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ROOTS OF STATES

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Dedicated to Professor K. R. Parthasarathy on the occasion of his 75th birthday

Abstract. Given a state ϕ on a unital C^*-algebra $\mathcal{A}$ we look at unital quantum dynamical semigroups $\{\tau_t\}_{t\geq 0}$ on $\mathcal{A}$ such that $\tau_{t_0}(\cdot) = \phi(\cdot)I$ for some $t_0 > 0$. We see that for the von Neumann algebra $\mathcal{B}(\mathcal{H})$, such quantum dynamical semigroups dilate to semigroups of unital $*$-endomorphisms (E_0-semigroups) in standard form and conversely all E_0-semigroups in standard form arise this way.

1. Introduction

Let $\mathcal{A}$ be a unital C^*-algebra and let ϕ be a state (positive linear functional of norm equal to one) on $\mathcal{A}$. We ask the question as to what are n-th roots of ϕ $(n \geq 1)$. As stated the question does not make sense as $\phi : \mathcal{A} \to \mathbb{C}$, and $\mathcal{A}$ can be different from $\mathbb{C}$. Instead we note that $\hat{\phi} : \mathcal{A} \to \mathcal{A}$ defined by

$$\hat{\phi}(a) = \phi(a)I, \ a \in \mathcal{A}$$

is a unital completely positive (CP) map. So the problem is to describe

$$R_n(\phi) := \{\tau : \tau \text{ is a unital CP map on } \mathcal{A}, \tau^n = \hat{\phi}\}.$$

Of course, $\hat{\phi}$ itself is an element of $R_n(\phi)$, but there can be other solutions.

Example 1.1. Let $\mathcal{A} = M_4(\mathbb{C})$ be the algebra of all 4×4 complex matrices, thought of as the algebra $\mathcal{B}(\mathbb{C}^4)$ of linear maps on $\mathbb{C}^4$. Let $\{e_i : 1 \leq i \leq 4\}$ be the standard ortho-normal basis of $\mathbb{C}^4$. Take $\phi(X) = \langle e_1, Xe_1\rangle$ for X in $\mathcal{A}$. Define $\tau : \mathcal{A} \to \mathcal{A}$ by,

$$\tau(X) = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{22} & a_{23} \\ 0 & 0 & a_{32} & a_{33} \end{bmatrix}$$

Then it is easily seen that τ is a unital completely positive satisfying $\tau^3(X) = \phi(X)I$ for all X.

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We may also ask whether there is a quantum Markov semigroup (one parameter semigroup of unital completely positive maps $\tau = \{\tau_t\}_{t\geq 0}$ such that $t \mapsto \tau_t$ is continuous in a suitable topology) such that

$$\tau_{t_0}(a) = \phi(a)I, \quad \forall\, a \in \mathcal{A}$$

for some fixed $t_0 > 0$. This can be considered as a kind of infinite divisibility of the state. But then we have the following simple observation.

Proposition 1.2. *Let $\mathcal{A}$ be a unital C^* algebra such that $\mathcal{A}$ is not isomorphic to $\mathbb{C}$. Let ϕ be a state on $\mathcal{A}$. Then there does not exist a uniformly continuous quantum Markov semigroup τ on $\mathcal{A}$ such that*

$$\tau_{t_0}(a) = \phi(a)I, \quad \forall\, a \in \mathcal{A}$$

for some fixed $t_0 > 0$.

Proof. As τ is uniformly continuous it has a bounded generator, that is, $\tau_t = e^{t\mathcal{L}}$ for some bounded linear map $\mathcal{L} : \mathcal{A} \to \mathcal{A}$. In particular, every τ_t is a invertible linear map, the inverse being $e^{-t\mathcal{L}}$. But clearly the map $a \mapsto \phi(a)I$ is not invertible if $\mathcal{A}$ is not one dimensional. □

Since the only reasonable notion of continuity in finite dimensions is uniform continuity, if $\mathcal{A}$ is finite dimensional (but not isomorphic $\mathbb{C}$), there are no quantum Markov semigroups solving our problem. The situation is different in infinite dimensions as the following example shows.

