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A VACUUM-ADAPTED APPROACH TO QUANTUM FEYNMAN–KAC FORMULAE

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Dedicated to Professor K. R. Parthasarathy on the occasion of his 75th birthday

Abstract. The vacuum-adapted formulation of quantum stochastic calculus is employed to perturb expectation semigroups *via* a Feynman–Kac formula. This gives an alternative perspective on the perturbation theory for quantum stochastic flows that has recently been developed by the authors.

1. Introduction

Let $\alpha = (\alpha_t)_{t\in\mathbb{R}}$ be an ultraweakly continuous group of normal $*$-automorphisms of a von Neumann algebra A acting faithfully on the Hilbert space $\mathfrak{h}$, and let δ be its ultraweak generator. Gaussian subordination may be used to construct an ultraweakly continuous semigroup $\mathcal{P}^0$ on A with ultraweak pre-generator $\frac{1}{2}\delta^2$ [17, Section 1] in the following manner. If $B = (B_t)_{t\geqslant 0}$ is standard Brownian motion on Wiener's probability space $\mathbb{W}$ then, by Itô's formula, the unital $*$-homomorphism

$$j_t : \mathsf{A} \to \mathsf{A} \overline{\otimes} L^\infty(\mathbb{W}) = L^\infty(\mathbb{W}; \mathsf{A}); \ a \mapsto \alpha_{B_t(\,\cdot\,)}(a) \qquad (t \geqslant 0)$$

satisfies the stochastic differential equation

$$j_t(x) = x + \int_0^t j_s(\delta(x))\,\mathrm{d}B_s + \frac{1}{2}\int_0^t j_s(\delta^2(x))\,\mathrm{d}s \qquad (x \in \operatorname{Dom}\delta^2) \tag{1.1}$$

in the strong sense on $L^2(\mathbb{W}; \mathfrak{h})$. Thus

$$\mathcal{P}^0_t(a)u := \mathbb{E}_{\mathbb{W}}[j_t(a)u] \qquad (a \in \mathsf{A},\, u \in \mathfrak{h} \subset L^2(\mathbb{W}; \mathfrak{h}))$$

defines an ultraweakly continuous semigroup $(\mathcal{P}^0_t)_{t\geqslant 0}$ of normal unital completely positive contractions on A whose ultraweak generator is as desired.

For the case where α is unitarily implemented, Lindsay and Sinha obtained an ultraweakly continuous semigroup $\mathcal{P}^b$ with Feynman–Kac representation

$$\mathcal{P}^b_t(a)u = \mathbb{E}_{\mathbb{W}}[j_t(a)m^b_t u] \qquad (t \geqslant 0,\, a \in \mathsf{A},\, u \in \mathfrak{h}) \tag{1.2}$$

whose ultraweak generator extends $\frac{1}{2}\delta^2 + \rho_b\delta$, where $\rho_b : a \mapsto ab$ is the operator on A of right multiplication by b [17, Theorem 3.2]. Here m^b is the exponential

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martingale such that

$$m_t^b = I + \int_0^t j_s(b) m_s^b \, \mathrm{d}B_s \qquad (t \geqslant 0),$$

where $b \in \mathsf{A}$ is self adjoint. For the Laplacian on $\mathbb{R}^{3d}$ and the commutative von Neumann algebra $L^\infty(\mathbb{R}^{3d})$, such vector-field perturbations were studied from this viewpoint by Parthasarathy and Sinha ([20]). Other works on quantum Feynman–Kac formulae include [1], [14], [2] and [5], all of which belong to the pre-quantum stochastic era. The classical Feynman-Kac formula for Schrödinger operators, which is closely related to instances of the Trotter product formula, is well described in the books [21] and [22].

The results of Lindsay and Sinha have been fully generalised in [11]. In that paper a general perturbation theory for quantum stochastic flows is developed, yielding a much wider class of quantum Feynman–Kac formulae. Here we take our inspiration from [6]. The semigroups defined in (1.2) will not, in general, be positive or even real (*i.e.,* $*$-preserving). In this light Bahn and Park investigate a more symmetric form of Feynman–Kac perturbation, using instead an operator process n^b such that

$$n_t^b f = f + \int_0^t j_s(b) \mathbb{E}_{\mathbb{W}}[n_s^b f | \mathcal{B}_s] \, \mathrm{d}B_s - \frac{1}{2} \int_0^t j_s(b^2) \mathbb{E}_{\mathbb{W}}[n_s^b f | \mathcal{B}_s] \, \mathrm{d}s \tag{1.3}$$

for all $f \in L^2(\mathbb{W}; \mathfrak{h})$, where $(\mathcal{B}_t)_{t \geqslant 0}$ is the canonical filtration of the Brownian motion B. In this case, letting

$$\mathcal{Q}_t^b(a) u := \mathbb{E}_{\mathbb{W}}[(n_t^b)^* j_t(a) n_t^b u] \qquad (a \in \mathsf{A},\, u \in \mathfrak{h})$$

gives an ultraweakly continuous completely positive semigroup $(\mathcal{Q}_t^b)_{t \geqslant 0}$ on A, which is contractive if n^b is and whose generator extends the map

$$\tfrac{1}{2}\delta^2 + \lambda_b \delta + \rho_b \delta + \lambda_b \rho_b - \tfrac{1}{2}\lambda_{b^2} - \tfrac{1}{2}\rho_{b^2}, \tag{1.4}$$

where λ_b denotes the operator on A given by left multiplication by b.

In this work we are guided by the form of (1.3); the conditional expectations make it reminiscent of a stochastic differential equation used by Alicki and Fannes for dilating quantum dynamical semigroups [4, Equation (12)]. As observed in [7], this type of equation may be profitably interpreted in the vacuum-adapted form of quantum stochastic calculus. In contrast to [11], where the standard identity-adapted (Hudson–Parthasarathy) theory is used, here the analysis is slightly easier although the algebra becomes a bit more complicated.

We describe the contents of the paper next, restricting our description here to the one-dimensional case, for simplicity. The requirement that α is unitarily implemented is removed; our primary object is a vacuum-adapted quantum stochastic flow. This is an ultraweakly continuous family $j = (j_t)_{t \geqslant 0}$ of normal $*$-homomorphisms which form a vacuum-adapted quantum stochastic cocycle on Boson Fock space over $L^2(\mathbb{R}_+)$ and which are as unital as vacuum adaptedness permits. The flow j is assumed to satisfy the quantum stochastic differential equation

$$\mathrm{d}j_t(x) = j_t(\delta_0(x)) \, \mathrm{d}A_t^\dagger + j_t(\pi_0(x)) \, \mathrm{d}\Lambda_t + j_t(\delta_0^\dagger(x)) \, \mathrm{d}A_t + j_t(\tau_0(x)) \, \mathrm{d}t \tag{1.5}$$

for all $x \in \mathsf{A}_0$, where A_0 is a subset of A, and the *structure maps*

$$\tau_0,\ \delta_0,\ \delta_0^\dagger,\ \pi_0 : \mathsf{A}_0 \to \mathsf{A}$$

must satisfy certain algebraic relations, thanks to the unital and $*$-homomorphic properties of j. Equation (1.5) generalises (1.1), which corresponds to the case where $\mathsf{A}_0 = \operatorname{Dom} \delta^2$, π_0 is the inclusion map,

$$\delta_0 = \delta_0^\dagger = \delta|_{\mathsf{A}_0} \qquad \text{and} \qquad \tau_0 = \tfrac{1}{2}\delta^2.$$

