathbb{X}$, $x_1 < \cdots < x_n$, denoting by $\mathcal{X} = \sum_{n=0}^{\infty} \mathcal{X}_n$ the set of all finite chains as the union of the sets

$$\mathcal{X}_n = \{\boldsymbol{s} \in \mathbb{X}^n : x_1 < \cdots < x_n\}$$

with $\mathcal{X}_0 = \{\emptyset\}$ containing the only one element $\emptyset \in \mathcal{X}_0$, the empty chain $\emptyset = \mathbb{X}^0$ identified with empty subset of $\mathbb{X}$. We introduce a measure 'element' $\mathrm{d}\vartheta = \prod_{x \in \vartheta} \mathrm{d}x$ on $\mathcal{X}$ induced by the direct sum $\oplus_{n=0}^{\infty} \lambda^{\otimes n}(\triangle_n), \triangle_n \in \mathfrak{F}_{\mathbb{X}}^{\otimes n}$ of product measures $\mathrm{d}\boldsymbol{s} = \prod_{i=1}^n \mathrm{d}x_i$ on $\mathbb{X}^n$ with the unit mass $\mathrm{d}\vartheta = 1$ at the only atomic point $\vartheta = \emptyset$.

Let $\{\mathfrak{k}_x : x \in \mathbb{X}\}$ be a family of Hilbert spaces $\mathfrak{k}_x$, let $\mathfrak{p}_0$ be an additive semigroup of nonnegative essentially measurable locally bounded functions $q : \mathbb{X} \to \mathbb{R}_+$ with zero included $0 \in \mathfrak{p}_0$, and let $\mathfrak{p}_1 = \{1 + q_0 : q_0 \in \mathfrak{p}_0\}$. We denote by $K_\star(q)$ the Hilbert space of essentially measurable vector-functions $\mathrm{k} : x \mapsto \mathrm{k}(x) \in \mathfrak{k}_x$ which are square integrable with the weight $q \in \mathfrak{p}_1$:

$$\|\mathrm{k}\|(q) = \left(\int \|\mathrm{k}(x)\|_x^2 \, q(x)\mathrm{d}x\right)^{1/2} < \infty.$$

With $q \geq 1$, any space $K_\star(q)$ can be embedded into the Hilbert space $\mathfrak{k} = K_\star(1)$, and the intersection $\cap_{q \in \mathfrak{p}_1} K_\star(q) \subseteq \mathfrak{k}$ can be identified with the projective limit $K_+ = \lim_{q \to \infty} K_\star(q)$. This follows from the facts that the function $\|\mathrm{k}\|(q)$ is increasing: $q \leq p \Rightarrow \|\mathrm{k}\|(q) \leq \|\mathrm{k}\|(p)$, and so $K_\star(p) \subseteq K_\star(q)$, and that the set $\mathfrak{p}_1$ is directed in the sense that for any $q = 1 + r$ and $p = 1 + s$, $r, s \in \mathfrak{p}_0$, there is a function in $\mathfrak{p}_1$ majorizing q and p (we can take for example $q + p - 1 = 1 + r + s \in \mathfrak{p}_1$).

The dual space $K_\star^-$ to K_+ is the space of generalized vector-functions $\mathrm{f}(x)$ defining the continuous functionals

$$\langle \mathrm{f} \mid \mathrm{k} \rangle = \int \langle \mathrm{f}(x) \mid \mathrm{k}(x) \rangle \, \mathrm{d}x, \quad \mathrm{k} \in K_+.$$

It is the inductive limit $K_- = \lim_{q \to 0} K_\star(q)$ in the opposite scale $\{K_\star(q) : q \in \mathfrak{p}_-\}$, where $\mathfrak{p}_-$ is the set of functions $q : \mathbb{X} \to (0, 1]$ such that $1/q \in \mathfrak{p}_1$, which is the union $\cup_{q \in \mathfrak{p}_-} K_\star(q)$ of the inductive family of Hilbert spaces $K_\star(q), q \in \mathfrak{p}_-$, with the norms $\|\mathrm{k}\|(q)$, containing as the minimal the space $\mathcal{K}_* = K_\star(1)$. Thus we obtain the Gel'fand chain

$$K_+ \subseteq K_\star(q_+) \subseteq \mathcal{K}_* \subseteq K_\star(q_-) \subseteq K_-$$

in the extended scale $\{K_\star(q) : q \in \mathfrak{p}\}$, where $\mathfrak{p} = \mathfrak{p}_- \cup \mathfrak{p}_1$, with $q_+ \in \mathfrak{p}_1$, $q_- \in \mathfrak{p}_-$. The dual space $K_+^\star = K^-$ is the space of the continuous linear functionals on K_+ containing the Hilbert space $\mathcal{K}$ called the rigged space with respect to the dense subspace $K^+ = K_-^\star$ of $\mathcal{K}$ equipped with the projective convergence in the scale $\|\mathrm{k}^*\|(q) = \|\mathrm{k}\|(q)$ for $q \in \mathfrak{p}_1$.

We can similarly define a Fock-Gel'fand triple $(F_+, \mathcal{F}_*, F_-)$ with

$$F_+ = \cap_{q\in\mathfrak{p}_1} F_\star(q),\ \mathcal{F}_* = F_\star(1),\ F_- = \cup_{q\in\mathfrak{p}_-} F_\star(q),$$

for the Hilbert scale $\{F_\star(q) : q \in \mathfrak{p}\}$ of the symmetric Fock spaces $F_\star(q) = \oplus_{n=0}^\infty K_\star^{(n)}(q)$ over $K_\star(q)$, where $K_\star^{(0)}(q) = \mathbb{C}$, $K_\star^{(1)}(q) = K_\star(q)$, and each $K_\star^{(n)}(q)$ for $n > 1$ is given by the product-weight $q_n(x_1, \dots, x_n) = \prod_{i=1}^n q(x_i)$ on $\mathbb{X}^n$. We shall consider the Guichardet [16] representation of the symmetric tensor-functions $\psi_n \in K_\star^{(n)}(q)$ regarding them as the restrictions $\psi|\mathcal{X}_n$ of the functions $\psi : \vartheta \mapsto \psi(\vartheta) \in K_\star^\otimes(\vartheta)$ with sections in the Hilbert products $K_\star^\otimes(\vartheta) = \otimes_{x\in\vartheta}\mathfrak{k}_x$, square integrable with the product weight $q(\vartheta) = \prod_{x\in\vartheta} q(x)$:

$$\|\psi\|(q) = \left(\int \|\psi(\vartheta)\|^2 q(\vartheta)\,\mathrm{d}\vartheta\right)^{1/2} < \infty.$$

The integral here is over all chains $\vartheta \in \mathcal{X}$ and defines the pairing on F_+ by

$$\langle\psi \mid \psi\rangle = \int \langle\psi(\vartheta) \mid \psi(\vartheta)\rangle\,\mathrm{d}\vartheta, \quad \psi \in F_+.$$

In more detail we can write this in the form

$$\int \|\psi(\vartheta)\|^2 q(\vartheta)\mathrm{d}\vartheta = \sum_{n=0}^\infty \int_{0\le t_1<} \cdots \int_{<t_n<\infty} \|\psi(x_1, \dots, x_n)\|^2 \prod_{i=1}^n q(x_i)\mathrm{d}x_i,$$

where the n-fold integrals for $\psi_n \in K_\star^{(n)}$ are taken over simplex domains $\mathcal{X}_n = \{\boldsymbol{s} \in \mathbb{X}^n : t(x_1) < \cdots < t(x_n)\}$.

One can easily establish an isomorphism between the space $F_\star(q)$ and the symmetric (or antisymmetric) Fock space over $K_\star(q)$ with a nonatomic measure $\mathrm{d}x$ in $\mathbb{X}$. It is defined by the isometry

$$\|\psi\|(q) = \left(\sum_{n=0}^\infty \frac{1}{n!}\int \cdots \int \|\psi(x_1, \dots, x_n)\|^2 \prod_{i=1}^n q(x_i)\mathrm{d}x_i\right)^{1/2},$$

where the functions $\psi(x_1, \dots x_n)$ can be extended to the whole of $\mathbb{X}^n$ in a symmetric (or antisymmetric) way uniquely up to the measure zero due to nonatomicity of $\mathrm{d}x$ on $\mathbb{X}$.

Finally let $\mathfrak{h}$ be a Hilbert space called the initial space for the Hilbert products $\mathcal{H}_* = \mathfrak{h} \otimes \mathcal{K}_*$ and $\mathcal{G}_* = \mathfrak{h} \otimes \mathcal{F}_*$. We consider the Hilbert scale $G_\star(q) = \mathfrak{h} \otimes F_\star(q)$, $q \in \mathfrak{p}$ of complete tensor products of $\mathfrak{h}$ and the Fock spaces over $K_\star(q)$, and we put

$$G_+ = \cap G_\star(q), \quad G_- = \cup G_\star(q)$$

which constitute the Gel'fand triple $G_+ \subseteq \mathcal{G}_* \subseteq G_-$ dual to $G^+ \subseteq \mathcal{G} \subseteq G^-$ of the Hermitian adjoint bra-spaces $G^+ = G_+^*$, $\mathcal{G} = \mathcal{G}_*^*$, $G^- = G_-^*$.

3. Triangular Kernels as Generalized Operators in Fock Space

Now we consider matrix chains $\vartheta^\cdot_\cdot = [\vartheta^\mu_\nu]^{\mu=-,\circ,+}_{\nu=-,\circ,+}$ and we define the triangular operator valued kernels

$$T(\vartheta^\cdot_\cdot) = 0 \text{ if } \vartheta^\mu_\nu \neq 0 \text{ for each } \mu > \nu,$$
$$T(\vartheta^\cdot_\cdot) = 1\left(\vartheta^-_-\right) T\left(\boldsymbol{\vartheta}\right) 1\left(\vartheta^+_+\right) \text{ otherwise,} \tag{3.1}$$

where $1\left(\vartheta^-_-\right) = 1 = 1\left(\vartheta^+_+\right)$ in $\mathbb{C}$, and

$$T(\boldsymbol{\vartheta}) = T\begin{pmatrix} \vartheta^-_+ & \vartheta^-_\circ \\ \vartheta^\circ_+ & \vartheta^\circ_\circ \end{pmatrix} : \mathfrak{k}^\otimes(\vartheta^-_\circ \sqcup \vartheta^\circ_\circ) \otimes \mathfrak{h} \to \mathfrak{k}^\otimes(\vartheta^\circ_\circ \sqcup \vartheta^\circ_+) \otimes \mathfrak{h}, \tag{3.2}$$

is an operator-valued function of $\boldsymbol{\vartheta} = (\vartheta^\mu_\nu)^{\mu=-,\circ}_{\nu=+,\circ}$ satisfying the integrability condition $\|T\|_q(r) < \infty$ for some $r^{-1} \in \mathfrak{p}_0$ and $q \in \mathfrak{p}_1$ with respect to the norms

$$\|T\|_q(r) = \int \mathrm{d}\vartheta^-_+ \left(\iint \operatorname*{ess\,sup}_{\vartheta^\circ_\circ} \left\{ \frac{\|T(\boldsymbol{\vartheta})\|}{q(\vartheta^\circ_\circ)} \right\}^2 r(\vartheta^\circ_+ \sqcup \vartheta^-_\circ) \mathrm{d}\vartheta^\circ_+ \mathrm{d}\vartheta^-_\circ \right)^{1/2}.$$

Note that $T\left(\boldsymbol{\vartheta}\right) \in \mathfrak{L}\left(\mathfrak{h}\right)$ in the scalar case $\mathfrak{k}_x = \mathbb{C}$.

We would like to consider the QS integral operators as the continuous maps $\mathrm{T} : G_+ \to G_-$ representing the triangular kernels $T(\vartheta^\cdot_\cdot)$. The representation, denoted by $\boldsymbol{\epsilon}$, is explicitly defined by

$$[\boldsymbol{\epsilon}(T)\chi](\vartheta) = \sum_{\vartheta^\circ_\circ \sqcup \vartheta^\circ_+ = \vartheta} \iint T\begin{pmatrix} \vartheta^-_+, & \vartheta^-_\circ \\ \vartheta^\circ_+, & \vartheta^\circ_\circ \end{pmatrix} \chi(\vartheta^\circ_\circ \sqcup \vartheta^-_\circ) \mathrm{d}\vartheta^-_\circ \mathrm{d}\vartheta^-_+ \tag{3.3}$$

on $\chi \in G_+$, which may be given by the operator-valued multiple integral [10]

$$[\boldsymbol{\imath}^t_0(\mathrm{M})\chi](\vartheta) = \sum_{\upsilon^\circ_\circ \sqcup \upsilon^\circ_+ \subseteq \vartheta^t} \int_{\mathcal{X}^t} \int_{\mathcal{X}^t} [\mathrm{M}(\boldsymbol{\upsilon})\mathring{\chi}(\upsilon^-_\circ \sqcup \upsilon^\circ_\circ)](\vartheta^\circ_-) \mathrm{d}\upsilon^-_+ \mathrm{d}\upsilon^-_\circ, \tag{3.4}$$

on $\mathcal{X}^\infty = \mathcal{X}$ of the multiple integrand $\mathrm{M}(\boldsymbol{\upsilon})$ as trivially vacuum adapted such that $\mathrm{M}(\boldsymbol{\upsilon}) = T(\boldsymbol{\upsilon}) \otimes \mathrm{P}_\emptyset$, where $[\mathrm{P}_\emptyset\chi](\vartheta) = \delta_\emptyset(\vartheta)\chi(\vartheta)$, and $\delta_\emptyset(\vartheta) = 1$ if $\vartheta = \emptyset$ and is 0 otherwise, such that $[\mathrm{M}(\boldsymbol{\upsilon})\mathring{\chi}(\upsilon^\circ_\circ \sqcup \upsilon^-_\circ)](\varkappa) = 0$ if $\varkappa \neq \emptyset$.