Example 1.3. Take $\mathcal{H} = \mathbb{C} \oplus L^2([0,1])$ and let $\mathcal{A}$ be the von Neumann algebra $\mathcal{B}(\mathcal{H})$. Define shift semigroup $S_t, t \geq 0$ on $L^2([0,1])$ by

$$S_t f(x) = \begin{cases} f(x-t) & \text{if } x - t \in [0,1]; \\ 0 & \text{otherwise.} \end{cases}$$

Note that $S = \{S_t\}_{t\geq 0}$ is a strongly continuous one parameter semigroup with $S_t = 0$ for $t \geq 1$. Now any operator in $\mathcal{B}(\mathcal{H})$ acts as

$$\begin{bmatrix} c & f^* \\ g & A \end{bmatrix} \begin{pmatrix} d \\ h \end{pmatrix} = \begin{pmatrix} cd + \langle f, h\rangle \\ dg + Ah \end{pmatrix}$$

where $c \in \mathcal{B}(\mathbb{C}), f, g \in L^2([0,1]), A \in \mathcal{B}(L^2([0,1]))$. Now define $\tau = \{\tau_t\}_{t\geq 0}$ by

$$\tau_t\left(\begin{bmatrix} c & f^* \\ g & A \end{bmatrix}\right) = \begin{bmatrix} c & (S_t f)^* \\ S_t g & S_t A S_t^* + c(I - S_t S_t^*) \end{bmatrix}$$

It is easily verified that τ is quantum Markov semigroup ($t \mapsto \tau_t(X)$ is continuous in strong operator topology for every $X \in \mathcal{B}(\mathcal{H})$). Moreover,

$$\tau_1(X) = \langle e_1, Xe_1\rangle I, \quad \forall\, X \in \mathcal{B}(\mathcal{H}).$$

where $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

In the next section we give a brief account of dilation theory of quantum Markov semigroups to E_0-semigroups (semigroups of endomorphisms). Then we show that the kind of quantum Markov semigroups considered above always dilate to E_0-semigroups which are somewhat special, namely they are E_0-semigroups in standard form (in the sense of Powers [13]). Conversely, every E_0-semigroup in standard form is minimal dilation of a quantum Markov semigroup of this form. In the last section we see the connection with the theory of tensor product systems of Hilbert spaces.

2. Dilation and Spatial E_0-semigroups

Let $\mathcal{H}$ be a complex, separable Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the von Neumann algebra of all bounded operators on $\mathcal{H}$. Here by a quantum Markov semigroup on $\mathcal{B}(\mathcal{H})$ we would mean a one parameter semigroup $\tau = \{\tau_t\}_{t\geq 0}$ of unital normal completely positive maps on $\mathcal{B}(\mathcal{H})$ such that the maps $t \mapsto \tau_t(X)$ is continuous in strong operator topology (equivalently in weak operator topology or $\sigma-$ weakly) for every X in $\mathcal{B}(\mathcal{H})$. A quantum Markov semigroup $\tau = \{\tau_t\}_{t\geq 0}$ is an E_0- semigroup if for every t, τ_t is a normal, unital $*$-endomorphism of $\mathcal{B}(\mathcal{H})$. Just as classical Markov semigroups have associated Markov processes by Kolmogorov-Daniel constructions, quantum Markov semigroups have associated non-commutative Markov processes (see [7], [8]). The notion of weak Markov flows was introduced in [8]. Looking at time-shift of the weak Markov flows leads to the minimal dilation theorem:

Theorem 2.1. *(Bhat,* [4], [6]*): Let $\tau = \{\tau_t\}_{t\geq 0}$ be a quantum Markov semigroup on $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a complex, separable Hilbert space. Then there exists a pair $(\mathcal{K}, \theta)$ where $\mathcal{K}$ is a complex, separable Hilbert space containing $\mathcal{H}$ and $\theta = \{\theta_t\}_{t\geq 0}$ is an E_0-semigroup on $\mathcal{B}(\mathcal{H})$ such that*

*(i) (*Dilation property*) For $X \in \mathcal{B}(\mathcal{H}), t \geq 0$,*

$$\tau_t(X) = P\theta_t(X)P.$$

[Here P is the orthogonal projection of $\mathcal{K}$ on to $\mathcal{H}$ and $X \in \mathcal{B}(\mathcal{H})$ is identified with $PXP \in \mathcal{B}(\mathcal{K})$.]