The appearance of the non-zero gauge term π_0 is due to the fact that we are working in the vacuum-adapted set-up: *cf.* [10, Theorem 7.3]. It follows from (1.5) that the quantum stochastic flow satisfies the equation

$$\langle u\Omega, j_t(x) v\Omega\rangle = \langle u, v\rangle + \int_0^t \langle u\Omega, j_s(\tau_0(x)) v\Omega\rangle \,\mathrm{d}s \qquad (u, v \in \mathfrak{h},\ t \geqslant 0,\ x \in \mathsf{A}_0),$$

where Ω denotes the Fock vacuum vector. The generator of the *vacuum-expectation semigroup* $\mathcal{P}^0 := (\mathbb{E} \circ j_t)_{t \geqslant 0}$ therefore extends the map τ_0. A natural assumption here is that τ_0 is a pre-generator of $\mathcal{P}^0$, however our results do not require it.

Starting with Evans and Hudson [13], several authors have used conjugation with a unitary process to perturb quantum stochastic flows. These works focused on the case of bounded structure maps, so that the vacuum-expectation semigroup $\mathcal{P}^0$ is norm continuous, and considered identity-adapted flows and processes. For $h = h^* \in \mathsf{A}$ and $l \in \mathsf{A}$ there exists a unitary process U such that

$$U_0 = I, \qquad \mathrm{d}U_t = j_t(l) U_t \,\mathrm{d}A_t^\dagger + j_t(-l^*) U_t \,\mathrm{d}A_t + j_t(-\mathrm{i}h - \tfrac{1}{2} l^* l) U_t \,\mathrm{d}t,$$

and the vacuum-expectation semigroup of the perturbed flow $(a \mapsto U_t^* j_t(a) U_t)_{t \geqslant 0}$ has generator

$$\tau_0 + \lambda_{l^*} \delta_0 + \rho_l \delta_0^\dagger + \lambda_{l^*} \rho_l \pi_0 + \mathrm{i}[h, \cdot\,] - \tfrac{1}{2}\{l^* l, \cdot\,\},$$

where $[\,\cdot\,,\cdot\,]$ and $\{\,\cdot\,,\cdot\,\}$ denote commutator and anticommutator. The main result obtained here includes this situation as a special case.

For any vacuum-adapted quantum stochastic flow j and any $c = \left[\begin{smallmatrix} c_0 \\ c_1 \end{smallmatrix}\right]$ in $\mathsf{A} \oplus \mathsf{A}$, Theorem 5.3 below gives a process M^c such that $M^c - I$ is vacuum adapted and the following quantum stochastic differential equation is satisfied:

$$\mathrm{d}(M^c - I)_t = j_t(c_0) M_t^c \,\mathrm{d}t + j_t(c_1) M_t^c \,\mathrm{d}A_t^\dagger.$$

Consequently, for any $d = \left[\begin{smallmatrix} d_0 \\ d_1 \end{smallmatrix}\right]$ in $\mathsf{A} \oplus \mathsf{A}$, there is an ultraweakly continuous semigroup $\mathcal{P}^{c,d}$ on A with

$$\langle u, \mathcal{P}_t^{c,d}(a) v\rangle = \langle u\Omega, (M_t^c)^* j_t(a) M_t^d v\Omega\rangle \qquad (u, v \in \mathfrak{h},\ t \geqslant 0,\ a \in \mathsf{A}).$$

When j satisfies (1.5), the ultraweak generator of $\mathcal{P}^{c,d}$ necessarily extends

$$\tau_0 + \lambda_{c_1^*} \delta_0 + \rho_{d_1} \delta_0^\dagger + \lambda_{c_1^*} \rho_{d_1} \pi_0 + \lambda_{c_0^*} + \rho_{d_0}. \tag{1.6}$$

This class of semigroups includes both the Lindsay–Sinha and the Bahn–Park examples, as well as those obtained by unitary conjugation; the generators of the latter correspond to the case

$$c = d = \begin{bmatrix} -\mathrm{i}h - \frac{1}{2} l^* l \\ l \end{bmatrix}, \quad \text{where } h = h^*.$$

1.1. Conventions. Hilbert spaces are complex with inner products linear in their second argument. The linear, Hilbert-space and ultraweak tensor products are denoted by $\underline{\otimes}$, $\otimes$ and $\overline{\otimes}$, respectively. For a Hilbert space H we adopt the Dirac-inspired notation $|\mathsf{H}\rangle$ for $B(\mathbb{C};\mathsf{H})$ and $\langle\mathsf{H}|$ for the topological dual $B(\mathsf{H};\mathbb{C})$, writing $|u\rangle$ for the operator $\lambda \mapsto \lambda u$ and $\langle u|$ for the functional $v \mapsto \langle u, v\rangle$, where $u \in \mathsf{H}$. Recall the E *notation*,

$$E_u := |u\rangle \otimes I \quad \text{and} \quad E^u := (E_u)^* = \langle u| \otimes I \qquad (u \in \mathsf{H}), \tag{1.7}$$

in which I denotes the identity operator on a Hilbert space determined by context. The following commutator and anticommutator notation is also used for elements of an algebra:

$$[a, b] := ab - ba \quad \text{and} \quad \{a, b\} := ab + ba. \tag{1.8}$$

2. Multipliers for Quantum Stochastic Flows

Fix now, and for the rest of the paper, Hilbert spaces $\mathfrak{h}$ and k, referred to as the *initial space* and *multiplicity space* or *noise dimension space*, respectively. Fix also a von Neumann algebra A acting faithfully on $\mathfrak{h}$. Set $\widehat{\mathsf{k}} := \mathbb{C} \oplus \mathsf{k}$,

$$\widehat{c} := \begin{pmatrix} 1 \\ c \end{pmatrix} \in \widehat{\mathsf{k}} \quad (c \in \mathsf{k}) \qquad \text{and} \qquad \omega := \widehat{0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{2.1}$$

Our basic reference for quantum stochastic calculus is [16].

For a subinterval J of $\mathbb{R}_+$, let $\mathcal{F}_J$ denote the Boson Fock space over $L^2(J;\mathsf{k})$ and let $\mathsf{N}_J := B(\mathcal{F}_J)$. For brevity, set $\mathcal{F} := \mathcal{F}_{\mathbb{R}_+}$, $\mathcal{F}_{t)} := \mathcal{F}_{[0,t)}$ and $\mathcal{F}_{[t} := \mathcal{F}_{[t,\infty)}$, with corresponding abbreviations for the noise algebra $\mathsf{N} = B(\mathcal{F})$. The identifications

$$\mathcal{F} = \mathcal{F}_{s)} \otimes \mathcal{F}_{[s} = \mathcal{F}_{s)} \otimes \mathcal{F}_{[s,t)} \otimes \mathcal{F}_{[t} \qquad (0 \leqslant s \leqslant t < \infty),$$

which arise from the exponential property of Fock space, entail the identifications

$$\mathsf{N} = \mathsf{N}_{s)} \overline{\otimes} \mathsf{N}_{[s} = \mathsf{N}_{s)} \overline{\otimes} \mathsf{N}_{[s,t)} \overline{\otimes} \mathsf{N}_{[t} \qquad (0 \leqslant s \leqslant t < \infty).$$

The notation Ω_J, I_J and id_J for the vacuum vector in $\mathcal{F}_J$, the identity operator on $\mathcal{F}_J$ and the identity map on N_J, respectively, is also useful, with corresponding abbreviations for other intervals, such as $\Omega_{[s}$, $I_{[s}$ and $\mathrm{id}_{[s}$, as above.