The integrand $\mathrm{M}(\boldsymbol{\upsilon})$ is not, however, a unique choice in (3.4), one can take any kernel-valued function $M\left(\boldsymbol{\upsilon}, \varkappa\right)$ satisfying $T\left(\boldsymbol{\vartheta}\right) = \sum_{\varkappa \sqcup \upsilon^\circ_\circ = \vartheta^\circ_\circ} M\left(\boldsymbol{\upsilon}, \varkappa\right)$ to obtain the operator $\boldsymbol{\epsilon}(T)$. We shall choose $M(\boldsymbol{\upsilon}, \varkappa) = M(\boldsymbol{\upsilon}) \otimes \mathrm{Q}^\otimes\left(\varkappa\right)$ as trivially Q-adapted multiple integrand [13] such that the operator-valued integrand of (3.4) is given instead as $\mathrm{M}(\boldsymbol{\upsilon}) = M(\boldsymbol{\upsilon}) \otimes \mathrm{Q}^\otimes$ to obtain

$$\left[\boldsymbol{\imath}^\infty_0(M \otimes \mathrm{Q}^\otimes)\chi\right](\vartheta) = [\boldsymbol{\epsilon}(T)\chi](\vartheta).$$

Here, the operator $\mathrm{Q}^\otimes$, given on chains $\vartheta \in \mathcal{X}$ as $\mathrm{Q}^\otimes(\vartheta) = \otimes_{x \in \vartheta}\mathrm{Q}(x)$ in the space $\mathfrak{k}^\otimes(\vartheta)$, is defined as a mapping $F_+ \to F_-$ such that there exists a $p \in \mathfrak{p}$ with

$$\|\mathrm{Q}^\otimes\|_p = \sup_{\chi \in F_\star(p)} \left\{ \frac{\|\mathrm{Q}^\otimes\chi\|(p^{-1})}{\|\chi\|(p)} \right\} < \infty.$$

In fact the vacuum projector corresponds to the case Q = O such that $\mathrm{P}_\emptyset = \mathrm{O}^\otimes$, and then of course $M(\boldsymbol{\upsilon}) = T(\boldsymbol{\upsilon})$. The kernel $M(\boldsymbol{\upsilon})$ is Maassen-Meyer kernel integrand generalized to the trivially Q-adapted case. Indeed, it is uniquely Q-related to $T(\boldsymbol{\vartheta})$ as it is given by the transformation

$$M\begin{pmatrix}\upsilon_+^-, & \upsilon_\circ^- \\ \upsilon_+^\circ, & \upsilon_\circ^\circ\end{pmatrix} = \sum_{\vartheta\subseteq\upsilon_\circ^\circ} T\begin{pmatrix}\upsilon_+^-, & \upsilon_\circ^- \\ \upsilon_+^\circ, & \vartheta\end{pmatrix} \otimes (-\mathrm{Q})^\otimes(\upsilon_\circ^\circ\setminus\vartheta),$$

that is the Meyer transformation of T defining the trivially adapted QS-multiple integrand for the integral representation $\mathrm{T} = \boldsymbol{\imath}_0^\infty(\mathrm{M})$ when Q = I. This simply follows from the definition of the action

$$[\mathrm{M}(\boldsymbol{\upsilon})\mathring{\chi}(\upsilon_\circ^-\sqcup\upsilon_\circ^\circ)](\vartheta_-^\circ) = M(\boldsymbol{\upsilon})\otimes\mathrm{Q}^\otimes(\vartheta_-^\circ)\chi(\upsilon_\circ^-\sqcup\upsilon_\circ^\circ\sqcup\vartheta_-^\circ)$$

on $\mathring{\chi}(\upsilon,\vartheta) = \chi(\upsilon\sqcup\vartheta)$, for the kernel

$$T\begin{pmatrix}\vartheta_+^-, & \vartheta_\circ^- \\ \vartheta_+^\circ, & \vartheta_\circ^\circ\end{pmatrix} = \sum_{\upsilon\subseteq\vartheta_\circ^\circ} M\begin{pmatrix}\vartheta_+^-, & \vartheta_\circ^- \\ \vartheta_+^\circ, & \upsilon\end{pmatrix} \otimes \mathrm{Q}^\otimes(\vartheta_\circ^\circ\setminus\upsilon)$$

that is the Möbius transformation of $M(\boldsymbol{\upsilon})$ when Q = I, inverting the Meyer transformation.

It was shown in [13], using the estimate for nonadapted integrals from [12], that if M is a q-contractive ampliation of M, such that $\mathrm{M} = M\otimes\mathrm{Q}^\otimes$ and $\|\mathrm{Q}^\otimes\|_q \le 1$, then $\|\mathrm{T}\|_p \le \|M\|_\infty^s(r)$ for $p \ge r^{-1}+q+s^{-1}$, where $\mathrm{T} = \boldsymbol{\imath}_0^\infty(\mathrm{M}) \equiv \boldsymbol{\epsilon}(T)$, and

$$\|M\|_t^s(r) = \int_{\mathcal{X}^t}\mathrm{d}\upsilon_+^-\left(\int_{\mathcal{X}^t}\mathrm{d}\upsilon_+^\circ\int_{\mathcal{X}^t}\mathrm{d}\upsilon_\circ^-\operatorname{ess}\sup_{\upsilon_\circ^\circ\in\mathcal{X}^t}\{s(\upsilon_\circ^\circ)\|M(\boldsymbol{\upsilon})\|\}^2 r(\upsilon_+^\circ\sqcup\upsilon_\circ^-)\right)^{1/2}.$$

However, using the equivalent representation (3.3) in the form of the multiple integral (3.4) of $\mathrm{M}(\boldsymbol{\upsilon}) = T(\boldsymbol{\upsilon})\otimes\mathrm{P}_\emptyset$, one may write $\|\mathrm{T}\|_p \le \|\mathrm{M}\|_{q,\infty}^s(r) \le \|T\|_{\frac{1}{s}}(r)$, where

$$\|\mathrm{M}\|_{q,t}^s(r) = \int_{\mathcal{X}^t}\left(\int_{\mathcal{X}^t}\int_{\mathcal{X}^t}\operatorname{ess}\sup_{\upsilon_\circ^\circ\in\mathcal{X}^t}(s(\upsilon_\circ^\circ)\|\mathrm{M}(\boldsymbol{\upsilon})\|_q)^2 r(\upsilon_+^\circ\sqcup\upsilon_\circ^-)\mathrm{d}\upsilon_+^\circ\mathrm{d}\upsilon_\circ^-\right)^{1/2}\mathrm{d}\upsilon_+^-,$$

taking into account the fact that $\|\mathrm{P}_\emptyset\|_q = 1$. This gives a more precise estimate, with $\|\mathrm{T}\|_p \le \|T\|_{\frac{1}{s}}(r)$ holding for $p \ge r^{-1}+s^{-1} = \lim_{q_0\searrow 0}(r^{-1}+q_0+s^{-1})$. From this estimate the previous one simply follows as

$$\begin{aligned}\|\sum_{\upsilon_\circ^\circ\subseteq\vartheta_\circ^\circ} M(\boldsymbol{\upsilon})\otimes\mathrm{Q}^\otimes(\vartheta_\circ^\circ\setminus\upsilon_\circ^\circ)\| &\le \sum_{\upsilon_\circ^\circ\subseteq\vartheta_\circ^\circ} q(\vartheta_\circ^\circ\setminus\upsilon_\circ^\circ)\|M(\boldsymbol{\upsilon})\| \\ &\le (q+s^{-1})(\vartheta_\circ^\circ)\operatorname{ess}\sup_{\upsilon_\circ^\circ\in\mathcal{X}}\{s(\upsilon_\circ^\circ)\|M(\boldsymbol{\upsilon})\|\end{aligned}$$

where $s(\upsilon_\circ^\circ) = \prod_{x\in\upsilon_\circ^\circ}s(x)$, $\quad q(\vartheta_\circ^\circ\setminus\upsilon_\circ^\circ) = \prod_{x\in\vartheta_\circ^\circ\setminus\upsilon_\circ^\circ}q(x)$, and

$$(q+s^{-1})(\vartheta_\circ^\circ) = \sum_{\upsilon_\circ^\circ\subseteq\vartheta_\circ^\circ} s^{-1}(\upsilon_\circ^\circ)q(\vartheta_\circ^\circ\setminus\upsilon_\circ^\circ) = \prod_{x\in\vartheta_\circ^\circ}(q(x)+s^{-1}(x)),$$

and consequently $\|T\|_p(r) \le \|M\|_\infty^s(r)$ for $p \ge q+1/s$. Hence in particular there follows the existence of the adjoint operator T^* bounded in norm $\|\mathrm{T}^*\|_p \le$

$\|T^\star\|_q(r) = \|T\|_q(r)$ as the representation

$$\boldsymbol{\epsilon}(T)^* = \boldsymbol{\epsilon}(T^\star),\ T^\star\begin{pmatrix}\vartheta_+^-, & \vartheta_\circ^- \\ \vartheta_+^\circ, & \vartheta_\circ^\circ\end{pmatrix} = T\begin{pmatrix}\vartheta_+^-, & \vartheta_+^\circ \\ \vartheta_\circ^-, & \vartheta_\circ^\circ\end{pmatrix}^* \tag{3.5}$$

of the $\star$-adjoint kernel $T^\star(\boldsymbol{\vartheta}) = T(\boldsymbol{\vartheta}')^*$, $(\vartheta_\nu^\mu)' = (\vartheta_{-\mu}^{-\nu})$.

4. The Inductive $\star$-Algebra of Relatively Bounded Kernels

In the next theorem we prove that the $\star$-map $\boldsymbol{\epsilon} : T \mapsto \boldsymbol{\epsilon}(T)$ is an operator representation of the $\star$-algebra of triangular kernels $T(\boldsymbol{\vartheta})$ satisfying the boundedness condition

$$\|T\|_{\boldsymbol{\alpha}} = \operatorname*{ess\,sup}_{\boldsymbol{\vartheta}=(\vartheta_\nu^\mu)}\{\|T(\boldsymbol{\vartheta})\| / \prod_{\mu\le\nu}\alpha_\nu^\mu(\vartheta_\nu^\mu)\} < \infty \tag{4.1}$$

relative to the product of the quadruple $\boldsymbol{\alpha} = (\alpha_\nu^\mu)_{\nu=\circ,+}^{\mu=-,\circ}$ of positive essentially measurable product functions $\alpha_\nu^\mu(\vartheta) = \prod_{x\in\vartheta}\alpha_\nu^\mu(x)$, $\vartheta \in \mathcal{X}$. These are defined by an L^1-integrable function $\alpha_+^- : \mathbb{X} \to \mathbb{R}_+$, by L^2-integrable functions $\alpha_+^\circ, \alpha_\circ^- : \mathbb{X} \to \mathbb{R}_+$ with a weight $r > 0$, $r^{-1} \in \mathfrak{p}_0$, and by an L^∞-function $\alpha_\circ^\circ : \mathbb{X} \to \mathbb{R}_+$, essentially bounded by unity relative to some $q \in \mathfrak{p}$:

$$\begin{aligned}\|\alpha_+^-\|^{(1)} &= \int |\alpha_+^-(x)|\mathrm{d}x < \infty,\\ \|\alpha\|^{(2)}(r) &= \Big(\int \alpha(x)^2 r(x)\mathrm{d}x\Big)^{1/2} < \infty, \quad \alpha = \alpha_\circ^-, \alpha_+^\circ \\ \|\alpha_\circ^\circ\|_q^{(\infty)} &= \operatorname*{ess\,sup}_x \frac{|\alpha_\circ^\circ(x)|}{q(x)} \le 1.\end{aligned} \tag{4.2}$$