*(ii) (*Minimality*)*

$$\mathcal{K} = \overline{\text{span}} \left\{ \begin{array}{r} \theta_{r_1}(X_1)\theta_{r_2}(X_2)\dots\theta_{r_n}(X_n)h : \quad r_1 \geq r_2 \geq \cdots \geq r_n \geq 0, \\ X_1, X_2, \dots, X_n \in \mathcal{B}(\mathcal{H}), n \geq 0, h \in \mathcal{H} \end{array} \right\}$$

*Furthermore (*Uniqueness up to unitary equivalence*), if $(\mathcal{K}', \theta')$ is another such pair then there exists a unitary $U : \mathcal{K} \to \mathcal{K}'$ such that $\theta'_t(Z) = U\theta_t(U^*ZU)U^*$ for all $Z \in \mathcal{B}(\mathcal{K}'), t \geq 0$.*

We call the pair $(\mathcal{K}, \theta)$ of this theorem as the minimal dilation of $(\mathcal{H}, \tau)$, or we may simply say that θ is a minimal dilation of τ. The following facts about minimal dilation are well-known and can be found in ([4], [6]):

(1) P is an increasing projection for θ, that is, $\theta_s(P) \leq \theta_t(P)$ if $0 \leq s \leq t < \infty$.
(2) θ is a primary dilation, that is, $\theta_t(P) \uparrow I$ as $t \uparrow \infty$.
(3) For $t \geq r_1 \geq r_2 \geq \cdots \geq r_n \geq 0, X, Y_1, Y_2, \dots, Y_n, Z_1, Z_2, \dots, Z_n \in \mathcal{B}(\mathcal{H}), n \geq 0$,

$$\begin{aligned} &\langle \theta_{r_1}(Y_1)\theta_{r_2}(Y_2)\dots\theta_{r_n}(Y_n)g, \theta_t(X)\theta_{r_1}(Z_1)\theta_{r_2}(Z_2)\dots\theta_{r_n}(Z_n)h\rangle \\ &= \langle g, \tau_{r_n}(Y_n^*(\cdots\tau_{r_2-r_3}(Y_2^*\tau_{r_1-r_2}(Y_1^*\tau_{t-r_1}(X)Z_1)Z_2)\cdots)Z_n)h\rangle. \end{aligned}$$

We have the following notion due to Powers.

Definition 2.2. An E_0-semigroup $\theta = \{\theta_t\}_{t\geq 0}$ of $\mathcal{B}(\mathcal{H})$ is said to be in standard form if there exists a unit vector $u \in \mathcal{H}$ such that $\theta_t(|u\rangle\langle u|) \uparrow I$ as $t \uparrow \infty$.

This is not quite the original definition given by Powers. The Definition 2.12 in [13] is that θ is in standard form if there exists a normal pure state ω of $\mathcal{B}(\mathcal{H})$ such that

$$\lim_{t\to\infty} \rho \circ \theta_t(\cdot) = \omega(\cdot).$$

for every normal state ρ of $\mathcal{B}(\mathcal{H})$. However, it is not very hard to see the equivalence of two definitions.

We briefly explain the importance of this notion of standard form. This relates to classification of E_0-semigroups. Two E_0-semigroups θ, ψ of $\mathcal{B}(\mathcal{H})$ are said to be *conjugate* if there exists a unitary M on $\mathcal{H}$ such that $\psi_t(X) = M\theta_t(M^*XM)M^*$ for all $X \in \mathcal{B}(\mathcal{H})$, and they are said to be *cocycle conjugate* if there exists a strongly continuous family $\{U_t\}_{t\geq 0}$ of unitaries, with $U_0 = I, U_{s+t} = U_s\theta_s(U_t)$, $\forall\, s,t$, such that

$$\psi_t(X) = U_t\theta_t(X)U_t^*, \quad \forall\, X \in \mathcal{B}(\mathcal{H}),\, t \geq 0.$$

Clearly, cocycle conjugacy is a weaker notion of equivalence for E_0-semigroups. There has been a lot of effort to classify E_0-semigroups up to cocycle conjugacy [3].