Denote by Δ any of the following projections:

$$P_{\mathsf{k}} \in B(\widehat{\mathsf{k}}), \quad P_{\mathsf{k}} \otimes \mathrm{id}_{\mathsf{A}} \in B(\widehat{\mathsf{k}}) \overline{\otimes} \mathsf{A}, \quad \text{and} \quad P_{\mathsf{k}} \otimes \mathrm{id}_{\mathsf{A}} \otimes I_{\mathcal{F}} \in B(\widehat{\mathsf{k}}) \overline{\otimes} \mathsf{A} \overline{\otimes} \mathsf{N}, \tag{2.2}$$

where $P_{\mathsf{k}} = \begin{bmatrix} 0 & 0 \\ 0 & I_{\mathsf{k}} \end{bmatrix} \in B(\widehat{\mathsf{k}})$ is the orthogonal projection onto k.

The right shift

$$s_t : L^2(\mathbb{R}_+;\mathsf{k}) \to L^2([t,\infty);\mathsf{k});\ f \mapsto f(\,\cdot - t) \qquad (t \geqslant 0)$$

has second quantisation

$$S_t : \mathcal{F} \to \mathcal{F}_{[t};\ \varepsilon(f) \mapsto \varepsilon(s_t f),$$

where $\varepsilon(g)$ denotes the exponential vector corresponding to the vector g, and the map

$$\sigma_t : \mathsf{A} \overline{\otimes} \mathsf{N} \to \mathsf{A} \overline{\otimes} \mathsf{N}_{[t};\ T \mapsto (I_{\mathfrak{h}} \otimes S_t) T (I_{\mathfrak{h}} \otimes S_t)^*$$

is a normal $*$-isomorphism for all $t \geqslant 0$.

Definition 2.1. A *vacuum-adapted quantum stochastic cocycle* k on A is a family of normal completely bounded maps $(k_t : \mathsf{A} \to \mathsf{A} \overline{\otimes} \mathsf{N})_{t \geqslant 0}$ such that, for all $a \in \mathsf{A}$ and $s, t \geqslant 0$,

(Ω-C i) $k_0(a) = a \otimes |\Omega\rangle\langle\Omega|$,

(Ω-C ii) $k_t(a) = k_{t)}(a) \otimes |\Omega_{[t}\rangle\langle\Omega_{[t}|$, where $k_{t)}(a) \in \mathsf{A} \overline{\otimes} \mathsf{N}_{t)}$,

(C iii) $k_{s+t} = \widehat{k}_s \circ \sigma_s \circ k_t$, where $\widehat{k}_s := k_{s)} \overline{\otimes} \mathrm{id}_{[s}$

and (C iv) $r \mapsto k_r(a)$ is ultraweakly continuous.

Such a family is a *flow* on A if each $k_{t)}$ is $*$-homomorphic and unital. Following tradition we use the letter j for quantum stochastic flows.

In the standard theory, (Ω-C i) and (Ω-C ii) are replaced by their identity-adapted counterparts,

(I-C i) $k_0(a) = a \otimes I_{\mathcal{F}}$

and (I-C ii) $k_t(a) = k_{t)}(a) \otimes I_{[t}$, where $k_{t)}(a) \in \mathsf{A} \overline{\otimes} \mathsf{N}_{t)}$.

Remark 2.2. The prescription

$$k^{(\Omega)} = \big(k_{t)}(\,\cdot\,) \otimes |\Omega_{[t}\rangle\langle\Omega_{[t}|\big)_{t \geqslant 0} \;\mapsto\; k^{(I)} = \big(k_{t)}(\,\cdot\,) \otimes I_{[t}\big)_{t \geqslant 0} \tag{2.3}$$

gives a bijective correspondence between the class of vacuum-adapted quantum stochastic cocycles and the class of identity-adapted quantum stochastic cocycles. Note that

$$k_{t)}(a) = E^{\Omega_{[t}} k_t(a) E_{\Omega_{[t}} \qquad (t \geqslant 0,\, a \in \mathsf{A}) \tag{2.4}$$

in both cases.

In terms of the orthogonal projection

$$P_t := I_{\mathfrak{h}} \otimes I_{t)} \otimes |\Omega_{[t}\rangle\langle\Omega_{[t}|, \tag{2.5}$$

condition (Ω-C ii) becomes

$$k_t(a) = P_t k_t(a) P_t,$$

whereas (I-C ii) only implies the weaker commutation relation

$$k_t(a) P_t = P_t k_t(a).$$

Let

$$\mathbb{E} := \mathrm{id}_{\mathsf{A}} \overline{\otimes} \omega_\Omega : \mathsf{A} \overline{\otimes} \mathsf{N} \to \mathsf{A}$$

denote the *vacuum expectation*, where ω_Ω is the state on N corresponding to the vacuum vector Ω.

Proposition 2.3. *Let k be a vacuum-adapted quantum stochastic cocycle on A. The ultraweakly continuous family of normal completely bounded maps $(\mathbb{E} \circ k_t)_{t \geqslant 0}$ on A forms a semigroup, called the* vacuum-expectation semigroup *of k.*

Proof. For all $t \geqslant 0$, the conditional expectation

$$\mathbb{E}_t^{\Omega} : \mathsf{A} \overline{\otimes} \mathsf{N} \to \mathsf{A} \overline{\otimes} \mathsf{N};\; T \mapsto \big(\mathrm{id}_{\mathsf{A} \overline{\otimes} \mathsf{N}_{t)}} \overline{\otimes} \omega_{\Omega_{[t}}\big)(T) \otimes |\Omega_{[t}\rangle\langle\Omega_{[t}| = P_t T P_t \tag{2.6}$$

has the tower property $\mathbb{E} \circ \mathbb{E}_t^{\Omega} = \mathbb{E}$. The claim follows since any vacuum-adapted quantum stochastic cocycle satisfies the identity

$$\mathbb{E}_t^{\Omega} \circ \widehat{k}_t \circ \sigma_t = k_t \circ \mathbb{E} \qquad (t \geqslant 0). \qquad \square$$

Quantum stochastic differential equations of the following form are a basic source of quantum stochastic cocycles.