The relative boundedness (4.1) ensures the projective boundedness of T by the inequality $\|T\|_q(r) \le$

$$\begin{aligned}&\le \int \mathrm{d}\vartheta_+^-\Big(\iint \operatorname*{esssup}_{\vartheta_\circ^\circ}\{\|T\|_{\boldsymbol{\alpha}}\prod\alpha_\nu^\mu(\vartheta_\nu^\mu)/q(\vartheta_\circ^\circ)\}^2 r(\vartheta_+^\circ\sqcup\vartheta_\circ^-)\mathrm{d}\vartheta_+^\circ\mathrm{d}\vartheta_\circ^-\Big)^{1/2}\\ &= \int\alpha_+^-(\vartheta)\mathrm{d}\vartheta\Big(\int\alpha_+^\circ(\vartheta)^2 r(\vartheta)\mathrm{d}\vartheta\int\alpha_\circ^-(\vartheta)^2 r(\vartheta)\mathrm{d}\vartheta\Big)^{1/2}\operatorname*{ess\,sup}_\vartheta\frac{\alpha_\circ^\circ(\vartheta)}{q(\vartheta)}\|T\|_{\boldsymbol{\alpha}} \\ &= \|T\|_{\boldsymbol{\alpha}}\exp\left\{\int(\alpha_+^-(x) + r(x)(\alpha_+^\circ(x)^2 + \alpha_\circ^-(x)^2)/2)\mathrm{d}x\right\},\end{aligned} \tag{4.3}$$

where we have taken account of the fact that $\int\alpha(\vartheta)\mathrm{d}\vartheta = \exp\left\{\int\alpha(x)\mathrm{d}x\right\}$ for $\alpha(\vartheta) = \prod_{x\in\vartheta}\alpha(x)$ and

$$\operatorname*{ess\,sup}_\vartheta\{\alpha_\circ^\circ(\vartheta)/q(\vartheta)\} = \sup_n \operatorname*{ess\,sup}_{x\in\mathbb{X}^n}\prod_{i=1}^n\{\alpha_\circ^\circ(x_i)/q(x_i)\} = 1 \text{ if } \alpha_\circ^\circ \le q.$$

Lemma 4.1. *Suppose that the multiple quantum-stochastic integral* $\mathrm{T}_t = \boldsymbol{\imath}_0^t(\mathrm{M})$ *is defined by a kernel operator-function* $\mathrm{M}(\boldsymbol{\upsilon}) = \boldsymbol{\epsilon}(M(\boldsymbol{\upsilon}))$ *with values in the operators of the form* (3.3) *for* $M(\boldsymbol{\upsilon}, \boldsymbol{\varkappa})$ *in terms of*

$$T_{\boldsymbol{\upsilon}}\begin{pmatrix}\varkappa_+^-, & \varkappa_\circ^- \\ \varkappa_+^\circ, & \varkappa_\circ^\circ\end{pmatrix} = M\begin{pmatrix}\upsilon_+^-, & \upsilon_\circ^-, & \varkappa_+^-, & \varkappa_\circ^- \\ \upsilon_+^\circ, & \upsilon_\circ^\circ, & \varkappa_+^\circ, & \varkappa_\circ^\circ\end{pmatrix}, \upsilon_\nu^\mu \in \mathcal{X},$$

for fixed $\boldsymbol{\upsilon}$ *and* $M(\boldsymbol{\upsilon}) : \varkappa \mapsto M(\boldsymbol{\upsilon},\varkappa)$ *is a kernel-valued integrand*

$$M(\boldsymbol{\upsilon},\varkappa) : \mathfrak{k}^{\otimes}(\upsilon_{\circ}^{-} \sqcup \varkappa_{\circ}^{-}) \otimes \mathfrak{k}^{\otimes}(\upsilon_{\circ}^{\circ} \sqcup \varkappa_{\circ}^{\circ}) \otimes \mathfrak{h} \to \mathfrak{k}^{\otimes}(\upsilon_{\circ}^{\circ} \sqcup \varkappa_{\circ}^{\circ}) \otimes \mathfrak{k}^{\otimes}(\upsilon_{+}^{\circ} \sqcup \varkappa_{+}^{\circ}) \otimes \mathfrak{h}.$$

Then $\mathrm{T}_t = \boldsymbol{\epsilon}(T_t)$ *for the kernel* $T_t(\boldsymbol{\vartheta}) = \boldsymbol{\nu}_0^t(\boldsymbol{\vartheta}, M)$ *given by the multiple counting integral on the kernel-integrands* M*, that is*

$$\boldsymbol{\imath}_0^t \circ \boldsymbol{\epsilon} = \boldsymbol{\epsilon} \circ \boldsymbol{\nu}_0^t \tag{4.4}$$

where

$$\boldsymbol{\nu}_0^t(\boldsymbol{\vartheta}, M) = \sum_{\boldsymbol{\upsilon} \subseteq \boldsymbol{\vartheta}^t} M(\boldsymbol{\upsilon}, \boldsymbol{\vartheta} \setminus \boldsymbol{\upsilon}), \tag{4.5}$$

with $\boldsymbol{\vartheta}^t = (\mathbb{X}^t \cap \vartheta_\nu^\mu)_{\nu=\circ,+}^{\mu=-,\circ}$ *such that the sum is taken over all possible* $\upsilon_\nu^\mu \subseteq \mathbb{X}^t \cap \vartheta_\nu^\mu$ *and* $\mu = -, \circ, \nu = \circ, +$.

If $M(\boldsymbol{\upsilon})$ *is relatively bounded in for each* $\boldsymbol{\upsilon} = (\upsilon_\nu^\mu)$ *such that*

$$\|M(\boldsymbol{\upsilon})\|_{\boldsymbol{\gamma}} \leq c \prod_{\mu,\nu} \beta_\nu^\mu(\upsilon_\nu^\mu), \quad \beta_\nu^\mu(\upsilon) = \prod_{x \in \upsilon} \beta_\nu^\mu(x)$$

for some $c > 0$ *and a pair of quadruples* $\boldsymbol{\beta} = (\beta_\nu^\mu)$ *,* $\beta_\nu^\mu \geq 0$ *and* $\boldsymbol{\gamma} = (\gamma_\nu^\mu)$*,* $\gamma_\nu^\mu \geq 0$ *satisfying the integrability conditions* (4.2) *for* $\boldsymbol{\gamma}$*, then the kernel* T *is relatively bounded:*

$$\|\boldsymbol{\nu}_0^t(M)\|_{\boldsymbol{\alpha}} \leq c$$

if $\alpha_\nu^\mu(x) \geq \beta_\nu^\mu(x) 1_{[0,t)}(x) + \gamma_\nu^\mu(x)$ *for all* μ*,* ν*, where* $1_{[0,t)}(x) = 1$ *if* $t(x) < t$ *and zero if* $t(x) \geq t$*. In particular, the generalized single integral* $\boldsymbol{\imath}_0^t(\mathbf{D})$ *of the triangular operator-integrand* $\mathbf{D}(x) = [\mathrm{D}_\nu^\mu(x)]$*, with* $\mathrm{D}_\nu^\mu(x) = \boldsymbol{\epsilon}(D_\nu^\mu(x))$*, is a representation*

$$\boldsymbol{\imath}_0^t \circ \boldsymbol{\epsilon} = \boldsymbol{\epsilon} \circ \boldsymbol{n}_0^t$$

of the single counting integral

$$\boldsymbol{n}_0^t(\boldsymbol{\vartheta}, D) = \sum_{\boldsymbol{x} \in \boldsymbol{\vartheta}^t} M(\mathbf{x}, \boldsymbol{\vartheta} \setminus \boldsymbol{x}), \quad M(\mathbf{x}_\nu^\mu, \varkappa) = D_\nu^\mu(x, \varkappa),$$

of the triangular kernel-integrand $D(x,\varkappa) = [D_\nu^\mu(x,\varkappa)]$*, where the sum is taken over all possible* $x \in \vartheta_\nu^\mu \cap \mathbb{X}^t$ *for* $\mu = -, \circ$ *and* $\nu = \circ, +$*, and* $\boldsymbol{x} = \boldsymbol{\upsilon}_\nu^\mu(x)$ *is one of the atomic tables*

$$\boldsymbol{x}_+^- = \begin{pmatrix} x & \emptyset \\ \emptyset & \emptyset \end{pmatrix}, \ \boldsymbol{x}_+^\circ = \begin{pmatrix} \emptyset & \emptyset \\ x & \emptyset \end{pmatrix}, \ \boldsymbol{x}_\circ^- = \begin{pmatrix} \emptyset & x \\ \emptyset & \emptyset \end{pmatrix}, \ \boldsymbol{x}_\circ^\circ = \begin{pmatrix} \emptyset & \emptyset \\ \emptyset & x \end{pmatrix}, \tag{4.6}$$

with indices $\mu(x) = \kappa$*,* $\nu(x) = \lambda$ *defined almost everywhere by the condition* $x \in \upsilon_\lambda^\kappa$*.*

Proof. If $M(\boldsymbol{\upsilon}, \varkappa)$ is an operator-valued integrand-kernel that is bounded relative to the pair $(\boldsymbol{\beta},\boldsymbol{\gamma})$ such that $\|M\|_{\boldsymbol{\beta},\boldsymbol{\gamma}} \leq c$, then the relatively bounded operator $\mathrm{T}_t = \boldsymbol{\epsilon}(T_t)$ is well-defined for $T_t = \boldsymbol{\nu}_0^t(M)$, since

$$\begin{aligned}\|T_t(\boldsymbol{\vartheta})\| &\leq \sum_{\upsilon_+^- \subseteq \vartheta_+^-}^{t(\upsilon_+^-)<t} \sum_{\upsilon_+^\circ \subseteq \vartheta_+^\circ}^{t(\upsilon_+^\circ)<t} \sum_{\upsilon_\circ^- \subseteq \vartheta_\circ^-}^{t(\upsilon_\circ^-)<t} \sum_{\upsilon_\circ^\circ \subseteq \vartheta_\circ^\circ}^{t(\upsilon_\circ^\circ)<t} \|M(\boldsymbol{\upsilon}, \boldsymbol{\vartheta} \setminus \boldsymbol{\upsilon})\| \\ &\leq c \prod_{\nu=\circ,+}^{\mu=-,\circ} \sum_{\upsilon_\nu^\mu \subseteq \vartheta_\nu^\mu}^{t(\upsilon_\nu^\mu)<t} \beta_\nu^\mu(\upsilon_\nu^\mu) \gamma_\nu^\mu(\vartheta_\nu^\mu \setminus \upsilon_\nu^\mu) = c \prod_{\nu=\circ,+}^{\mu=-,\circ} \alpha_\nu^\mu(\vartheta_\nu^\mu),\end{aligned}$$

where $\alpha^\mu_\nu(\vartheta) = \prod^{t(x)<t}_{x\in\vartheta}[\beta^\mu_\nu(x) + \gamma^\mu_\nu(x)] \cdot \prod^{t(x)\geq t}_{x\in\vartheta} \gamma^\mu_\nu(x)$ for $\beta^\mu_\nu(\upsilon) = \prod_{x\in\upsilon} \beta^\mu_\nu(x)$ and $\gamma^\mu_\nu(\varkappa) = \prod_{x\in\varkappa} \gamma^\mu_\nu(x)$. Applying the representation (3.3) to $T_t(\boldsymbol{\vartheta}) = \boldsymbol{\nu}^t_0(\boldsymbol{\vartheta}, M)$ it is easy to obtain the representation of the operator $\boldsymbol{\epsilon}(T_t)$ in the form of the generalized multiple integral (3.4) of $\mathrm{M}(\boldsymbol{\upsilon}) = \boldsymbol{\epsilon}(M(\boldsymbol{\upsilon}))$. Indeed, $[\mathrm{T}_t\chi](\vartheta) =$

$$= \sum_{\vartheta^\circ_\circ \sqcup \vartheta^\circ_+ = \vartheta} \iint \sum_{\boldsymbol{\upsilon}\subseteq\boldsymbol{\vartheta}^t} M(\boldsymbol{\upsilon}, \boldsymbol{\vartheta}\setminus\boldsymbol{\upsilon})\chi(\vartheta^\circ_\circ \sqcup \vartheta^-_\circ)\mathrm{d}\vartheta^-_\circ \mathrm{d}\vartheta^-_+$$

$$= \sum_{\upsilon^\circ_\circ \sqcup \upsilon^\circ_+ \subseteq \vartheta^t} \iint_{\mathcal{X}^t\times\mathcal{X}^t} \mathrm{d}\upsilon^-_\circ \mathrm{d}\upsilon^-_+ \sum_{\varkappa^\circ_\circ \sqcup \varkappa^\circ_+ = \vartheta^\circ_-} \iint M(\boldsymbol{\upsilon}, \varkappa)\mathring{\chi}(\upsilon^\circ_\circ \sqcup \upsilon^-_\circ, \varkappa^\circ_\circ \sqcup \varkappa^-_\circ)\mathrm{d}\varkappa^-_\circ \mathrm{d}\varkappa^-_+,$$

where $\vartheta^\circ_- = \vartheta \setminus (\upsilon^\circ_\circ \sqcup \upsilon^\circ_+)$, $\mathring{\chi}(\upsilon, \varkappa) = \chi(\varkappa \sqcup \upsilon)$. Consequently, $\mathrm{T}_t = \boldsymbol{\imath}^t_0(\mathrm{M})$, where

$$[\mathrm{M}(\boldsymbol{\upsilon})\mathring{\chi}(\upsilon^\circ_\circ \sqcup \upsilon^-_\circ)](\vartheta) = \sum_{\varkappa^\circ_\circ \sqcup \varkappa^\circ_+ = \vartheta} \iint M(\boldsymbol{\upsilon}, \varkappa)\mathring{\chi}(\upsilon^\circ_\circ \sqcup \upsilon^-_\circ,\ \varkappa^\circ_\circ \sqcup \varkappa^-_\circ)\mathrm{d}\varkappa^-_\circ \mathrm{d}\varkappa^-_+,$$

that is, we have proved that $\boldsymbol{\epsilon} \circ \boldsymbol{\nu}^t_0 = \boldsymbol{\imath}^t_0 \circ \boldsymbol{\epsilon}$.