An E_0-semigroup θ is said to be spatial if there exists a one parameter semigroup of isometries $\{V_t\}_{t\geq 0}$, which intertwine with θ: $\theta_t(X)V_t = V_tX$, $\forall\, X \in \mathcal{B}(\mathcal{H})$. Non-spatial E_0-semigroups are said to be of type III. Spatial semigroups are further classified into type I (or completely spatial) and type II depending upon the richness of the space of intertwining semigroups [3].

Observe that every E_0-semigroup θ in standard form is spatial, indeed if θ, u are as in Definition 2.2, then $\{V_t\}_{t\geq 0}$ defined by

$$V_tx = \theta_t(|x\rangle\langle u|)u$$

is an intertwining semigroup of isometries. Powers [13] has shown that if two type I E_0-semigroups are in standard form and are cocycle conjugate then they are conjugate. In other words the standard form is unique for type I E_0- semigroups. Strengthening this Alevras[1] shows that two cocycle conjugate spatial E_0-semigroups are conjugate if and only if the group of local unitary cocycles acts transitively on normalized units (see [13], [11], [5] for definitions). This transitivity condition called as amenability in [5] is satisfied by type I E_0-semigroups and was conjectured to hold for all spatial E_0-semigroups. See Liebscher [10] for comments regarding this question (page 97) and a lot of other useful information about spatial E_0-semigroups. However, an example of Tsirelson [16] settles this conjecture in the negative.

Now we come to one of the main results of this article.

Theorem 2.3. *Let $\tau = \{\tau_t\}_{t\geq 0}$ be a quantum Markov semigroup of $\mathcal{B}(\mathcal{H})$ such that*

$$\tau_{t_0}(X) = \langle u, Xu\rangle I, \quad \forall\, X \in \mathcal{B}(\mathcal{H}),$$

for some $t_0 > 0$, where $u \in \mathcal{H}$ is a unit vector. Let $(\mathcal{K}, \theta)$ be the minimal dilation of $(\mathcal{H}, \tau)$. Then θ is an E_0-semigroup in standard form with $\theta_t(|u\rangle\langle u|) \uparrow I$ as $t \uparrow \infty$.

Proof. We break up the theorem into following elementary steps.

(i) For all $t \geq t_0, X \in \mathcal{B}(\mathcal{H})$, $\tau_t(X) = \langle u, Xu\rangle I$;

(ii) For all $t \geq 0$, $\langle u, \tau_t(|u\rangle\langle u|)u\rangle = 1$;

(iii) For all $t \geq 0$, $\tau_t(|u\rangle\langle u|)u = u$;

(iv) For all $t \geq 0$, $\theta_t(|u\rangle\langle u|)u = u$;

(v) For all $s, t \geq 0$, $\theta_{s+t}(|u\rangle\langle u|) \geq \theta_s(|u\rangle\langle u|)$;

(vi) $\theta_t(|u\rangle\langle u|) \uparrow I$ as $t \uparrow \infty$.

For $t \geq t_0$, $\tau_t(X) = \tau_{t-t_0}(\tau_{t_0}(X)) = \tau_{t-t_0}(\langle u, Xu\rangle I) = \langle u, Xu\rangle\tau_{t-t_0}(I) = \langle u, Xu\rangle I$. This proves (i). We already have the result (ii) for $t \geq t_0$ from (i). Now for $0 \leq t < t_0$,

$$\begin{aligned} 1 &= \langle u, \tau_{t+t_0}(|u\rangle\langle u|)u\rangle = \langle u, \tau_{t_0}(\tau_t(|u\rangle\langle u|))u\rangle \\ &= \langle u, \{\langle u, \tau_t(|u\rangle\langle u|)u\rangle I\}u\rangle = \langle u, \tau_t(|u\rangle\langle u|)u\rangle. \end{aligned}$$

Now (iii) follows from (ii) as τ_t is a contraction, and hence $\tau_t(|u\rangle\langle u|)$ is a contraction. Then by dilation property, $\langle u, \theta_t(|u\rangle\langle u|)u\rangle = \langle u, \tau_t(|u\rangle\langle u|)u\rangle = 1$. By contractivity of $\theta_t(|u\rangle\langle u|)$, $\theta_t(|u\rangle\langle u|)u = u$ for all $t \geq 0$. But θ_t being a $*$-endomorphism, $\theta_t(|u\rangle\langle u|)$ is a projection. Then it is clear that $\theta_t(|u\rangle\langle u|) \geq |u\rangle\langle u|$. Applying θ_s on this inequality we have (v).