Remark 2.4. Under the correspondence (2.3), $k^{(\Omega)}$ satisfies a quantum stochastic differential equation of the form

$$k_0(a) = a \otimes |\Omega\rangle\langle\Omega|, \qquad \mathrm{d}k_t = \widetilde{k}_t\big(\psi(a)\big)\,\mathrm{d}\Lambda_t \tag{2.7}$$

on a subset A_0 of A, where $\widetilde{k}_t := \mathrm{id}_{B(\widehat{\mathsf{k}})} \overline{\otimes}\, k_t$, if and only if $k^{(I)}$ satisfies a quantum stochastic differential equation of the form

$$k_0(a) = a \otimes I_{\mathcal{F}}, \qquad \mathrm{d}k_t = \widetilde{k}_t\big(\phi(a)\big)\,\mathrm{d}\Lambda_t \tag{2.8}$$

on A_0, where the maps ψ, $\phi : \mathsf{A}_0 \to B(\widehat{\mathsf{k}}) \overline{\otimes} \mathsf{A}$ are related by the following identity:

$$\psi(a) = \phi(a) + \Delta \otimes a \qquad (a \in \mathsf{A}_0).$$

This is proved in [10, Theorem 7.3]. Here Λ is the matrix of fundamental quantum stochastic integrators [15]; see [16].

Remark 2.5 ([18, Section 6]). Let the map $\phi : \mathsf{A} \to B(\widehat{\mathsf{k}}) \overline{\otimes} \mathsf{A}$ have the block-matrix form

$$\phi(a) = \begin{bmatrix} \mathrm{i}[h,a] - \frac{1}{2}\{r^*r, a\} + r^*\pi(a)r & ar^* - r^*\pi(a) \\ ra - \pi(a)r & \pi(a) - I_{\mathsf{k}} \otimes a \end{bmatrix} \qquad (a \in \mathsf{A}), \tag{2.9}$$

where $h \in \mathsf{A}$ is self adjoint, $r \in |\mathsf{k}\rangle \overline{\otimes} \mathsf{A}$ and $\pi : \mathsf{A} \to B(\mathsf{k}) \overline{\otimes} \mathsf{A}$ is a normal unital $*$-homomorphism. Then the quantum stochastic differential equation (2.8) has a unique solution and this is an identity-adapted quantum stochastic flow. Conversely, if an identity-adapted quantum stochastic flow satisfies (2.8) for some normal bounded map $\phi : \mathsf{A} \to B(\widehat{\mathsf{k}}) \overline{\otimes} \mathsf{A}$ then ϕ has the form (2.9).

Definition 2.6. Let j be a vacuum-adapted quantum stochastic flow on A. A family of operators $M = (M_t)_{t \geqslant 0}$ in $\mathsf{A} \overline{\otimes} \mathsf{N}$ is a *multiplier* for j if, for all s, $t \geqslant 0$,

(M i) $M_0 = I_{\mathfrak{h} \otimes \mathcal{F}}$,
(M ii) $M_t P_t = P_t M_t$,
(M iii) $M_{s+t} = J_s(M_t) M_s$, where $J_s := \widehat{\jmath}_s \circ \sigma_s$
and (M iv) $r \mapsto M_r$ is strongly continuous.

The Banach–Steinhaus Theorem and condition (M iv) imply that M is locally bounded.

Theorem 2.7 (*Cf.* [6, Theorem 2.1])**.** *Let M and N be multipliers for the vacuum-adapted quantum stochastic flow j. The ultraweakly continuous normal completely bounded family*

$$\mathcal{P} := \big(a \mapsto \mathbb{E}[M_t^* j_t(a) N_t]\big)_{t \geqslant 0}$$

forms a semigroup, which is completely contractive if M and N are contractive and is completely positive if $M = N$.

Proof. To prove the semigroup property, let $a \in \mathsf{A}$ and $s, t \geqslant 0$. By the tower property for the conditional expectation $\mathbb{E}_s^\Omega$ defined in (2.6), it follows that

$$\begin{aligned}
\mathcal{P}_{s+t}(a) &= \mathbb{E}\big[\mathbb{E}_s^\Omega[M_s^* J_s(M_t^*) J_s(j_t(a)) J_s(N_t) N_s]\big] && \text{by (C iii) and (M iii)}\\
&= \mathbb{E}[M_s^* \mathbb{E}_s^\Omega[J_s(M_t^* j_t(a) N_t)] N_s] && \text{by (M ii)}\\
&= \mathbb{E}\big[M_s^* j_s(\mathbb{E}[M_t^* j_t(a) N_t]) N_s\big] && (2.10)\\
&= \mathcal{P}_s(\mathcal{P}_t(a)).
\end{aligned}$$

For the equality (2.10), note that if $a \in \mathsf{A}$ and $b \in \mathsf{N}$ then

$$\mathbb{E}_s^\Omega[J_s(a \otimes b))\big] = \langle \Omega, b\Omega \rangle\, j_s(a) = j_s(\mathbb{E}[a \otimes b]);$$

thus $\mathbb{E}_s^\Omega \circ J_s = j_s \circ \mathbb{E}$, by linearity and ultraweak continuity. □

Remark 2.8. Some of the ideas in this section go back to early work of Accardi [1, Sections 2 and 4]; see also [3, Section 2.3].

3. A Vacuum-adapted Quantum Stochastic Differential Equation

Let $(u_t)_{t\in\mathbb{R}}$ be a strongly continuous one-parameter unitary group in A, let $B = (B_t)_{t\geqslant 0}$ be the canonical Brownian motion on Wiener's probability space $\mathbb{W}$ and, taking $\mathsf{k} = \mathbb{C}$, identify $L^2(\mathbb{W})$ with $\mathcal{F}$ *via* the Wiener–Itô–Segal isomorphism. If the unitary operator $U_t \in \mathsf{A} \overline{\otimes} \mathsf{N}$ is such that

$$U_t\xi : \omega \mapsto u_{B_t(\omega)}\xi(\omega) = u_{\omega(t)}\xi(\omega) \qquad (\xi \in L^2(\mathbb{W}; \mathfrak{h}))$$

then the family of maps $\big(j_t^B : a \mapsto U_t(a \otimes I_\mathcal{F})U_t^*\big)_{t\geqslant 0}$ is an identity-adapted quantum stochastic flow on A ([6, Lemma 3.1], *cf.* [16, Section 5]).

Bahn and Park considered the operator stochastic differential equation

$$M_0^a = I_{\mathfrak{h}\otimes\mathcal{F}}, \qquad \mathrm{d}M_t^a = j_t^B(a) P_t M_t^a \,\mathrm{d}B_t - \tfrac{1}{2} j_t^B(a^2) P_t M_t^a \,\mathrm{d}t, \tag{3.1}$$

where $a \in \mathsf{A}$, and obtained a solution pointwise in $L^2(\mathbb{W}; \mathfrak{h})$ [6, Proposition 3.2]. They showed that the collection of operators $(M_t^a)_{t\geqslant 0}$ forms a multiplier for the quantum stochastic flow j^B [6, Proposition 3.3].