In particular, if $M(\boldsymbol{\upsilon}, \varkappa) = 0$ for $\sum |\upsilon^\mu_\nu| \neq 1$, then, obviously

$$\boldsymbol{\nu}^t_0(\boldsymbol{\vartheta}, M) = \boldsymbol{n}^t_0(\boldsymbol{\vartheta}, D), \quad \boldsymbol{\imath}^t_0(\mathrm{M}) = \boldsymbol{i}^t_0(\mathbf{D}),$$

where $M^\mu_\nu(x, \varkappa) = M(\mathbf{x}^\mu_\nu, \varkappa)$ and $\mathrm{M}(\boldsymbol{\upsilon}) = 0$ for $\sum |\upsilon^\mu_\nu| \neq 1$, $\mathrm{D}^\mu_\nu(x) = \mathrm{M}(\mathbf{x}^\mu_\nu)$. This yields the representation $\boldsymbol{\epsilon} \circ \boldsymbol{n}^t_0 = \boldsymbol{i}^t_0 \circ \boldsymbol{\epsilon}$ for the single generalized non-adapted integral $\boldsymbol{i}^t_0(\mathbf{D}) = \int_{\mathbb{X}^t} \Lambda(\mathbf{D}, \mathrm{d}x)$, $\Lambda(\mathbf{D}, \triangle) = \Lambda^\nu_\mu(\mathrm{D}^\mu_\nu, \triangle)$ for $\triangle = \mathbb{X}^t$, in the form of the sum

$$\sum_{\mu,\nu} \Lambda^\nu_\mu(\boldsymbol{\epsilon}(D^\mu_\nu), \triangle) = \boldsymbol{\epsilon}\Big(\sum_{\mu,\nu} N^\nu_\mu(D^\mu_\nu, \triangle)\Big), \quad N^\nu_\mu(\boldsymbol{\vartheta}, D, \triangle) = \sum_{x\in\vartheta^\mu_\nu\cap\triangle} D(x, \boldsymbol{\vartheta}\setminus\boldsymbol{x}^\mu_\nu)$$

of representations of four kernel measures $N^\mu_\nu(\boldsymbol{\vartheta}, D^\mu_\nu, \triangle)$ that define kernel representations $\boldsymbol{\epsilon} \circ N(\triangle) = \Lambda(\triangle) \circ \boldsymbol{\epsilon}$ of the canonical measures $\Lambda(\mathbf{D}, \triangle)$ with $\mathrm{D}^\mu_\nu(x) = \boldsymbol{\epsilon}(D^\mu_\nu(x))$. □

One may now realize the commutative diagram of the quantum stochastic calculus (Fig. 1.).

5. Formulation of The Generalized Itô Formula

In the following theorem, which generalizes the Itô formula to non-commutative and non-adapted quantum stochastic processes $\mathrm{T}_t = \boldsymbol{\epsilon}(T_t)$ given by an operator-valued kernel $T_t(\boldsymbol{\vartheta})$, we use the following triangular-matrix notation

$$\mathbf{T}(x) = [\mathrm{T}(\mathbf{x}^\mu_\nu)], \quad \mathrm{T}(\mathbf{x}) = \nabla_{\mathbf{x}}(\mathrm{T}_t)|_{t=t(x)}$$

for the *QS point derivative* $\nabla_{\mathbf{x}}(\mathrm{T}) = \boldsymbol{\epsilon}\left(\dot{T}(\mathbf{x})\right)$ given by the QS point split [13, 9] of the kernel $\dot{T}(\mathbf{x}, \varkappa) = T(\varkappa \sqcup \boldsymbol{x})$, with $\mathrm{T}^\mu_\nu(x) = \mathrm{T}(\mathbf{x}^\mu_\nu)$ equal to zero for $\mu > \nu$ and $\mathrm{T}^-_-(x) = \mathrm{T}_{t(x)} = \mathrm{T}^+_+(x)$.

We notice that if $T_t(\boldsymbol{\vartheta}) = T_0(\boldsymbol{\vartheta}) + \boldsymbol{n}^t_0(\boldsymbol{\vartheta}, D)$, corresponding to the single-integral representation $\mathrm{T}_t - \mathrm{T}_0 = \boldsymbol{i}^t_0(\mathbf{D})$ with $\mathrm{D}^\mu_\nu(x) = \boldsymbol{\epsilon}(D^\mu_\nu(x))$, then $\dot{T}_t(\mathbf{x}, \varkappa) =$

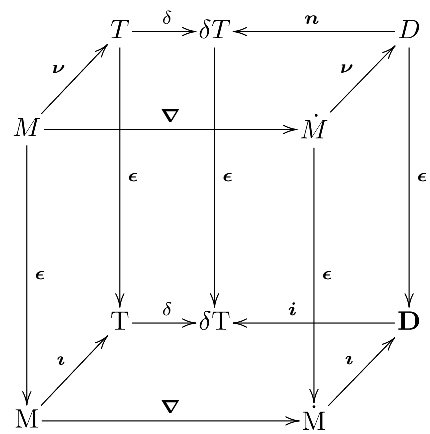

FIGURE 1. QS calculus commutative diagram. Here δ has been used to denote the variation $\delta\mathrm{T} = \mathrm{T}_t - \mathrm{T}_0$ of the operator T, and ∇ is the single point split operator [13, 9].

$T_t(\varkappa \sqcup \boldsymbol{x})$ may be given by

$$\dot{T}_t(\mathbf{x}, \varkappa) = \dot{T}_{t \wedge t_+(x)}(\mathbf{x}, \varkappa) + \sum_{\boldsymbol{z} \in \varkappa}^{t_+(x) \le t(z) < t} D(\mathbf{z}, (\varkappa \setminus \boldsymbol{z}) \sqcup \boldsymbol{x}).$$

Indeed, if $t \in (t(x), t_+(x)]$, then $\dot{T}_t(\mathbf{x}, \varkappa) = \dot{T}_{t_+(x)}(\mathbf{x}, \varkappa)$ and is therefore not dependent on t, where

$$t_+(x) = \min\{t(x') > t(x) : x' \in \sqcup \varkappa_\nu^\mu\},$$

and therefore the point-wise right limit

$$\dot{T}_{t_+(x)}(\mathbf{x}, \varkappa) := \lim_{t \searrow t(x)} \dot{T}_t(\mathbf{x}, \varkappa) = \dot{T}_{t(x)}(\mathbf{x}, \varkappa) + D(\mathbf{x}, \varkappa)$$

trivially exists in any uniform topology for each $\mathbf{x} \in \{\mathbf{x}_\nu^\mu : \mu, \nu = -, \circ, +\}$ and $\varkappa$; with

$$\dot{T}_{t_+(x)}(\mathbf{x}_-^-, \varkappa) = T_{t(x)}(\varkappa) = \dot{T}_{t_+(x)}(\mathbf{x}_+^+, \varkappa)$$

for $\dot{T}_t(\mathbf{x}_-^-, \varkappa) = T_t(\varkappa) = \dot{T}_t(\mathbf{x}_+^+, \varkappa)$ due to the independentcy of $T(\vartheta_\cdot^\cdot)$ on ϑ_-^- and ϑ_+^+.

Now we assume that the QS point split $\nabla_{\mathbf{x}}(\mathrm{T}_t) = \boldsymbol{\epsilon}\left(\dot{T}_t(\mathbf{x})\right)$, for $t > t(x)$, has the right limit

$$\mathrm{T}_+(\mathbf{x}) := \lim_{t \searrow t(x)} \nabla_{\mathbf{x}}(\mathrm{T}_t) \equiv \boldsymbol{\epsilon}\left(\dot{T}_{t_+(x)}(\mathbf{x})\right)$$

under the continuity of $\boldsymbol{\epsilon}$ in an appropriate topology on T_t, as they have trivially the limits

$$\mathrm{T}_+(\mathbf{x}_-^-) = \boldsymbol{\epsilon}\left(\dot{T}_{t(x)}(\mathbf{x}_-^-)\right) = \mathrm{T}_{t(x)} = \boldsymbol{\epsilon}\left(\dot{T}_{t(x)}(\mathbf{x}_+^+)\right) = \mathrm{T}_+(\mathbf{x}_+^+)$$

for $\mathbf{x} \in \{\mathbf{x}_-^-, \mathbf{x}_+^+\}$. The operator-valued triangular matrix function $\mathbf{T}_+(x) = [\mathrm{T}_+(\mathbf{x}_\nu^\mu)]$ of these limits is called the *QS germ* of the process T as an operator-valued function $t \mapsto \mathrm{T}_t$. As it is proved in the main theorem, these germ-limits are given as $\mathrm{T}_+(\mathbf{x}) = \mathrm{T}(\mathbf{x}) + \mathrm{D}(\mathbf{x})$ by the matrix elements $\mathrm{D}(\mathbf{x}_\nu^\mu)$ of the QS-derivatives $\mathbf{D} = [\mathrm{D}(\mathbf{x}_\nu^\mu)]$ and when the kernels T_t are given as the multiple counting integrals $T_t = \boldsymbol{\nu}_0^t(M)$ (4.5) we obtain the multiple QS integral representation $\mathrm{T}_t = \boldsymbol{\imath}_0^t(\mathrm{M})$ [13] with $\mathrm{M}(\boldsymbol{\upsilon}) = \boldsymbol{\epsilon}(M(\boldsymbol{\upsilon}))$ defining the matrix elements $\mathrm{D}_\nu^\mu(x)$ as

$$\mathrm{D}(\mathbf{x}_\nu^\mu) = \boldsymbol{\imath}_0^{t(x)}\left(\dot{\mathrm{M}}(\mathbf{x}_\nu^\mu)\right) = \boldsymbol{\epsilon}(D(\mathbf{x}_\nu^\mu))$$

for $x \in \mathbb{X}$ from each atomic table $\boldsymbol{x}_\nu^\mu \ni x$ in (4.6).

In the following Main Theorem taken from [10],[12] regarding the Itô product formula in terms of the kernels T we shall adopt the convention that given a chain $\varkappa = \varkappa_+^- \sqcup \varkappa_\circ^- \sqcup \varkappa_\circ^\circ \sqcup \varkappa_+^\circ$, we have sub-chains $\varkappa^\mu := \varkappa_\circ^\mu \sqcup \varkappa_+^\mu$, and $\varkappa_\nu := \varkappa_\nu^- \sqcup \varkappa_\nu^\circ$.

Definition 5.1. Let the kernels X, Y be given by the continuous operators $X(\boldsymbol{\sigma}) : \mathfrak{k}^\otimes(\sigma_\circ) \otimes \mathfrak{h} \to \mathfrak{k}^\otimes(\sigma^\circ) \otimes \mathfrak{h}$ and $Y(\boldsymbol{\tau}) : \mathfrak{k}^\otimes(\tau_\circ) \otimes \mathfrak{h} \to \mathfrak{k}^\otimes(\tau^\circ) \otimes \mathfrak{h}$, then the associative product $X \cdot Y$ [9] is given by

$$[X \cdot Y](\boldsymbol{\vartheta}) = \sum_{\upsilon_\circ^- \sqcup \tau_\circ^- = \vartheta_\circ^-}^{\sigma_+^\circ \sqcup \upsilon_+^\circ = \vartheta_+^\circ} \sum_{\sigma_+^- \sqcup \upsilon_+^- \sqcup \tau_+^- = \vartheta_+^-} X(\boldsymbol{\sigma}) Y(\boldsymbol{\tau}), \tag{5.1}$$

as a mapping of $\mathfrak{k}^\otimes(\tau_\circ) \otimes \mathfrak{h}$ into $\mathfrak{k}^\otimes(\sigma^\circ) \otimes \mathfrak{h}$, where $\upsilon_\circ^\circ = \vartheta_\circ^\circ$, $\vartheta^\circ = \sigma^\circ$, and $\tau_\circ = \vartheta_\circ$, and

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_+^- & \upsilon^- \\ \sigma_+^\circ & \upsilon^\circ \end{pmatrix}, \quad \boldsymbol{\tau} = \begin{pmatrix} \tau_+^- & \tau_\circ^- \\ \upsilon_+ & \upsilon_\circ \end{pmatrix},$$

with unit kernel $\hat{I}$ given by operators $\hat{I}(\boldsymbol{\vartheta}) = \hat{\mathrm{I}}(\vartheta_\circ^\circ) \otimes \delta_\emptyset(\vartheta \setminus \vartheta_\circ^\circ)$ such that $\hat{I} \cdot X = X = X \cdot \hat{I}$.