Now by the property (3) listed above for minimal dilation (with $X = |u\rangle\langle u|, t \geq r_1 + t_0$), for $r_1 \geq r_2 \geq \cdots \geq r_n \geq 0, Y_1, Y_2, \ldots, Y_n, Z_1, Z_2, \ldots, Z_n \in \mathcal{B}(\mathcal{H}), n \geq 0$,

$$\begin{aligned} &\langle \theta_{r_1}(Y_1)\theta_{r_2}(Y_2)\ldots\theta_{r_n}(Y_n)g, \theta_t(|u\rangle\langle u|)\theta_{r_1}(Z_1)\theta_{r_2}(Z_2)\ldots\theta_{r_n}(Z_n)h\rangle \\ &= \langle g, \tau_{r_n}(Y_n^*(\cdots\tau_{r_2-r_3}(Y_2^*\tau_{r_1-r_2}(Y_1^*\tau_{t-r_1}(|u\rangle\langle u|)Z_1)Z_2)\cdots)Z_n)h\rangle \\ &= \langle g, \tau_{r_n}(Y_n^*(\cdots\tau_{r_2-r_3}(Y_2^*\tau_{r_1-r_2}(Y_1^*P_{\mathcal{H}}Z_1)Z_2)\cdots)Z_n)h\rangle \\ &= \langle \theta_{r_1}(Y_1)\theta_{r_2}(Y_2)\ldots\theta_{r_n}(Y_n)g, \theta_{r_1}(Z_1)\theta_{r_2}(Z_2)\ldots\theta_{r_n}(Z_n)h\rangle. \end{aligned}$$

So $\theta_t(|u\rangle\langle u|) \uparrow I$ as $t \uparrow \infty$. □

Now we claim that the converse of this Theorem also holds in the following sense.

Theorem 2.4. *Let $\theta = \{\theta_t\}_{t\geq 0}$ be an E_0-semigroup in standard form of $\mathcal{B}(\mathcal{K})$ so that $\theta_t(|u\rangle\langle u|) \uparrow I$ for a unit vector $u \in \mathcal{K}$. Then for $t_0 > 0$, there exists a pair $(\mathcal{H}, \tau)$, where $\mathcal{H}$ is a subspace of $\mathcal{K}$ containing u and τ is a quantum Markov semigroup of $\mathcal{B}(\mathcal{H})$ such that*

(i) $\tau_{t_0}(X) = \langle u, Xu\rangle I$ for $X \in \mathcal{B}(\mathcal{H})$.

(ii) $(\mathcal{K}, \theta)$ is the minimal dilation of $(\mathcal{H}, \tau)$.

Proof. Take $\mathcal{H} = \theta_{t_0}(|u\rangle\langle u|)(\mathcal{K})$. Then as the projection onto $\mathcal{H}$, $P_{\mathcal{H}} = \theta_{t_0}(|u\rangle\langle u|)$, is an increasing projection for θ, $\{\tau_t\}_{t\geq 0}$ on $\mathcal{B}(\mathcal{H})$ defined by

$$\tau_t(X) = P_{\mathcal{H}}\theta_t(X)P_{\mathcal{H}},\ X \in \mathcal{B}(\mathcal{H}) = P_{\mathcal{H}}(\mathcal{B}(\mathcal{K}))P_{\mathcal{H}}$$

is a unital quantum dynamical semigroup. Further, for $X \in \mathcal{B}(\mathcal{H})$

$$\begin{aligned} \tau_{t_0}(X) &= P_{\mathcal{H}}\theta_{t_0}(X)P_{\mathcal{H}} \\ &= \theta_{t_0}(|u\rangle\langle u|)\theta_{t_0}(X)\theta_{t_0}(|u\rangle\langle u|) \\ &= \langle u, Xu\rangle\theta_{t_0}(|u\rangle\langle u|) \\ &= \langle u, Xu\rangle P_{\mathcal{H}}. \end{aligned}$$