Fix $a \in \mathsf{A}$ and set $N_t := M_t^a - I_{\mathfrak{h}\otimes\mathcal{F}}$ for all $t \geqslant 0$, so that

$$N_t\xi = \int_0^t Q_s\xi \,\mathrm{d}B_s - \frac{1}{2}\int_0^t R_s\xi \,\mathrm{d}s + \int_0^t Q_s N_s \xi \,\mathrm{d}B_s - \frac{1}{2}\int_0^t R_s N_s \xi \,\mathrm{d}s$$

for all $\xi \in L^2(\mathbb{W}; \mathfrak{h})$, where

$$Q_t := j_t^B(a) P_t \qquad \text{and} \qquad R_t := j_t^B(a^2) P_t.$$

As $\big(j_t^B(b)\big)_{t\geqslant 0}$ is identity adapted for all $b \in \mathsf{A}$, the processes Q and R are vacuum adapted. By [9, Theorem 2.2], the process N above is the unique vacuum-adapted solution of the quantum stochastic differential equation

$$N_0 = 0, \qquad \mathrm{d}N_t = Q_t \,\mathrm{d}A_t^\dagger - \tfrac{1}{2} R_t \,\mathrm{d}t + Q_t N_t \,\mathrm{d}A_t^\dagger - \tfrac{1}{2} R_t N_t \,\mathrm{d}t. \tag{3.2}$$

To see that (3.2) is the correct quantum stochastic generalisation of (3.1), for simplicity take $\mathfrak{h} = \mathbb{C}$ and let $\mathfrak{z}(f)$ denote the Brownian exponential corresponding to $f \in L^2(\mathbb{R}_+)$, *i.e.,* the unique element of $L^2(\mathbb{W})$ such that

$$\mathfrak{z}(f)_t := \mathbb{E}_{\mathbb{W}}[\mathfrak{z}(f)|\mathcal{B}_t] = 1 + \int_0^t f(s)\mathbb{E}_{\mathbb{W}}[\mathfrak{z}(f)|\mathcal{B}_s]\,\mathrm{d}B_s \qquad (t \geqslant 0),$$

where $(\mathcal{B}_t)_{t\geqslant 0}$ is the canonical filtration generated by the Brownian motion B. (Recall that $\mathfrak{z}(f)$ corresponds to $\varepsilon(f)$ and $\mathbb{E}_{\mathbb{W}}[\,\cdot\,|\mathcal{B}_t]$ to P_t.) If $(X_t)_{t\geqslant 0}$ is a process of bounded operators on $\mathcal{F}$ with locally bounded norm and such that $X_tP_t = P_tX_t$ for all $t \geqslant 0$ then, by the (classical) Itô product formula,

$$\begin{aligned}\mathbb{E}_{\mathbb{W}}\Big[\overline{\mathfrak{z}(f)}\int_0^t X_sP_s\mathfrak{z}(g)\,\mathrm{d}B_s\Big] &= \mathbb{E}_{\mathbb{W}}\Big[\int_0^t \overline{f(s)\mathfrak{z}(f)_s}X_s\mathfrak{z}(g)_s\,\mathrm{d}s\Big] \\ &= \Big\langle \varepsilon(f), \int_0^t X_sP_s\,\mathrm{d}A_s^\dagger\varepsilon(g)\Big\rangle \qquad (f, g \in L^2(\mathbb{R}_+)).\end{aligned}$$

Definition 3.1. For a Hilbert space H, a *bounded process in* $B(\mathsf{H})\,\overline{\otimes}\,\mathsf{A}$ is a family of operators $Z = (Z_t)_{t\geqslant 0}$ in $B(\mathsf{H})\,\overline{\otimes}\,\mathsf{A}\,\overline{\otimes}\,\mathsf{N}$ such that

$$t \mapsto \langle \zeta', Z_t\zeta\rangle \text{ is measurable} \qquad (\zeta, \zeta' \in \mathsf{H} \otimes \mathfrak{h} \otimes \mathcal{F});$$

such a process is *vacuum adapted* if

$$Z_t = (I_{\mathsf{H}} \otimes P_t)Z_t(I_{\mathsf{H}} \otimes P_t) \qquad (t \geqslant 0)$$

or, equivalently,

$$Z_t = Z_{t)} \otimes |\Omega_{[t}\rangle\langle\Omega_{[t}| \quad \text{for some } Z_{t)} \in B(\mathsf{H})\,\overline{\otimes}\,\mathsf{A}\,\overline{\otimes}\,\mathsf{N}_{[0,t)} \quad (t \geqslant 0).$$

A vacuum-adapted bounded process G in $B(\widehat{\mathsf{k}})\,\overline{\otimes}\,\mathsf{A}$ is an *integrand* process if its block-matrix form $\left[\begin{smallmatrix} k & m \\ l & n\end{smallmatrix}\right]$ is such that

$$\|G\|_t := \|k\|_{1,t} + \|l\|_{2,t} + \|m\|_{2,t} + \|n\|_{\infty,t} < \infty \quad (t \geqslant 0),$$

where, for $p = 1, 2$ or ∞, $\|f\|_{p,t}$ denotes the L^p norm of the function $1_{[0,t)}f$.

The following result is the coordinate-independent version of [8, Proposition 37], with non-trivial initial space. Recall the notation (1.7) and (2.1).

Proposition 3.2. *Let G be an integrand process. There is a unique bounded vacuum-adapted process $\int G\,\mathrm{d}\Lambda = \big(\int_0^t G_s\,\mathrm{d}\Lambda_s\big)_{t\geqslant 0}$ in A such that*

$$\langle u\varepsilon(f), \int_0^t G_s\,\mathrm{d}\Lambda_s v\varepsilon(g)\rangle = \int_0^t \langle u\varepsilon(f), E^{\widehat{f(s)}}G_sE_{\widehat{g(s)}}v\varepsilon(g)\rangle\,\mathrm{d}s \qquad (t \geqslant 0)$$

for all u, $v \in \mathfrak{h}$ and f, $g \in L^2(\mathbb{R}_+; \mathsf{k})$. Moreover, the following inequality holds:

$$\|\int_0^t G_s\,\mathrm{d}\Lambda_s\| \leqslant \|G\|_t \qquad (t \geqslant 0).$$

We shall need to pass suitably adapted operators inside quantum stochastic integrals. The next lemma takes care of this.

Lemma 3.3. *Let G be an integrand process such that $G\Delta \equiv 0$ and let X be a bounded vacuum-adapted process in* A. *Then*

$$\int_s^t G_r \,\mathrm{d}\Lambda_r\, X_s = \int_s^t G_r(I_{\widehat{\mathsf{k}}} \otimes X_s)\,\mathrm{d}\Lambda_r \qquad (0 \leqslant s \leqslant t). \tag{3.3}$$

Proof. Let $u, v \in \mathfrak{h}$ and $f, g \in L^2(\mathbb{R}_+; \mathsf{k})$; note that

$$\langle u\varepsilon(f), \int_0^t G_r \,\mathrm{d}\Lambda_r\, v\varepsilon(g)\rangle = \int_0^t \langle u\varepsilon(f), E^{\widehat{f(r)}} G_r E_\omega v\varepsilon(g)\rangle \,\mathrm{d}r,$$

since $\Delta^\perp E_{\widehat{c}} = E_\omega$ for all $c \in \mathsf{k}$. If $A \in B(\mathfrak{h} \otimes \mathcal{F}_{s)})$ and $\xi \in \mathfrak{h} \otimes \mathcal{F}$ then, setting $P_{[s} := |\Omega_{[s}\rangle\langle\Omega_{[s}|$ for brevity, it follows that

$$\begin{aligned}\langle u\varepsilon(f), \int_s^t G_r \,\mathrm{d}\Lambda_r (A \otimes P_{[s})\xi\rangle &= \int_s^t \langle u\varepsilon(f), E^{\widehat{f(r)}} G_r E_\omega (A \otimes P_{[s})\xi\rangle \,\mathrm{d}r \\ &= \int_s^t \langle u\varepsilon(f), E^{\widehat{f(r)}} G_r (I_{\widehat{\mathsf{k}}} \otimes A \otimes P_{[s}) E_\omega \xi\rangle \,\mathrm{d}r \\ &= \langle u\varepsilon(f), \int_s^t G_s (I_{\widehat{\mathsf{k}}} \otimes A \otimes P_{[s}) \,\mathrm{d}\Lambda_r \xi\rangle. \qquad \square\end{aligned}$$

The following existence and uniqueness theorem is sufficiently general for present purposes.