Main Theorem 5.2. (i) *If kernel $T(\boldsymbol{\vartheta})$ is relatively bounded, then the same is true for the kernel $T^\star(\boldsymbol{\vartheta}) : \|T^\star\|_{\boldsymbol{\gamma}} = \|T\|_{\boldsymbol{\gamma}'}$, where*

$$\boldsymbol{\gamma} = \begin{pmatrix} \gamma_+^- & \gamma_\circ^- \\ \gamma_+^\circ & \gamma_\circ^\circ \end{pmatrix}, \quad \boldsymbol{\gamma}' = \begin{pmatrix} \gamma_+^- & \gamma_+^\circ \\ \gamma_\circ^- & \gamma_\circ^\circ \end{pmatrix},$$

and the operator $\mathrm{T}^* = \boldsymbol{\epsilon}(T^\star)$, as well as the operator $\mathrm{T} = \boldsymbol{\epsilon}(T)$, is p-bounded by the estimate (4.3) *for $p \geq q + 1/r$. For any such kernels $T(\boldsymbol{\upsilon})$ and $T^\star(\boldsymbol{\upsilon})$, bounded relative to the quadruples $\boldsymbol{\alpha} = (\alpha_\nu^\mu)$ and $\boldsymbol{\gamma} = (\gamma_\nu^\mu)$ of functions $\alpha_\nu^\mu(x)$ and $\gamma_\nu^\mu(x)$ satisfying* (4.2), *the operator*

$$\boldsymbol{\epsilon}(T)\boldsymbol{\epsilon}(T)^* = \boldsymbol{\epsilon}(T \cdot T^\star), \quad \boldsymbol{\epsilon}(\hat{I}) = \hat{\mathrm{I}}, \textit{ where } \hat{I} = \mathbf{1}_\mathfrak{h} \otimes I^\otimes,$$

is well-defined as a $\star$-representation of kernel product (5.1) *having the estimate $\|T \cdot T^\star\|_{\boldsymbol{\beta}} \leq \|T\|_{\boldsymbol{\alpha}} \|T^\star\|_{\boldsymbol{\gamma}}$ if $\beta_\nu^\mu \geq (\boldsymbol{\alpha} \cdot \boldsymbol{\gamma})_\nu^\mu$, where $(\boldsymbol{\alpha} \cdot \boldsymbol{\gamma})_\nu^\mu(x) = \sum \alpha_\kappa^\mu(x) \gamma_\nu^\kappa(x)$ is defined by the product of triangular matrices*

$$\begin{bmatrix} 1 & \alpha_\circ^- & \alpha_+^- \\ 0 & \alpha_\circ^\circ & \alpha_+^\circ \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \gamma_\circ^- & \gamma_+^- \\ 0 & \gamma_\circ^\circ & \gamma_+^\circ \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \alpha_\circ^- \gamma_\circ^\circ + \gamma_\circ^-, & \gamma_+^- + \alpha_\circ^- \gamma_+^\circ + \alpha_+^- \\ 0, & \alpha_\circ^\circ \gamma_\circ^\circ, & \alpha_\circ^\circ \gamma_+^\circ + \gamma_+^\circ \\ 0, & 0, & 1 \end{bmatrix}.$$

(ii) *Let* $\mathrm{T}_t = \boldsymbol{\epsilon}(T_t)$ *with* $\dot{T}_t(\mathbf{x},\varkappa) = T_t(\boldsymbol{x} \sqcup \varkappa)$ *having the QS right limit at* $t \searrow t(x)$. *Let* $\mathbf{T}(x) = \left[\nabla_{\mathbf{x}^\mu_\nu}\left(\mathrm{T}_{t(x)}\right)\right]$ *and* $\mathbf{T}_+(x) = [\mathrm{T}_+(\mathbf{x}^\mu_\nu)]$ *denote triangular matrices at* $\mathbf{x}$ *with* $t = t(x)$ *having the operator-valued matrix elements*

$$\mathrm{T}(\mathbf{x}^\mu_\nu) = \boldsymbol{\epsilon}(\dot{T}_{t(x)}(\mathbf{x}^\mu_\nu)) \equiv \mathrm{T}^\mu_\nu(x), \quad \mathrm{T}_+(\mathbf{x}^\mu_\nu) = \boldsymbol{\epsilon}(\dot{T}_{t+(x)}(\mathbf{x}^\mu_\nu)) \equiv \mathrm{G}^\mu_\nu(x) \tag{5.2}$$

corresponding to the single point split $\dot{T}_t(\mathbf{x}^\mu_\nu)$ *at* $x \in \mathbf{x}^\mu_\nu$ *with* $t(x) \le t$. *Then the operator-functions* $\mathrm{D}^\mu_\nu(x) = \mathrm{G}^\mu_\nu(x) - \mathrm{T}^\mu_\nu(x)$ *are quantum-stochastic derivatives of the function* $t \mapsto \mathrm{T}_t$ *which define the QS differential* $\mathrm{dT}_t = \mathrm{d}\boldsymbol{i}^t_0(\mathbf{D})$ *in the difference form so that* $\mathrm{T}_t - \mathrm{T}_0 = \boldsymbol{i}^t_0(\mathbf{T}_+ - \mathbf{T})$. *Moreover,* $\mathrm{T}^*_t - \mathrm{T}^*_0 = \boldsymbol{i}^t_0(\mathbf{T}^\ddagger_+ - \mathbf{T}^\ddagger)$, *and we have the generalized non-adapted Itô formula*

$$\mathrm{T}_t\mathrm{T}^*_t - \mathrm{T}_0\mathrm{T}^*_0 = \boldsymbol{i}^t_0(\mathbf{T}\mathbf{D}^\ddagger + \mathbf{D}\mathbf{T}^\ddagger + \mathbf{D}\mathbf{D}^\ddagger) = \boldsymbol{i}^t_0(\mathbf{T}_+\mathbf{T}^\ddagger_+ - \mathbf{T}\mathbf{T}^\ddagger), \tag{5.3}$$

where $\mathbf{D} \mapsto \mathbf{D}^\ddagger$ *is the pseudo-Euclidean conjugation* $[\mathrm{D}^\mu_\nu(x)]^\ddagger = [\mathrm{D}^{-\nu}_{-\mu}(x)]^*$ *of the triangular operators*

$$\mathbf{T} = \begin{bmatrix} \mathrm{T} & \mathrm{T}^-_\circ & \mathrm{T}^-_+ \\ 0 & \mathrm{T}^\circ_\circ & \mathrm{T}^\circ_+ \\ 0 & 0 & \mathrm{T} \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 & \mathrm{D}^-_\circ & \mathrm{D}^\circ_+ \\ 0 & \mathrm{D}^\circ_\circ & \mathrm{D}^\circ_+ \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{T}_+ = \begin{bmatrix} \mathrm{T} & \mathrm{G}^-_\circ & \mathrm{G}^-_+ \\ 0 & \mathrm{G}^\circ_\circ & \mathrm{G}^\circ_+ \\ 0 & 0 & \mathrm{T} \end{bmatrix} \equiv \mathbf{G}$$

with the standard block-matrix multiplication $(\mathbf{XY})^\mu_\nu = \mathrm{X}^\mu_\kappa \mathrm{Y}^\kappa_\nu$, *where* κ *is summed over the indices* $-, \circ, +$.

Proof. (i) The adjoint operators $\boldsymbol{\epsilon}(T)$ and $\boldsymbol{\epsilon}(T^\star)$, defining the $\star$-representation (3.3) with respect to the kernels T, bounded in the sense of (4.1) and (4.2), are p-bounded for $p \ge q + 1/r$ by the estimate $\|\boldsymbol{\epsilon}(T)\|_p \le \|T\|_q(r)$ and inequality (4.3); this leads to the exponential estimate

$$\|\boldsymbol{\epsilon}(T)\|_p \le \|T\|_{\boldsymbol{\alpha}} \exp\left\{\|\alpha^-_+\|^{(1)} + \tfrac{1}{2}\left(\|\alpha^\circ_+\|^{(2)}(r)^2 + \|\alpha^-_\circ\|^{(2)}(r)^2\right)\right\}.$$

The formula for the kernel multiplication $T^\star \cdot T$ which is given above (5.1) is $\boldsymbol{\beta}$-bounded for $\boldsymbol{\beta} = \boldsymbol{\alpha}\cdot\boldsymbol{\gamma}$, since $\|[T \cdot T^\star](\boldsymbol{\vartheta})\| \le$

$$\begin{aligned}
&\le \sum \left\| T\begin{pmatrix} \vartheta^-_+ \setminus \sigma^-_+, & \varkappa^-_\circ \sqcup \varkappa^-_+ \\ \vartheta^\circ_+ \setminus \varkappa^\circ_+, & \vartheta^\circ_\circ \sqcup \varkappa^\circ_+ \end{pmatrix} \right\| \cdot \left\| T^\star\begin{pmatrix} \vartheta^-_+ \setminus \tau^-_+, & \vartheta^-_\circ \setminus \varkappa^-_\circ \\ \varkappa^-_+ \sqcup \varkappa^\circ_+, & \vartheta^\circ_\circ \sqcup \varkappa^-_\circ \end{pmatrix} \right\| \\
&\le \|T\|_{\boldsymbol{\alpha}} \|T^\star\|_{\boldsymbol{\gamma}} \sum \boldsymbol{\alpha}^\otimes \begin{pmatrix} \vartheta^-_+ \setminus \sigma^-_+, & \varkappa^-_\circ \sqcup \varkappa^-_+ \\ \vartheta^\circ_+ \setminus \varkappa^\circ_+, & \vartheta^\circ_\circ \sqcup \varkappa^\circ_+ \end{pmatrix} \boldsymbol{\gamma}^\otimes \begin{pmatrix} \vartheta^-_+ \setminus \tau^-_+, & \vartheta^-_\circ \setminus \varkappa^-_\circ \\ \varkappa^-_+ \sqcup \varkappa^\circ_+, & \vartheta^\circ_\circ \sqcup \varkappa^-_\circ \end{pmatrix} \\
&= \|T\|_{\boldsymbol{\alpha}} \|T^\star\|_{\boldsymbol{\gamma}} (\boldsymbol{\alpha}\cdot\boldsymbol{\gamma})^\otimes(\vartheta); \qquad (\boldsymbol{\alpha}\cdot\boldsymbol{\gamma})^\mu_\nu = \sum_{\mu \le \kappa \le \nu} \alpha^\mu_\kappa \gamma^\kappa_\nu,
\end{aligned}$$

where we have employed the multiplication formula $\boldsymbol{\alpha}^\otimes \cdot \boldsymbol{\gamma}^\otimes = (\boldsymbol{\alpha}\cdot\boldsymbol{\gamma})^\otimes$ for scalar exponential kernels

$$\boldsymbol{\beta}^\otimes(\boldsymbol{\vartheta}) = \prod \beta^\mu_\nu(\vartheta^\mu_\nu); \; \beta^\mu_\nu(\vartheta) = \prod_{x \in \vartheta} \beta^\mu_\nu(x) : \; (\boldsymbol{\alpha}\cdot\boldsymbol{\gamma})^\mu_\nu(x) = \sum \alpha^\mu_\kappa(x)\gamma^\kappa_\nu(x).$$