Clearly θ is a dilation of τ. The result that it is a minimal dilation follows from Theorem 3.7 in [5]. We sketch the proof here for completeness. Let $\{e_k\}_{k\geq 1}$ be a complete ortho-normal basis of $\mathcal{K}$. Now for $t \geq t_0$ and $z \in \mathcal{K}$ we consider the vector $\theta_t(|u\rangle\langle u|)z$. Let n be the natural number satisfying $0 \leq t - nt_0 < t_0$. By normality and unitality of θ, we have

$$\begin{aligned}\theta_t(|u\rangle\langle u|)z &= \sum_{k_1,\ldots,k_n} \theta_t(|u\rangle\langle u|)\theta_{t-t_0}(|e_{k_1}\rangle\langle e_{k_1}|)\cdots\theta_{t-nt_0}(|e_{k_n}\rangle\langle e_{k_n}|)z \\ &= \sum_{k_1,\ldots,k_n} \theta_{t-t_0}(|y_1\rangle\langle e_{k_1}|)\cdots\theta_{t-nt_0}(|e_{k_n}\rangle\langle e_{k_n}|)z\end{aligned}$$

where $y_{k_1} = \theta_{t_0}(|u\rangle\langle u|)e_{k_1} \in \mathcal{H}$. Further,

$$\begin{aligned}&\theta_t(|u\rangle\langle u|)z \\ =& \sum_{k_1,\ldots,k_n} \theta_{t-t_0}(|y_1\rangle\langle u|)\theta_{t-t_0}(|u\rangle\langle e_{k_1}|)\theta_{t-2t_0}(|e_{k_2}\rangle\langle e_{k_2}|)\cdots\theta_{t-nt_0}(|e_{k_n}\rangle\langle e_{k_n}|)z \\ =& \sum_{k_1,\ldots,k_n} \theta_{t-t_0}(|y_1\rangle\langle u|)\theta_{t-2t_0}(|y_{k_1,k_2}\rangle\langle e_{k_2}|)\cdots\theta_{t-nt_0}(|e_{k_n}\rangle\langle e_{k_n}|)z\end{aligned}$$

where $y_{k_1,k_2} = \theta_{t_0}(|u\rangle\langle e_{k_1}|)e_{k_2} \in \mathcal{H}$. Continuing this way, we get

$$\theta_t(|u\rangle\langle u|)z = \sum_{k_1,\ldots,k_n} \theta_{t-t_0}(|y_1\rangle\langle u|)\theta_{t-2t_0}(|y_{k_1,k_2}\rangle\langle u|)\cdots\theta_{t-nt_0}(|y_{k_{n-1},k_n}\rangle\langle e_{k_n}|)z$$

Note that $|y_1\rangle\langle u|, |y_{k_1,k_2}\rangle\langle u|$, etc. are in $\mathcal{B}(\mathcal{H})$. □

It is to be noted that the projection described in this Theorem may not be the only way of getting a suitable increasing projection. For instance, for the semigroup in Example 1.3, the minimal dilation space can be taken to be the symmetric Fock space over $L^2([0,\infty))$ and the dilation E_0-semigroup θ as the standard CCR flow (second quantization of shift). The space $\mathcal{H} = \mathbb{C} \oplus L^2([0,1])$ is identified with $\mathbb{C}\omega \oplus L^2([0,1])$, where ω is the vacuum vector and $L^2([0,1])$ is a portion of one particle sector in the Fock space. Clearly the projection onto this subspace is not equal to $\theta_t(|\omega\rangle\langle\omega|)$ for any t.

3. Spatial Product Systems

A tensor product system of Hilbert spaces is a pair $(\mathcal{E},\mathcal{U})$ where $\mathcal{E} = \{\mathcal{E}_t\}_{t\geq 0}$ is a family of Hilbert spaces and $\mathcal{U}$ is a family of unitaries $\{U_{s,t} : \mathcal{E}_s \otimes \mathcal{E}_t \to \mathcal{E}_{s+t}\}_{s,t\geq 0}$, satisfying the associativity condition:

$$U_{r+s,t}(U_{r,s} \otimes I_t) = U_{r,s+t}(I_r \otimes U_{s,t}), \quad \forall\, r,s,t.$$

(Here I_t denotes the identity operator on $\mathcal{E}_t$.) Often some additional measurability conditions are put (see [2]). While bringing in measurability, the space $\mathcal{E}_0$ is excluded in [2] for purely technical reasons. A 'measurable' family $u = \{u_t\}$ of vectors $u_t \in \mathcal{E}_t$ is said to be a *unit* if $U_{s,t}(u_s \otimes u_t) = u_{s+t}$ for all s,t. Such a unit is said to be normalized if $\|u_t\| = 1$ for all t. The product systems are classified looking at their units.