Theorem 3.4. *Let G and X be as in Lemma 3.3, with X locally bounded in norm. Then there is a unique vacuum-adapted process Z in* A *such that*

$$Z_t = X_t + \int_0^t G_s(I_{\widehat{\mathsf{k}}} \otimes Z_s)\,\mathrm{d}\Lambda_s \qquad (t \geqslant 0). \tag{3.4}$$

Furthermore,

$$\|Z\|_{\infty,t} \leqslant \sqrt{2}\,\|X\|_{\infty,t} \exp(2\|l\|_{2,t}^2 + 2\|k\|_{1,t}^2) \qquad (t \geqslant 0),$$

where $\begin{bmatrix} k & 0 \\ l & 0 \end{bmatrix}$ is the block-matrix form of G, and Z is norm continuous if and only if X is.

Proof. Define a sequence of processes $(X^{(n)})_{n\geqslant 0}$ inductively by letting $X^{(0)} := X$ and

$$X_t^{(n+1)} := \int_0^t G_s(I_{\widehat{\mathsf{k}}} \otimes X_s^{(n)})\,\mathrm{d}\Lambda_s \qquad (t \geqslant 0).$$

This process is well defined and such that

$$\|X_t^{(n+1)}\| \leqslant \|k\,X^{(n)}\|_{1,t} + \|l\,X^{(n)}\|_{2,t} \qquad (t \geqslant 0),$$

so, integrating by parts,

$$\|X^{(n+1)}\|_{\infty,t}^2 \leqslant 2\|k\,X^{(n)}\|_{1,t}^2 + 2\|l\,X^{(n)}\|_{2,t}^2 \leqslant \int_0^t c(s)\|X^{(n)}\|_{\infty,s}^2\,\mathrm{d}s,$$

where

$$c(s) := 4\|k_s\| \int_0^s \|k_r\|\,\mathrm{d}r + 2\|l_s\|^2.$$

It follows that

$$\|X^{(n+1)}\|^2_{\infty,t} \leqslant \frac{1}{n!}\Big(\int_0^t c(s)\,\mathrm{d}s\Big)^n \|X\|^2_{\infty,t} \qquad (n \geqslant 0,\, t \geqslant 0),$$

so $Z_t := \sum_{n=0}^\infty X_t^{(n)}$ exists for all $t \geqslant 0$, the series being convergent in norm. A dominated-convergence argument shows that Z satisfies (3.4) and, since

$$\|Z_t\|^2 \leqslant 2\|X_t\|^2 + 2\int_0^t c(s)\|Z_s\|^2\,\mathrm{d}s \qquad (t \geqslant 0),$$

the inequality and so uniqueness follow from Gronwall's lemma. The final claim is immediate. □

4. Multipliers *via* Quantum Stochastic Differential Equations

Fix a vacuum-adapted quantum stochastic flow j on A and let

$$J_t := \widehat{\jmath}_t \circ \sigma_t : \mathsf{A} \overline{\otimes} \mathsf{N} \to \mathsf{A} \overline{\otimes} \mathsf{N}_{t)} \overline{\otimes} \mathsf{N}_{[t} = \mathsf{A} \overline{\otimes} \mathsf{N}$$

and $\widetilde{J}_t := \mathrm{id}_{B(\widehat{\mathsf{k}})} \overline{\otimes} J_t$, for all $t \geqslant 0$. The ultraweakly continuous family of normal unital $*$-homomorphisms $(J_t)_{t\geqslant 0}$ form a semigroup (*cf.* [19, Proposition 4.3]).

The following result is a vacuum-adapted version of [11, Lemma 5.1] which suffices here.

Lemma 4.1. *If the integrand process G is norm continuous then the family of operators $\big(1_{[s,\infty)}(r)\widetilde{J}_s(G_{r-s})\big)_{r\geqslant 0}$, where 1_A denotes the indicator function of the set A, defines an integrand process such that*

$$J_s\Big(\int_0^t G_r\,\mathrm{d}\Lambda_r\Big) = \int_s^{s+t} \widetilde{J}_s(G_{r-s})\,\mathrm{d}\Lambda_r \qquad (t \geqslant 0).$$

Sketch proof. Apply the ampliation of the vector functional $A \mapsto \langle \varepsilon(f), A\varepsilon(g)\rangle$ to the left-hand side, then consider suitable Riemann sums. □

With this technical lemma we can construct multipliers of j by solving quantum stochastic differential equations with coefficients driven by j.

Lemma 4.2. *For all $c \in |\widehat{\mathsf{k}}\rangle \overline{\otimes} \mathsf{A}$ there is a unique process $M^c = (M^c_t)_{t\geqslant 0}$ in A such that $M^c - I = (M^c_t - I_{\mathfrak{h}\otimes\mathcal{F}})_{t\geqslant 0}$ is vacuum adapted and*

$$M^c_t = I_{\mathfrak{h}\otimes\mathcal{F}} + \int_0^t \widetilde{\jmath}_s(cE^\omega)(I_{\widehat{\mathsf{k}}} \otimes M^c_s)\,\mathrm{d}\Lambda_s,$$

where $\widetilde{\jmath}_s := \mathrm{id}_{B(\widehat{\mathsf{k}})} \overline{\otimes} j_s$ and $\omega := \binom{1}{0} \in \widehat{\mathsf{k}}$, i.e.,

$$\langle u\varepsilon(f), (M^c - I)_t v\varepsilon(g)\rangle = \int_0^t \langle u\varepsilon(f), j_s(E^{\widehat{f(s)}}c)M^c_s v\varepsilon(g)\rangle\,\mathrm{d}s \qquad (t \geqslant 0)$$

for all u, $v \in \mathfrak{h}$ and f, $g \in L^2(\mathbb{R}_+;\mathsf{k})$. The process M^c is norm continuous.