Using the main formula

$$\int \sum_{\upsilon \subseteq \vartheta} f(\upsilon, \vartheta \setminus \upsilon)\mathrm{d}\vartheta = \iint f(\upsilon, \varkappa)\mathrm{d}\upsilon\mathrm{d}\varkappa, \quad \forall f \in L^1(\mathcal{X} \times \mathcal{X}), \tag{5.4}$$

of the scalar integration [13, 9], we write the scalar square of the action (3.3) in the form $\|\boldsymbol{\epsilon}(T)\chi\|^2 =$

$$
\begin{aligned}
&= \int \left\| \sum_{\vartheta_\circ^\circ \sqcup \vartheta_+^\circ = \vartheta^\circ} \iint T(\boldsymbol{\vartheta})\chi(\vartheta_\circ)\mathrm{d}\vartheta_+^- \mathrm{d}\vartheta_\circ^- \right\|^2 \mathrm{d}\vartheta^\circ \\
&= \iint \iiint \int \sum_{\sigma_\circ^\circ \sqcup \sigma_\circ^- = \vartheta^\circ} \sum_{\tau_\circ^\circ \sqcup \tau_+^\circ = \vartheta^\circ} \left\langle T(\boldsymbol{\sigma'})\chi(\sigma^\circ) \middle| T(\boldsymbol{\tau})\chi(\tau_\circ) \right\rangle \mathrm{d}\vartheta^\circ \mathrm{d}\sigma_+^- \mathrm{d}\sigma_+^\circ \mathrm{d}\tau_+^- \mathrm{d}\tau_\circ^- \\
&= \iint \iiint \int \left\langle \chi(\sigma^\circ) \middle| T^\star(\boldsymbol{\sigma})\, T(\boldsymbol{\tau})\, \chi(\tau_\circ^- \sqcup \varkappa_\circ) \right\rangle \mathrm{d}\varkappa \mathrm{d}\sigma_+^- \mathrm{d}\sigma_+^\circ\, \mathrm{d}\tau_+^- \mathrm{d}\tau_\circ^- \\
&= \int \left\langle \chi(\vartheta^\circ) \middle| \sum_{\vartheta_\circ^\circ \sqcup \vartheta_+^\circ = \vartheta^\circ} \iint (T^\star \cdot T)(\boldsymbol{\vartheta})\chi(\vartheta_\circ)\mathrm{d}\vartheta_+^- \mathrm{d}\vartheta_\circ^- \right\rangle \mathrm{d}\vartheta^\circ \\
&= \langle \chi | \epsilon(T^\star \cdot T)\chi \rangle,
\end{aligned}
$$

where $\varkappa_\circ^\circ = \sigma_\circ^\circ \cap \tau_\circ^\circ$, $\varkappa_+^\circ = \sigma_\circ^\circ \cap \tau_+^\circ$, $\varkappa_\circ^- = \tau_\circ^\circ \cap \sigma_\circ^-$, $\varkappa_+^- = \sigma_\circ^- \cap \tau_+^\circ$,

$$
\boldsymbol{\sigma} = \begin{pmatrix} \sigma_+^- & \varkappa^- \\ \sigma_+^\circ & \varkappa^\circ \end{pmatrix}, \quad \boldsymbol{\tau} = \begin{pmatrix} \tau_+^- & \tau_\circ^- \\ \varkappa_+ & \varkappa_\circ \end{pmatrix}
$$

and the integral over $\mathrm{d}\vartheta^\circ$ of the double sum

$$
\sum_{\sigma_\circ^\circ \sqcup \sigma_\circ^- = \vartheta^\circ} \sum_{\tau_\circ^\circ \sqcup \tau_+^\circ = \vartheta^\circ} = \sum_{\varkappa_\circ^\circ \sqcup \varkappa_+^\circ \sqcup \varkappa_\circ^- \sqcup \varkappa_+^- = \vartheta^\circ}
$$

is replaced by the quadruple integral over $\mathrm{d}\varkappa = \mathrm{d}\varkappa_\circ^\circ \mathrm{d}\varkappa_+^\circ \mathrm{d}\varkappa_\circ^- \mathrm{d}\varkappa_+^-$. In the last line we have substituted back $\vartheta_\circ^- = \tau_\circ^- \sqcup \varkappa_\circ^-$, $\vartheta_+^\circ = \sigma_+^\circ \sqcup \varkappa_+^\circ$, $\vartheta_+^- = \sigma_+^- \sqcup \varkappa_+^- \sqcup \tau_+^-$, $\varkappa_\circ^\circ = \vartheta_\circ^\circ$, and also $\vartheta^\circ = \sigma^\circ$, $\vartheta_\circ = \tau_\circ$. Indeed it follows that $\epsilon(T)^* \epsilon(T) = \epsilon(T^\star \cdot T)$, from which we also obtain $\epsilon(\hat{I}) = \hat{\mathrm{I}}$ that may also be trivially verified from (3.3). □

(ii) We shall now consider the stochastic differential dT_t of the multiple integral $\mathrm{T}_t = \boldsymbol{\imath}_0^t(\mathrm{M})$ of the operator function $\mathrm{M}(\boldsymbol{\upsilon}) = \boldsymbol{\epsilon}(M(\boldsymbol{\upsilon}))$ defined by the quantum-stochastic derivatives

$$
\mathrm{D}_\nu^\mu(x) = \boldsymbol{\imath}_0^{t(x)} \circ \boldsymbol{\epsilon}\left(\dot{M}(\mathbf{x}_\nu^\mu)\right) = \boldsymbol{\epsilon} \circ \boldsymbol{\nu}_0^{t(x)}\left(\dot{M}(\mathbf{x}_\nu^\mu)\right) = \boldsymbol{\epsilon}(D_\nu^\mu(x)),
$$

representing the differences of the kernels

$$
D(\mathbf{x}_\nu^\mu, \varkappa) = \boldsymbol{\nu}_0^{t(x)}\left(\varkappa, \dot{M}(\mathbf{x}_\nu^\mu)\right) = \dot{T}_{t_+(x)}(\mathbf{x}_\nu^\mu, \varkappa) - \dot{T}_{t(x)}(\mathbf{x}_\nu^\mu, \varkappa).
$$

Here $\boldsymbol{\nu}_0^t\left(\varkappa, \dot{M}(\mathbf{x})\right) = \sum_{\boldsymbol{\upsilon} \subseteq \varkappa^t} M(\boldsymbol{\upsilon} \sqcup \boldsymbol{x}, \varkappa \setminus \boldsymbol{\upsilon})$, $\boldsymbol{x}$ is one of the atomic tables (4.6),

$$
\dot{T}_{t(x)}(\mathbf{x}, \varkappa) = \sum_{\boldsymbol{\upsilon} \subseteq \varkappa^{t(x)}} M(\boldsymbol{\upsilon}, (\varkappa \sqcup \boldsymbol{x}) \setminus \boldsymbol{\upsilon}) = T_{t(x)}(\varkappa \sqcup \boldsymbol{x}),
$$

where $\varkappa^{t(x)} = \varkappa \cap [0, t(x))$, and

$$\dot{T}_{t+(x)}(\mathbf{x}, \varkappa) = \sum_{\boldsymbol{\upsilon} \subseteq \varkappa^{t(x)} \sqcup \boldsymbol{x}} M(\boldsymbol{\upsilon}, (\varkappa \sqcup \boldsymbol{x}) \backslash \boldsymbol{\upsilon})$$
$$= T_{t(x)}(\varkappa \sqcup \boldsymbol{x}) + \sum_{\boldsymbol{\upsilon} \subseteq \varkappa^{t(x)}} M(\boldsymbol{\upsilon} \sqcup \boldsymbol{x}, \varkappa \setminus \boldsymbol{\upsilon})$$
$$= \dot{T}_{t(x)}(\mathbf{x}, \varkappa) + \boldsymbol{\nu}_0^{t(x)} \left(\varkappa, \dot{M}(\mathbf{x}) \right).$$

We note that $T_{t_+}(\boldsymbol{\vartheta}) = \sum_{\boldsymbol{\upsilon} \subseteq \boldsymbol{\vartheta}^{t_+}} M(\boldsymbol{\upsilon}, \boldsymbol{\vartheta} \setminus \boldsymbol{\upsilon})$, where $t_+ = \min\{t(\boldsymbol{x}) > t : \boldsymbol{x} \in \boldsymbol{\vartheta}\}$, $\boldsymbol{\vartheta}^{t_+} = \{\boldsymbol{x} \in \boldsymbol{\vartheta} : t(x) \leq t\}$, so that $\dot{T}_{t+(x)}(\mathbf{x}, \varkappa) = \dot{T}_t(\mathbf{x}, \varkappa)$ for any $t \in (t(\boldsymbol{x}), t_+(\boldsymbol{x})]$. Thus the derivatives $\mathrm{D}_\nu^\mu(x)$, $x \in \mathbb{X}^t$, defining the increment $\mathrm{T}_t - \mathrm{T}_0 = \boldsymbol{i}_0^t(\mathbf{D})$, can be written in the form of the differences

$$\mathrm{D}_\nu^\mu(x) = \boldsymbol{\epsilon} \left(\dot{T}_{t+(x)}(\mathbf{x}_\nu^\mu) \right) - \boldsymbol{\epsilon} \left(\dot{T}_{t(x)}(\mathbf{x}_\nu^\mu) \right)$$

of the operators (5.2). If we consider $\dot{T}_t(\mathbf{x})$ as one of the four entries $\dot{T}_t(x)_\nu^\mu = \dot{T}_t(\mathbf{x}_\nu^\mu)$ in the matrix-kernel $\left[\dot{T}_t(x)_\nu^\mu \right] \equiv \dot{\mathbf{T}}_t(x)$ with $\dot{T}_t(x)_-^- = T_{t(x)} = \dot{T}_t(x)_+^+$, we can define the triangular matrix-functions

$$\mathbf{T}(x) = \boldsymbol{\epsilon} \left(\dot{\mathbf{T}}_{t(x)}(x) \right), \quad \mathbf{T}_+(x) = \boldsymbol{\epsilon} \left(\dot{\mathbf{T}}_{t_+(x)}(x) \right).$$

This allows us to obtain the quantum non-adapted Itô formula in the form

$$\mathrm{T}_t \mathrm{T}_t^* - \mathrm{T}_0 \mathrm{T}_0^* = \boldsymbol{i}_0^t (\mathbf{T}\mathbf{D}^\ddagger + \mathbf{D}\mathbf{T}^\ddagger + \mathbf{D}\mathbf{D}^\ddagger),$$

where $\mathbf{D}(x) = \mathbf{T}_+(x) - \mathbf{T}(x)$ as a consequence of the fact that the map (3.3) is a $\star$-homomorphism $\mathrm{T}_t \mathrm{T}_t^* = \boldsymbol{\epsilon}(T_t \cdot T_t^\star)$, and of the formula (5.1) for the product of the operator-valued kernels T_t and $T_t^\star$ which can be written in the form

$$\left[T_t \cdot T_t^\star \right] (\boldsymbol{\vartheta} \sqcup \boldsymbol{x}_\nu^\mu) = \sum_{\kappa=\mu}^{\nu} \left(\dot{T}_t(x)_\kappa^\mu \cdot \dot{T}_t^\star(x)_\nu^\kappa \right) (\boldsymbol{\vartheta}) = \left(\dot{\mathbf{T}}_t \cdot \dot{\mathbf{T}}_t^\ddagger \right)_\nu^\mu (x, \boldsymbol{\vartheta}).$$

Here the right-hand side is computed as an entry in the product of the kernel-valued triangular operator $\dot{\mathbf{T}}_t(x) = \dot{T}_t(\mathbf{x}_\cdot^\cdot)$ with $\dot{\mathbf{T}}_t^\ddagger(x) = \dot{T}_t^\star(\mathbf{x}_\cdot^\cdot)$ which defines the multiplication of the entries in terms of the product of the triangular operator-valued kernel $\left[\dot{T}_t(\mathbf{x}_\nu^\mu) \right]$ and $\left[\dot{T}_t^\star(\mathbf{x}_\nu^\mu) \right] = \left[\dot{T}_t(\mathbf{x}_{-\mu}^{-\nu})^* \right]$ with $\dot{T}_t(\mathbf{x}_-^-, \boldsymbol{\vartheta})^* = T_t(\boldsymbol{\vartheta})^* = \dot{T}_t(\mathbf{x}_+^+, \boldsymbol{\vartheta})^*$ and $\dot{T}(\mathbf{x}, \boldsymbol{\vartheta})^* = T(\boldsymbol{\vartheta} \sqcup \boldsymbol{x})^*$. Indeed, from (5.1) we obtain

$$[T \cdot T^\star](\boldsymbol{\vartheta} \sqcup \boldsymbol{x}_\circ^\circ) = [\dot{T}(\mathbf{x}_\circ^\circ) \cdot \dot{T}^\star(\mathbf{x}_\circ^\circ)](\boldsymbol{\vartheta}),$$
$$[T \cdot T^\star](\boldsymbol{\vartheta} \sqcup \boldsymbol{x}_+^\circ) = [\dot{T}(\mathbf{x}_\circ^\circ) \cdot \dot{T}^\star(\mathbf{x}_+^\circ) + \dot{T}(\mathbf{x}_+^\circ) \cdot T^\star](\boldsymbol{\vartheta}),$$
$$[T \cdot T^\star](\boldsymbol{\vartheta} \sqcup \boldsymbol{x}_\circ^-) = [T \cdot \dot{T}^\star(\mathbf{x}_\circ^-) + \dot{T}(\mathbf{x}_\circ^-) \cdot \dot{T}^\star(\mathbf{x}_\circ^\circ)](\boldsymbol{\vartheta}),$$
$$[T \cdot T^\star](\boldsymbol{\vartheta} \sqcup \boldsymbol{x}_+^-) = [T \cdot \dot{T}^\star(\mathbf{x}_+^-) + \dot{T}(\mathbf{x}_\circ^-) \cdot \dot{T}^\star(\mathbf{x}_+^\circ) + \dot{T}(\mathbf{x}_+^-) \cdot T^\star](\boldsymbol{\vartheta}),$$

which are the matrix elements of the kernel-valued triangular operator

$$\left[[T \cdot T^\star] (\boldsymbol{\vartheta} \sqcup \boldsymbol{x}_\nu^\mu) \right] = \left[\dot{T}(\mathbf{x}_\lambda^\mu) \cdot \dot{T}^\star(\mathbf{x}_\nu^\lambda) \right] (\boldsymbol{\vartheta}) = \left(\dot{\mathbf{T}} \cdot \dot{\mathbf{T}}^\ddagger \right) (x, \boldsymbol{\vartheta}).$$