Arveson associated tensor product system of Hilbert spaces with E_0- semigroups of $\mathcal{B}(\mathcal{H})$ through the space of intertwiners as follows. Suppose $\theta = \{\theta_t\}_{t\geq 0}$ is an E_0-semigroup of $\mathcal{B}(\mathcal{H})$. For every $t \geq 0$, let

$$\mathcal{E}_t = \{Y \in \mathcal{B}(\mathcal{H}) : \theta_t(X)Y = YX, \quad \forall\, X \in \mathcal{B}(\mathcal{H})\}.$$

In other words, $\mathcal{E}_t$ is the space of intertwiners between representation θ_t and the identity representation. For Y, Z in $\mathcal{E}_t$, the intertwining property implies that Y^*Z is in the commutant of $\mathcal{B}(\mathcal{H})$ and hence it is a scalar. This allows to consider $\mathcal{E}_t$ as a Hilbert space with inner product defined by $\langle Y, Z\rangle I = Y^*Z$. Further, with mappings $V_{s,t} : \mathcal{E}_s \otimes \mathcal{E}_t \to \mathcal{E}_{s+t}$ we have a product system of Hilbert spaces $(\mathcal{E}, V)$. It is clear that a normalized unit here consists of an intertwining semigroup of isometries.

An alternative way of getting a product system from an E_0-semigroup θ of $\mathcal{B}(\mathcal{H})$ is described in ([4], [6]). It is as follows. Fix a unit vector $a \in \mathcal{H}$. For $t \geq 0$, let $\mathcal{P}_t = \theta_t(|a\rangle\langle a|)(\mathcal{H})$. Define $U_{s,t} : \mathcal{P}_s \otimes \mathcal{P}_t \to \mathcal{P}_{s+t}$ by,

$$U_{s,t}(y \otimes z) = \theta_t(|y\rangle\langle a|)z.$$

Then $U_{s,t}$'s are unitaries and $(\mathcal{P}, U)$ is a tensor product system of Hilbert spaces. This is actually isomorphic to the opposite product system of $(\mathcal{E}, V)$.

Let $(\mathcal{P}_t, U_{s,t})_{s,t\geq 0}$ be a spatial product system and suppose $u = \{u_t\}_{t\geq 0}$ is a normalized unit of this product system. Then we can describe the quantum dynamical semigroup which appears in the last section as follows: Fix $t_0 > 0$. Take $\mathcal{H} = \mathcal{P}_{t_0}$. Define isometries $V_t : \mathcal{H} \to \mathcal{P}_t \otimes \mathcal{P}_{t_0}$ by

$$V_t(x) = u_t \otimes x, \quad \forall\, x \in \mathcal{H},$$

and define completely positive maps τ_t on $\mathcal{B}(\mathcal{H})$ by

$$\tau_t(X) = B_t^*(X \otimes I_t)B_t, \quad \forall\, X \in \mathcal{B}(\mathcal{H}),$$

where $B_t = U^*_{t_0,t}U_{t,t_0}V_t$.

Theorem 3.1. *Let $(\mathcal{P}_t, U_{s,t})_{s,t\geq 0}$ be a spatial product system with a normalized unit $u = \{u_t\}_{t\geq 0}$. Then $\tau = \{\tau_t\}_{t\geq 0}$, defined above is a unital quantum dynamical semigroup such that $\tau_{t_0}(X) = \langle u_{t_0}, Xu_{t_0}\rangle I$ for all $X \in \mathcal{B}(\mathcal{H})$.*