Proof. Define an integrand process G by setting $G_t := \widetilde{\jmath}_t(cE^\omega)$ for all $t \geqslant 0$. In view of the identity $\widetilde{\jmath}(\,\cdot\,)\Delta = \widetilde{\jmath}(\,\cdot\,\Delta)$, which exploits the abuse of notation (2.2), and the fact that $E^\omega \Delta = 0$, Theorem 3.4 gives a vacuum-adapted process N in A which is norm continuous and such that

$$N_t = \int_0^t G_s \,\mathrm{d}\Lambda_s + \int_0^t G_s(I_{\widehat{\mathsf{k}}} \otimes N_s)\,\mathrm{d}\Lambda_s \qquad (t \geqslant 0). \tag{4.1}$$

Hence $M^c_t := I_{\mathfrak{h}\otimes\mathcal{F}} + N_t$ is a norm-continuous process as required; uniqueness holds because the solution of (4.1) is unique. □

Theorem 4.3. *For all $c \in |\widehat{\mathsf{k}}\rangle \overline{\otimes} \mathsf{A}$ the process M^c given by Lemma 4.2 is a multiplier for j.*

Proof. It suffices to verify that condition (M iii) of Definition 2.6 holds. Fix $s \geqslant 0$ and let

$$M_t := \begin{cases} M^c_t & \text{if } t \in [0,s), \\ J_s(M^c_{t-s})M^c_s & \text{if } t \in [s,\infty). \end{cases}$$

Now $\widetilde{J}_s \circ \widetilde{\jmath}_{r-s} = \widetilde{\jmath}_r$ for all $r \geqslant s$, by (C iii) of Definition 2.1, so Lemma 3.3 and Proposition 4.1 imply that

$$\begin{aligned} M_{s+t} &= J_s\Big(I_{\mathfrak{h}\otimes\mathcal{F}} + \int_0^t \widetilde{\jmath}_r(cE^\omega)(I_{\widehat{\mathsf{k}}} \otimes M^c_r)\,\mathrm{d}\Lambda_r\Big)M^c_s \\ &= M^c_s + \int_s^{s+t} \widetilde{J}_s\big(\widetilde{\jmath}_{r-s}(cE^\omega)(I_{\widehat{\mathsf{k}}} \otimes M^c_{r-s})\big)(I_{\widehat{\mathsf{k}}} \otimes M^c_s)\,\mathrm{d}\Lambda_r \\ &= M^c_s + \int_s^{s+t} \widetilde{\jmath}_r(cE^\omega)\big(I_{\widehat{\mathsf{k}}} \otimes (J_s(M^c_{r-s})M^c_s)\big)\,\mathrm{d}\Lambda_r \\ &= I_{\mathfrak{h}\otimes\mathcal{F}} + \int_0^s \widetilde{\jmath}_r(cE^\omega)(I_{\widehat{\mathsf{k}}} \otimes M^c_r)\,\mathrm{d}\Lambda_r + \int_s^{s+t} \widetilde{\jmath}_r(cE^\omega)(I_{\widehat{\mathsf{k}}} \otimes M_r)\,\mathrm{d}\Lambda_r \\ &= I_{\mathfrak{h}\otimes\mathcal{F}} + \int_0^{s+t} \widetilde{\jmath}_r(cE^\omega)(I_{\widehat{\mathsf{k}}} \otimes M_r)\,\mathrm{d}\Lambda_r \qquad (t \geqslant 0). \end{aligned}$$

By Lemma 4.2, $M \equiv M^c$ and $M^c_{t+s} = M_{t+s} = J_s(M^c_t)M^c_s$, as required. □

5. Semigroup Perturbation

For vacuum-adapted integrands the quantum Itô product formula takes the following form [8, Section 5.4].

Lemma 5.1. *Let $Z := \int G\,\mathrm{d}\Lambda$ and $Z' := \int G'\,\mathrm{d}\Lambda$ for integrand processes G and G'. Then*

$$H := (I_{\widehat{\mathsf{k}}} \otimes Z)\Delta^\perp G' + G\Delta^\perp(I_{\widehat{\mathsf{k}}} \otimes Z') + G\Delta G'$$

defines an integrand process such that $ZZ' = \int H\,\mathrm{d}\Lambda$.

The product of three integrals gives the following.

Corollary 5.2. *Let G, G' and G'' be integrand processes and let $Z := \int G \,\mathrm{d}\Lambda$, $Z' := \int G' \,\mathrm{d}\Lambda$ and $Z'' := \int G'' \,\mathrm{d}\Lambda$. Then*

$$\begin{aligned} H := (I_{\widehat{\mathsf{k}}} \otimes ZZ')\Delta^\perp G'' + (I_{\widehat{\mathsf{k}}} \otimes Z)\Delta^\perp G' \Delta^\perp (I_{\widehat{\mathsf{k}}} \otimes Z'') + G\Delta^\perp (I_{\widehat{\mathsf{k}}} \otimes Z'Z'') \\ + (I_{\widehat{\mathsf{k}}} \otimes Z)\Delta^\perp G' \Delta G'' + G\Delta G' \Delta^\perp (I_{\widehat{\mathsf{k}}} \otimes Z'') + G\Delta G' \Delta G'' \end{aligned}$$

is an integrand process such that $ZZ'Z'' = \int H \,\mathrm{d}\Lambda$.

We may now give the main result.

Theorem 5.3. *Let $\psi : \mathsf{A}_0 \to \mathsf{A} \mathbin{\overline{\otimes}} B(\widehat{\mathsf{k}})$, where A_0 is a subset of A, and suppose j satisfies the vacuum-adapted quantum stochastic differential equation*

$$j_0(x) = x \otimes |\Omega\rangle\langle\Omega|, \qquad \mathrm{d}j_t(x) = \widetilde{\jmath}_t\big(\psi(x)\big) \,\mathrm{d}\Lambda_t \qquad (x \in \mathsf{A}_0).$$

For each c, $d \in |\widehat{\mathsf{k}}\rangle \mathbin{\overline{\otimes}} \mathsf{A}$, the generator τ of the pointwise ultraweakly continuous semigroup $\mathcal{P} := \big(\mathbb{E}[(M_t^c)^* \, j_t(\,\cdot\,) \, M_t^d]\big)_{t \geqslant 0}$ satisfies $\operatorname{Dom} \tau \supset \mathsf{A}_0$ and, for all $x \in \mathsf{A}_0$,

$$\begin{aligned} \tau(x) = E^\omega \psi(x) E_\omega + c^* \Delta \psi(x) E_\omega + E^\omega \psi(x) \Delta d \\ + c^* \Delta \psi(x) \Delta d + c^* E_\omega x + x E^\omega d. \end{aligned} \tag{5.1}$$

Proof. Let $x \in \mathsf{A}_0$ and $t \geqslant 0$; note that $(M_t^c)^* j_t(x) M_t^d - j_t(x)$ equals

$$\begin{aligned} (M^c - I)_t^* (j_t - j_0)(x)(M^d - I)_t + (M^c - I)_t^* (j_t - j_0)(x) \\ + (j_t - j_0)(x)(M^d - I)_t + (M^c - I)_t^* j_0(x)(M^d - I)_t \\ + (M^c - I)_t^* j_0(x) + j_0(x)(M^d - I)_t. \end{aligned} \tag{5.2}$$