This allows us to write the operator-valued triangular matrix

$$\left[\nabla_{\mathbf{x}^{\mu}_{\nu}} \boldsymbol{\epsilon}\left(T \cdot T^{\star}\right)\right] = \sum_{\kappa=\mu}^{\nu} \boldsymbol{\epsilon}\left[\dot{T}(x)^{\mu}_{\kappa} \cdot \dot{T}^{\star}(x)^{\kappa}_{\nu}\right] = \boldsymbol{\epsilon}\left(\dot{\mathbf{T}} \cdot \dot{\mathbf{T}}^{\ddagger}\right)(x)$$

as the block-matrix product

$$\boldsymbol{\epsilon}\left(\dot{\mathbf{T}}(x) \cdot \dot{\mathbf{T}}^{\ddagger}(x)\right) = \boldsymbol{\epsilon}\left(\dot{\mathbf{T}}(x)\right)\boldsymbol{\epsilon}\left(\dot{\mathbf{T}}^{\ddagger}(x)\right) = \boldsymbol{\epsilon}\left(\dot{\mathbf{T}}(x)\right)\boldsymbol{\epsilon}\left(\dot{\mathbf{T}}(x)\right)^{\ddagger}$$

of $\mathbf{T} = \boldsymbol{\epsilon}\left(\dot{\mathbf{T}}\right)$ and $\mathbf{T}^{\ddagger} = \boldsymbol{\epsilon}\left(\dot{\mathbf{T}}\right)^{\ddagger}$. So we have proved that

$$\boldsymbol{\nabla}_x\left(\boldsymbol{\epsilon}(T \cdot T^{\star})\right) = \mathbf{T}(x)\,\mathbf{T}^{\ddagger}(x) \equiv \boldsymbol{\nabla}_x\left(\boldsymbol{\epsilon}(T)\right)\boldsymbol{\nabla}_x\left(\boldsymbol{\epsilon}(T^{\star})\right)$$

in terms of the germ-matrices $\mathbf{T}(x) = \boldsymbol{\nabla}_x\left(\boldsymbol{\epsilon}(T)\right)$, $\mathbf{T}^{\ddagger}(x) = \boldsymbol{\nabla}_x\left(\boldsymbol{\epsilon}(T^{\star})\right)$ having the operator entries

$$\mathbf{T}(x)^{\mu}_{\nu} = \boldsymbol{\epsilon}\left(\dot{T}\left(\mathbf{x}^{\mu}_{\nu}\right)\right),\ \mathbf{T}^{\ddagger}(x)^{\mu}_{\nu} = \boldsymbol{\epsilon}\left(\dot{T}\left(\mathbf{x}^{-\nu}_{-\mu}\right)\right)^{*}.$$

Thus we evaluate the germ of $\mathrm{T}_t\mathrm{T}^*_t$ at $t = t(x)$ and $t = t_+(x)$ and obtain the difference formula

$$\boldsymbol{\epsilon}\left(\left(\dot{\mathbf{T}}_{t_+(x)} \cdot \dot{\mathbf{T}}^{\ddagger}_{t_+(x)}\right)(x) - \left(\dot{\mathbf{T}}_{t(x)} \cdot \dot{\mathbf{T}}^{\ddagger}_{t(x)}\right)(x)\right) = \mathbf{T}_+(x)\mathbf{T}^{\ddagger}_+(x) - \mathbf{T}(x)\mathbf{T}^{\ddagger}(x),$$

which allows us to write the stochastic derivative of the quantum non-adapted process $\mathrm{T}_t\mathrm{T}^*_t$ in the form

$$\mathrm{d}\left(\mathrm{T}_t\mathrm{T}^*_t\right) = \mathrm{d}\boldsymbol{i}^t_0\left(\mathbf{T}_+\mathbf{T}^{\ddagger}_+ - \mathbf{T}\mathbf{T}^{\ddagger}\right),$$

corresponding to (5.3). The theorem has been proved. □

6. Weak Form and Q-Adapted Quantum Itô Formula

Using the non-adapted table of stochastic multiplication,

$$\begin{aligned}
\mathbf{T}^{\ddagger}_+\mathbf{T}_+ - \mathbf{T}^{\ddagger}\mathbf{T} &= \mathbf{D}^{\ddagger}\mathbf{T} + \mathbf{T}^{\ddagger}\mathbf{D} + \mathbf{D}^{\ddagger}\mathbf{D} \\
&= \begin{bmatrix} 0, & \mathrm{T}^*\mathrm{D}^-_\circ, & \mathrm{T}^*\mathrm{D}^-_+ + \mathrm{D}^{-*}_+\mathrm{T} \\ 0, & 0, & \mathrm{D}^{-*}_\circ\mathrm{T} \\ 0, & 0, & 0 \end{bmatrix} + \begin{bmatrix} 0, & \mathrm{D}^{\circ*}_+\mathrm{D}^\circ_\circ, & \mathrm{D}^{\circ*}_+\mathrm{D}^\circ_+ \\ 0, & \mathrm{D}^{\circ*}_\circ\mathrm{D}^\circ_\circ, & \mathrm{D}^{\circ*}_\circ\mathrm{D}^\circ_+ \\ 0, & 0, & 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} 0, & \mathrm{D}^{\circ*}_+\mathrm{T}^\circ_\circ + \mathrm{T}^{\circ*}_+\mathrm{D}^\circ_\circ, & \mathrm{D}^{\circ*}_+\mathrm{T}^\circ_+ + \mathrm{T}^{\circ*}_+\mathrm{D}^\circ_+ \\ 0, & \mathrm{D}^{\circ*}_\circ\mathrm{T}^\circ_\circ + \mathrm{T}^{\circ*}_\circ\mathrm{D}^\circ_\circ, & \mathrm{D}^{\circ*}_\circ\mathrm{T}^\circ_+ + \mathrm{T}^{\circ*}_\circ\mathrm{D}^\circ_+ \\ 0, & 0, & 0 \end{bmatrix},
\end{aligned}$$

we can write (5.3) in the weak form $\|\mathrm{T}_t\chi\|^2 - \|\mathrm{T}_0\chi\|^2 =$

$$\begin{aligned}
= \int_{\mathbb{X}^t} 2\Re\left\langle \mathrm{T}_{t(x)}\chi \mid \mathrm{D}^-_+(x)\chi + \mathrm{D}^-_\circ(x)\mathring{\chi}(x)\right\rangle \mathrm{d}x + \int_{\mathbb{X}^t}\left\|\mathrm{D}^\circ_+(x)\chi + \mathrm{D}^\circ_\circ(x)\mathring{\chi}(x)\right\|^2 \mathrm{d}x \\
+ \int_{\mathbb{X}^t} 2\Re\left\langle \nabla_x\mathrm{T}_{t(x)}\chi \mid \mathrm{D}^\circ_+(x)\chi + \mathrm{D}^\circ_\circ(x)\mathring{\chi}(x)\right\rangle \mathrm{d}x, \qquad (6.1)
\end{aligned}$$

where $\nabla_x\mathrm{T}_{t(x)}\chi = \mathrm{T}^\circ_+(x)\chi + \mathrm{T}^\circ_\circ(x)\mathring{\chi}(x)$. This formula is valid for any non-adapted single integral $\mathrm{T}_t = \mathrm{T}_0 + \boldsymbol{i}^t_0(\mathbf{D})$ with square integrable values $\mathrm{T}_t\chi$ for all $\chi \in G_+$ with ∇_x understood as the Malliavin derivative [22] at the point $x \in \mathbb{X}$ represented in Fock space by $[\nabla_x\mathrm{T}_{t(x)}\chi](\vartheta) = [\mathrm{T}_{t(x)}\chi](\vartheta \sqcup x)$.

Indeed, taking into account that

$$\left\langle \chi \mid \boldsymbol{i}_0^t(\mathbf{D})\chi \right\rangle = \int_{\mathbb{X}^t} \left[\left\langle \chi \mid \mathrm{D}_+^-(x)\chi + \mathrm{D}_\circ^- \mathring{\chi}(x) \right\rangle + \left\langle \mathring{\chi}(x) \mid \mathrm{D}_+^\circ(x)\chi + \mathrm{D}_\circ^\circ(x)\mathring{\chi}(x) \right\rangle \right] \mathrm{d}x,$$

we readily obtain the weak form of the non-adapted Itô formula if we substitute $\mathbf{D}^\ddagger\mathbf{T} + \mathbf{D}^\ddagger\mathbf{D} + \mathbf{T}^\ddagger\mathbf{D}$ in place of $\mathbf{D}$. But first notice that we may write

$$\left\langle \chi \mid \boldsymbol{i}_0^t(\mathbf{D})\chi \right\rangle = \int_{\mathbb{X}^t} \langle \chi \mid \boldsymbol{\nabla}_x^\ddagger \mathbf{D}(x) \boldsymbol{\nabla}_x \chi \rangle \mathrm{d}x,$$

where $\langle \chi \mid \boldsymbol{\nabla}_x^\ddagger = (\chi^*, \mathring{\chi}^*(x), 0)$, and $\boldsymbol{\nabla}_x \chi$ is its pseudo-adjoint. Then we may write the weak form of the non-adapted Itô formula as

$$\begin{aligned}
\left\| \mathrm{T}_t \chi \right\|^2 - \left\| \mathrm{T}_0 \chi \right\|^2 &= \int_{\mathbb{X}^t} \langle \chi \mid \boldsymbol{\nabla}_x^\ddagger \left(\mathbf{D}^\ddagger(x)\mathbf{T}(x) + \mathbf{D}^\ddagger(x)\mathbf{D}(x) + \mathbf{T}^\ddagger(x)\mathbf{D}(x) \right) \boldsymbol{\nabla}_x \chi \rangle \mathrm{d}x \\
&= 2\Re \int_{\mathbb{X}^t} \langle \chi \mid \boldsymbol{\nabla}_x^\ddagger \mathbf{T}^\ddagger(x) \mathbf{D}(x) \boldsymbol{\nabla}_x \chi \rangle \mathrm{d}x \\
&\qquad + \int_{\mathbb{X}^t} \langle \chi \mid \boldsymbol{\nabla}_x^\ddagger \mathbf{D}^\ddagger(x) \mathbf{D}(x) \boldsymbol{\nabla}_x \chi \rangle \mathrm{d}x,
\end{aligned}$$

giving us a non-adapted generalization of the Itô term of the Hudson-Parthasarathy formula for the adapted integrals in the form

$$\int_{\mathbb{X}^t} \langle \chi \mid \boldsymbol{\nabla}_x^\ddagger \mathbf{D}^\ddagger(x) \mathbf{D}(x) \boldsymbol{\nabla}_x \chi \rangle \mathrm{d}x = \int_{\mathbb{X}^t} \left\| \mathrm{D}_+^\circ(x)\chi + \mathrm{D}_\circ^\circ(x)\mathring{\chi}(x) \right\|^2 \mathrm{d}x,$$

and indeed

$$\begin{aligned}
\int_{\mathbb{X}^t} \langle \chi \mid \boldsymbol{\nabla}_x^\ddagger \mathbf{T}^\ddagger(x) \mathbf{D}(x) \boldsymbol{\nabla}_x \chi \rangle \mathrm{d}x &= \int_{\mathbb{X}^t} \left\langle \mathrm{T}_{t(x)}\chi \mid \mathrm{D}_+^-(x)\chi + \mathrm{D}_\circ^-(x)\mathring{\chi}(x) \right\rangle \mathrm{d}x \\
&\quad + \int_{\mathbb{X}^t} \left\langle \nabla_x \mathrm{T}_{t(x)}\chi \mid \mathrm{D}_+^\circ(x)\chi + \mathrm{D}_\circ^\circ(x)\mathring{\chi}(x) \right\rangle \mathrm{d}x,
\end{aligned}$$

as long as $\nabla_x \mathrm{T}_{t(x)}\chi = \mathrm{T}_+^\circ(x)\chi + \mathrm{T}_\circ^\circ(x)\mathring{\chi}(x)$.