Proof. As each B_t is an isometry, τ_t is a normal, unital completely positive map for every t. Now to see the semigroup property, first note that the operators $V_t \in \mathcal{B}(\mathcal{P}_{t_0}, \mathcal{P}_t \otimes \mathcal{P}_{t_0})$ have the following intertwining property: For any $Z \in \mathcal{B}(\mathcal{P}_s \otimes \mathcal{P}_{t_0}, \mathcal{P}_{t_0} \otimes \mathcal{P}_s)$, $x \in \mathcal{P}_{t_0}$,

$$\begin{aligned}(V_t \otimes I_s)ZV_sx &= (V_t \otimes I_s)Z(u_s \otimes x)\\ &= u_t \otimes Z(u_s \otimes x)\\ &= (I_t \otimes Z)(u_t \otimes u_s \otimes x).\end{aligned}$$

Making use of this property and associativity of the product system unitaries (applications of this are indicated below in curly braces),

$$\begin{aligned}(B_t \otimes I_s)B_s x &= [U^*_{t_0,t}U_{t,t_0}V_t \otimes I_s][U^*_{t_0,s}U_{s,t_0}V_s]x \\ &= [U^*_{t_0,t}U_{t,t_0} \otimes I_s][I_t \otimes U^*_{t_0,s}U_{s,t_0}](u_t \otimes u_s \otimes x) \\ &= (U^*_{t_0,t} \otimes I_s)\{(U_{t,t_0} \otimes I_s)(I_t \otimes U^*_{t_0,s})\}(I_t \otimes U^*_{s,t_0}(u_t \otimes u_s \otimes x) \\ &= \{U^*_{t_0,t} \otimes I_s)U^*_{t+t_0,s}\}\{U_{t,t_0+s}(I_t \otimes U_{s,t_0})\}(u_t \otimes u_s \otimes x) \\ &= [(I_{t_0} \otimes U^*_{t,s})U^*_{t_0,t+s}U_{t+s,t_0}(U_{t,s} \otimes I_{t_0})](u_t \otimes u_s \otimes x) \\ &= (I_{t_0} \otimes U^*_{t,s})U^*_{t_0,t+s}U_{t+s,t_0}(u_{s+t} \otimes x) \\ &= (I_{t_0} \otimes U^*_{t,s})B_{s+t}x.\end{aligned}$$

Then it is clear that $\tau_{s+t}(X) = \tau_s(\tau_t(X)), \forall X \in \mathcal{B}(\mathcal{H}), s, t \geq 0.$ □

We have not taken care here to verify the continuity of this quantum dynamical semigroup. Continuity is actually a consequence of measurability assumptions on the product system and the unit. One way to do this is to write down the dilation. For all s, t, $\mathcal{P}_s$ imbeds in $\mathcal{P}_{s+t}$ in a consistent way, through isometries $i_{s,s+t}(x) = U_{t,s}(u_t \otimes x)$. Taking inductive limit with respect to these maps we get a Hilbert space $\mathcal{K} = \text{indlim}_{t\to\infty}(\mathcal{P}_t)$ with embeddings $i_s : \mathcal{P}_s \to \mathcal{K}$ satisfying $i_s(x_s) = i_{s+t}(U_{t,s}(u_t \otimes x_s))$ for all $x_s \in \mathcal{P}_s$. Then we define unitaries $W_t : \mathcal{K} \otimes \mathcal{P}_t \to \mathcal{K}$ by

$$W_t(i_s(x_s) \otimes y_t) = i_{s+t}(U_{s,t}(x_s \otimes y_t)), \quad \forall\, x_s \in \mathcal{P}_s,\, y_t \in \mathcal{P}_t$$

and extending linearly. Then one obtains an E_0-semigroup of $\mathcal{B}(\mathcal{K})$ by setting

$$\theta_t(Z) = W_t(Z \otimes I_t)W_t^*, \quad \forall\, Z \in \mathcal{B}(\mathcal{K}),\, t \geq 0.$$

This construction has been carried out in the Appendix of [2] (albeit using a different language), in particular the continuity has been verified. Now the semigroup τ described above is simply the compression of θ to $\mathcal{B}(\mathcal{P}_{t_0})$, on identifying $\mathcal{P}_{t_0}$ with its image $i_{t_0}(\mathcal{P}_{t_0})$ in $\mathcal{K}$.

In this article we have considered only pure states. It is not clear to us as to how to treat mixed states. Although here we have looked at only spatial product systems (product systems with units), this work was motivated by an effort to understand [14], [15], [9], which treat general product systems. The situation for general product systems is somewhat more complicated and their study is being postponed.

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