If u, $v \in \mathfrak{h}$ and f, $g \in L^2(\mathbb{R}_+; \mathsf{k})$ then, writing P_Ω for $|\Omega\rangle\langle\Omega| \in \mathsf{N}$,

$$\begin{aligned} \langle u\varepsilon(f), j_0(x)(M^d - I)_t v\varepsilon(g)\rangle &= \langle (x^* u)\Omega, \int_0^t \widetilde{\jmath}_s(dE^\omega)(I_{\widehat{\mathsf{k}}} \otimes M_s^d) \,\mathrm{d}\Lambda_s v\varepsilon(g)\rangle \\ &= \int_0^t \langle (x^* u)\Omega, E^\omega \widetilde{\jmath}_s(dE^\omega)(I_{\widehat{\mathsf{k}}} \otimes M_s^d) E_{\widehat{g(s)}} v\varepsilon(g)\rangle \,\mathrm{d}s \\ &= \int_0^t \langle u\varepsilon(f), (x \otimes P_\Omega) j_s(E^\omega d) M_s^d v\varepsilon(g)\rangle \,\mathrm{d}s, \end{aligned}$$

therefore

$$j_0(x)(M^d - I)_t = \int_0^t (x \otimes P_\Omega) j_s(E^\omega d) M_s^d \,\mathrm{d}s,$$

$$(M^c - I)_t^* j_0(x) = \int_0^t (M_s^c)^* j_s(c^* E_\omega)(x \otimes P_\Omega) \,\mathrm{d}s$$

and

$$\begin{aligned} (M^c - I)_t^* j_0(x)(M^d - I)_t &= \int_0^t (M^c - I)_s^* (x \otimes P_\Omega) j_s(E^\omega d) M_s^d \,\mathrm{d}s \\ &+ \int_0^t (M_s^c)^* j_s(c^* E_\omega)(x \otimes P_\Omega)(M^d - I)_s \,\mathrm{d}s. \end{aligned}$$

This implies that the sum of the last three terms in (5.2) equals

$$\int_0^t (M_s^c)^* \big((x \otimes P_\Omega) j_s(E^\omega d) + j_s(c^* E_\omega)(x \otimes P_\Omega)\big) M_s^d \,\mathrm{d}s$$
$$= \int_0^t (\widetilde{M}_s^c)^* \big((I_{\widehat{\mathsf{k}}} \otimes x \otimes P_\Omega) \widetilde{\jmath}_s(\Delta^\perp dE^\omega) + \widetilde{\jmath}_s(E_\omega c^* \Delta^\perp)(I_{\widehat{\mathsf{k}}} \otimes x \otimes P_\Omega)\big) \widetilde{M}_s^d \,\mathrm{d}\Lambda_s,$$

where $\widetilde{M}_s^e := I_{\widehat{\mathsf{k}}} \otimes M_s^e$ for $e = c,\, d$.

After some working, with the aid of Lemma 5.1 and Corollary 5.2, it follows that $(M_t^c)^* j_t(x) M_t^d - j_0(x)$ equals

$$\int_0^t \big(\widetilde{\jmath}_s(A_1) + \widetilde{\jmath}_s(A_2)\widetilde{M}_s^d + (\widetilde{M}_s^c)^* \widetilde{\jmath}_s(A_3) + (\widetilde{M}_s^c)^* \widetilde{\jmath}_s(A_4) \widetilde{M}_s^d\big) \,\mathrm{d}\Lambda_s,$$

where

$$A_1 := \Delta\psi(x)\Delta,$$
$$A_2 := \Delta\psi(x)\Delta^\perp + \Delta\psi(x)\Delta d E^\omega,$$
$$A_3 := \Delta^\perp\psi(x)\Delta + E_\omega c^* \Delta\psi(x)\Delta$$

and
$$A_4 := \Delta^\perp\psi(x)\Delta^\perp + E_\omega c^*\Delta\psi(x)\Delta^\perp + \Delta^\perp\psi(x)\Delta d E^\omega + E_\omega c^* \Delta\psi(x)\Delta d E^\omega + E_\omega c^* \Delta^\perp (I_{\widehat{\mathsf{k}}} \otimes x) + (I_{\widehat{\mathsf{k}}} \otimes x)\Delta^\perp d E^\omega.$$

Hence

$$\langle u, (\mathcal{P}_t(x) - x)v\rangle = \langle u\Omega, \big((M_t^c)^* j_t(x) M_t^d - j_0(x)\big) v\Omega\rangle$$
$$= \int_0^t \langle u\Omega, \big(j_s(E^\omega A_1 E_\omega) + j_s(E^\omega A_2 E_\omega) M_s^d + (M_s^c)^* j_s(E^\omega A_3 E_\omega) + (M_s^c)^* j_s(E^\omega A_4 E_\omega) M_s^d\big) v\Omega\rangle \,\mathrm{d}s$$
$$= \int_0^t \langle u\Omega, (M_s^c)^* j_s(E^\omega A_4 E_\omega) M_s^d v\Omega\rangle \,\mathrm{d}s$$
$$= \int_0^t \langle u, \mathcal{P}_s(y) v\rangle \,\mathrm{d}s,$$

where

$$y = E^\omega \psi(x) E_\omega + c^*\Delta\psi(x)E_\omega + E^\omega\psi(x)\Delta d + c^*\Delta\psi(x)\Delta d + c^* E_\omega x + x E^\omega d,$$

as required. □

Remark 5.4. In terms of the direct-sum decomposition $\widehat{\mathsf{k}} = \mathbb{C} \oplus \mathsf{k}$, if

$$\psi = \begin{bmatrix} \tau_0 & \delta_0^\dagger \\ \delta_0 & \pi_0 \end{bmatrix}, \qquad c = \begin{bmatrix} k_1 \\ l_1 \end{bmatrix} \quad \text{and} \quad d = \begin{bmatrix} k_2 \\ l_2 \end{bmatrix}$$

then (5.1) becomes

$$\tau(x) = \tau_0(x) + l_1^* \delta_0(x) + \delta_0^\dagger(x) l_2 + l_1^* \pi_0(x) l_2 + k_1^* x + x k_2 \qquad (x \in \mathsf{A}_0).$$

When ψ is bounded and $\mathsf{A}_0 = \mathsf{A}$, the map δ_0 is a bounded π_0-derivation. Since $\delta_0(\mathsf{A}_0) \subset \mathsf{A} \,\overline{\otimes}\, |\mathsf{k}\rangle$, it follows that δ_0 is implemented ([12], see [16, Chapter 6]) and so

$$\tau(x) = \mathrm{i}[h, x] - \tfrac{1}{2}\{r^* r, x\} + r^* \pi_0(x) r + (x r^* - r^* \pi_0(x)) l_2 \\ + l_1^* (r x - \pi_0(x) r) + l_1^* \pi_0(x) l_2 + k_1^* x + x k_2$$

for some $h = h^* \in \mathsf{A}$ and $r \in |\mathsf{k}\rangle \,\overline{\otimes}\, \mathsf{A}$. Equivalently,

$$\tau(x) = d_1^* \pi_0(x) d_2 + e_1^* x + x e_2,$$

where $d_i = l_i - r$ and $e_i = k_i + r^* l_i - \frac{1}{2} r^* r - \mathrm{i}h$ for $i = 1, 2$.

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