Note that if $\mathrm{T}_t = \boldsymbol{\epsilon}(T_t)$ is the representation (3.3) of the kernel (3.2), then obviously

$$[\boldsymbol{\epsilon}(T_t)\chi](\vartheta \sqcup x) = \left[\boldsymbol{\epsilon}\left(\dot{T}_t(\mathbf{x}_+^\circ) \right) \chi + \boldsymbol{\epsilon}\left(\dot{T}(\mathbf{x}_\circ^\circ) \right) \mathring{\chi}(x) \right](\vartheta),$$

and therefore $\nabla_x \mathrm{T}_{t(x)}\chi = \mathrm{T}_+^\circ(x)\chi + \mathrm{T}_\circ^\circ(x)\mathring{\chi}(x)$ is satisfied. Also notice that

$$\int_{\mathbb{X}^t} \langle \chi \mid \boldsymbol{\nabla}_x^\ddagger \mathbf{T}^\ddagger(x) \mathbf{D}(x) \boldsymbol{\nabla}_x \chi \rangle \mathrm{d}x = \int_{\mathbb{X}^t} \langle \chi \mid \mathrm{T}_{t(x)}^* \mathrm{d}\mathrm{T}_{t(x)} \chi \rangle.$$

In the scalar case $\mathfrak{k}_x = \mathbb{C}$ for $\mathrm{D}_+^- = 0 = \mathrm{D}_\circ^\circ$, $\mathrm{D}_\circ^-(x) = \mathrm{D}(x) = \mathrm{D}_+^\circ(x)$, and $\mathrm{T}_\circ^\circ(x) = \mathrm{T}_{t(x)}$, $\mathrm{T}_\circ^-(x) = \mathrm{T}_+^\circ(x) \equiv [\nabla_x, \mathrm{T}_{t(x)}] := \partial\mathrm{T}(x)$ we obtain

$$\left\| \mathrm{T}_t \chi \right\|^2 - \left\| \mathrm{T}_0 \chi \right\|^2 = \int_{\mathbb{X}^t} 2\Re \left\langle \mathrm{T}_{t(x)}\chi \mid \mathrm{d}\mathrm{T}_{t(x)}\chi \right\rangle + \int_{\mathbb{X}^t} \left\| \mathrm{D}(x)\chi \right\|^2 \mathrm{d}x,$$

where the first term may be decomposed into the form

$$\int_{\mathbb{X}^t} 2\Re\left\langle \mathrm{T}_{t(x)}\chi \mid \mathrm{dT}_{t(x)}\chi\right\rangle = \int_{\mathbb{X}^t} 2\Re\langle \chi \mid \boldsymbol{\nabla}_x^{\ddagger}\Big(\mathrm{T}^*_{t(x)} \otimes \mathbf{I}(x)\Big)\mathbf{D}(x)\boldsymbol{\nabla}_x\chi\rangle \mathrm{d}x + \int_{\mathbb{X}^t} 2\Re\left\langle \partial\mathrm{T}(x)\chi \mid \mathrm{D}(x)\chi\right\rangle . \quad (6.2)$$

The second term of (6.2) vanishes in the adapted case, and the first term is the adapted contribution corresponding to $\mathbf{T}(x) = \mathrm{T}_{t(x)} \otimes \mathbf{I}(x)$. This gives the Itô formula for the normally-ordered non-adapted integral

$$\mathrm{T}_t - \mathrm{T}_0 = \int_{\mathbb{X}^t} (\Lambda^+_\circ(\mathrm{d}x)\mathrm{D}(x) + \mathrm{D}(x)\Lambda^\circ_-(\mathrm{d}x)) = \int_{\mathbb{X}^t} \mathrm{dT}_{t(x)}$$

with respect to the Wiener stochastic measure $w(\triangle)$, $\triangle \in \mathfrak{F}_{\mathbb{X}}$, which is represented in $\mathcal{G}_*$ by commuting operators $\widehat{w}(\triangle) = \Lambda^+_\circ(\triangle) + \Lambda^\circ_-(\triangle)$. Consider a particular case when the operators T_0, $\mathbf{D}(x)$, and consequently T_t are multiplications by anticipating functions $T_0(w)$, $D(x,w)$, and $T_t(w)$, of w, that is, $\mathrm{T}_0 = T_0(\widehat{w})$, $\mathrm{D}(x) = D(x,\widehat{w})$, and $\mathrm{T}_t = T_t(\widehat{w})$. Then the operators $\partial\mathrm{T}\,(x) = [\nabla_x,\, \mathrm{T}_{t(x)}] = \boldsymbol{\epsilon}(\dot{T}_{t(x)}(x))$ are defined by the Malliavin derivative $\dot{T}_t(x,w)|_{t=t(x)}$ as the Wiener representation of the point split $\dot{T}_{t(x)}(x,\vartheta) = T_{t(x)}(x \sqcup \vartheta)$ of operator-valued kernels in the multiple stochastic integral

$$T_t(w) = \int T_t(\vartheta)w(\mathrm{d}\vartheta) \equiv I_w(T_t).$$

In this particular case (6.1) was also obtained by Nualart and Pardoux in [24]. Note that in the weak form we can write $\partial\mathrm{T}(x)^*\mathrm{D}(x) = \boldsymbol{\nabla}_x^{\ddagger}\boldsymbol{\partial}\mathrm{T}(x)^{\ddagger}\mathbf{D}(x)\boldsymbol{\nabla}_x$, where

$$\boldsymbol{\partial}\mathrm{T}(x) := \begin{bmatrix} 0 & \partial\mathrm{T}(x) & 0 \\ 0 & 0 & \partial\mathrm{T}(x) \\ 0 & 0 & 0 \end{bmatrix}.$$

We note that in the Q-adapted case we always have $\mathbf{T}\,(x) = \mathrm{T}_{t(x)} \otimes \mathbf{Q}\,(x)$ with $\mathrm{Q}^\circ_\circ(x) = \mathrm{Q}(x)$, $\mathrm{Q}^-_-\,(x) = 1 = \mathrm{Q}^+_+\,(x)$ and otherwise $\mathrm{Q}^\mu_\nu(x) = 0$. Obviously, the product $\mathrm{T}^*_t\mathrm{T}_t$ for Q-adapted process T_t remains Q-adapted iff Q is orthoprojector $\mathrm{Q} = \mathrm{Q}^*\mathrm{Q} = \mathrm{Q}^*$. Note that if $\mathrm{Q} \neq \mathrm{I}$ then the operator $\partial\mathrm{T} = [\nabla_x, \mathrm{T}_{t(x)}] \neq 0$. However, we may replace this commutator with the Q-commutator

$$[\nabla_x, \mathrm{T}_{t(x)}]_{\mathrm{Q}} := \nabla_x\mathrm{T}_{t(x)} - \mathrm{Q}(x)\mathrm{T}_{t(x)}\nabla_x,$$

which vanishes when T_t is Q-adapted.

Corollary 6.1. *The quantum stochastic process* $\mathrm{T}_t = \boldsymbol{\epsilon}(T_t)$ *is* Q*-adapted if and only if the kernel process* T_t *is* Q*-adapted in the sense that*

$$T_t(\sigma,\varkappa,\tau) = \int T_t\begin{pmatrix} \vartheta, & \tau \\ \sigma, & \varkappa \end{pmatrix}\mathrm{d}\vartheta = T_t(\sigma^t,\varkappa^t,\tau^t) \otimes \delta_\emptyset(\sigma_{[t})\mathrm{Q}^{\otimes}(\varkappa_{[t})\delta_\emptyset(\tau_{[t}),$$

where $\delta_\emptyset(\varkappa) = 1$ *if* $\varkappa = \emptyset$, $\delta_\emptyset(\varkappa) = 0$ *if* $\varkappa \neq \emptyset$, $\varkappa^t = \varkappa \cap \mathbb{X}^t$, $\varkappa_{[t} = \{x \in \varkappa : t(x) \geq t\}$. *The quantum-stochastic Itô formula* (5.3) *for such processes can be written in* *the strong form*

$$\mathrm{T}_t^*\mathrm{T}_t - \mathrm{T}_0^*\mathrm{T}_0 = \int_{\mathbb{X}^t} (\mathrm{T}_{t(x)}^*\mathrm{dT}(x) + \mathrm{dT}^*(x)\mathrm{T}_{t(x)} + \mathrm{dT}^*(x)\mathrm{dT}(x))$$
$$= \boldsymbol{i}_0^t(\mathbf{T}_+^{\ddagger}\mathbf{T}_+ - \mathrm{T}^*\mathrm{T} \otimes \mathbf{Q}^{\ddagger}\mathbf{Q}),$$

where $\mathbf{Q}(x) = [\mathrm{Q}_\nu^\mu(x)]$ *is the block-diagonal operator* $\mathrm{Q}_-^- = 1 = \mathrm{Q}_+^+$, $\mathrm{Q}_\circ^\circ = \mathrm{Q}$, *and*

$$\mathrm{dT}(x) = \Lambda(\mathbf{D}, \mathrm{d}x),\ \mathrm{dT}^*(x) = \Lambda(\mathbf{D}^{\ddagger}, \mathrm{d}x),$$
$$\mathrm{dT}^*(x)\mathrm{dT}(x) = \Lambda(\mathbf{D}^{\ddagger}\mathbf{D}, \mathrm{d}x).$$

This can be written in the weak form (6.1), *where* $\nabla_x \mathrm{T}_{t(x)}\chi = [\mathrm{T}_{t(x)} \otimes \mathrm{Q}(x)]\mathring{\chi}(x)$.

Remark 6.2. Let the quantum stochastic process $\mathrm{T}_t = \boldsymbol{\epsilon}(T_t)$ be given by the multiple QS integral of the kernel $\mathrm{M} = \boldsymbol{\epsilon}(M)$ such that $T_t = \boldsymbol{\nu}_0^t(M)$, then T_t is Q-adapted if and only if $\mathrm{M}(\boldsymbol{\upsilon}) = M(\boldsymbol{\upsilon}) \otimes \mathrm{Q}^{\otimes}$ up to equivalence with respect to some $\star$-kernel N satisfying null condition $\boldsymbol{\nu}_0^t(\boldsymbol{\vartheta}, N) = 0\ \forall\, \vartheta \in \mathcal{X}$, such that

$$T_t(\boldsymbol{\vartheta}) = T_t(\boldsymbol{\vartheta}^t) \otimes Q(\boldsymbol{\vartheta}_{[t}) \quad \Leftrightarrow \quad M(\boldsymbol{\upsilon}, \boldsymbol{\varkappa}) = M(\boldsymbol{\upsilon}) \otimes Q(\boldsymbol{\varkappa}) + N(\boldsymbol{\upsilon}, \boldsymbol{\varkappa}), \tag{6.3}$$

for all $t > 0$, and for all chain-tables $\boldsymbol{\upsilon} \sqcup \boldsymbol{\varkappa} = \boldsymbol{\vartheta} = \boldsymbol{\vartheta}^t \sqcup \boldsymbol{\vartheta}_{[t}$ with $\boldsymbol{\vartheta}^t = \boldsymbol{\vartheta} \cap \mathbb{X}^t$, $\boldsymbol{\vartheta}_{[t} = \boldsymbol{\vartheta} \cap \mathbb{X}_{[t}$, and $\boldsymbol{\vartheta}_{[t} = \boldsymbol{\varkappa}_{[t} \subseteq \boldsymbol{\varkappa}$, and we have identified $M(\boldsymbol{\upsilon}) \equiv M(\boldsymbol{\upsilon}, \boldsymbol{\emptyset})$.

If the integral kernel is of the form $M \otimes Q$ then it follows trivially that the process T_t is Q-adapted. Now consider the Q-Meyer transform of T_t given as

$$M_t(\boldsymbol{\upsilon}) = \sum_{\boldsymbol{\sigma} \sqcup \boldsymbol{\tau} = \boldsymbol{\upsilon}} T_t(\boldsymbol{\sigma}) \otimes [-Q](\boldsymbol{\tau}),$$

where $\boldsymbol{\sigma} = (\sigma_\nu^\mu)$, $\boldsymbol{\tau} = (\tau_\nu^\mu)$, $\sigma_\nu^\mu, \tau_\nu^\mu \in \mathcal{X}$, and $[-Q](\boldsymbol{\varkappa}) := \otimes_{\boldsymbol{x} \in \boldsymbol{\varkappa}} - Q(\boldsymbol{x})$, then it follows that T_t is given by the Q-Möbius transform

$$T_t(\boldsymbol{\vartheta}) = \sum_{\boldsymbol{\upsilon} \subseteq \boldsymbol{\vartheta}^t} M_t(\boldsymbol{\upsilon}) \otimes Q(\boldsymbol{\vartheta} \setminus \boldsymbol{\upsilon}),$$

that is $T_t = \boldsymbol{\nu}_0^t(M_t \otimes Q)$. Now suppose that there is another kernel M' such that we have $T_t = \boldsymbol{\nu}_0^t(M')$, then by linearity of ν_0^t we find that $\boldsymbol{\nu}_0^t(M_t \otimes Q - M') = 0$, thus indeed $M' = M_t \otimes Q + N$, where $\boldsymbol{\nu}_0^t(N) = 0$. Notice however that we now have the Maassen-Meyer kernel M_t depending on the terminal time t.

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