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AN INTERPOLATING FAMILY OF MEANS

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Dedicated to Professor K. R. Parthasarathy on the occasion of his 75th birthday

Abstract. This paper is concerned with a new family of binary symmetric means M_p of two positive numbers a and b:

$$\frac{1}{\mathrm{M}_p(a,b)} = c_p \int_0^\infty \frac{dx}{[(x^p+a^p)(x^p+b^p)]^{1/p}}, \quad 0 < p < \infty,$$

where the constant c_p, depending on p, is chosen to have $\mathrm{M}_p(a,a) = a$. Two distinctive members in the family are the well-known logarithmic mean ($p=1$) and arithmetic-geometric mean ($p=2$). Different expressions for M_p are obtained to establish its other properties, including $\mathrm{M}_2(a,b) \le \mathrm{M}_\infty(a,b)$ and the relation between M_p and the power difference mean. Through investigating the induced operator norm of the integral operator with M_p^{-1} as its kernel, a generalization of the Hilbert inequality is obtained. Finally positive definiteness of certain matrices as implications of inequalities between two means is also investigated.

1. Introduction

Let a and b be positive numbers. The *logarithmic mean* $\mathrm{L}(a,b)$ of a and b defined as

$$\mathrm{L}(a,b) := \frac{a-b}{\ln a - \ln b} \tag{1.1}$$

has long been used in problems related to heat flow [16] and electrical conduction [17]. More recently it has been employed in differential geometry [2, 5]. The well-known *arithmetic-geometric mean* $\mathrm{AG}(a,b)$ of Gauss is defined as follows: the sequences $\{a_n\}$ and $\{b_n\}$ defined inductively as

$$a_0 = a, \qquad b_0 = b,$$
$$a_{n+1} = \frac{a_n+b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n},$$

have a common limit, and

$$\mathrm{AG}(a,b) := \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n. \tag{1.2}$$

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This mean, introduced by Legendre and then by Gauss, is related to the evaluation of elliptic integrals, and several other problems in analysis [11, 8].

The expressions (1.1) and (1.2) do not carry any hint that these two means could belong to a common family. There are alternative descriptions for both. It can be seen that

$$\frac{1}{\mathrm{L}(a,b)} = \int_0^\infty \frac{dx}{(x+a)(x+b)}, \tag{1.3}$$

and an ingenious calculation, due to Gauss [12], is used to show

$$\frac{1}{\mathrm{AG}(a,b)} = \frac{2}{\pi}\int_0^\infty \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)}}. \tag{1.4}$$

This similarity between the expressions (1.3) and (1.4) is the motivation for us to introduce a family of means $\mathrm{M}_p(a,b)$, $0 \le p \le \infty$, defined by the relation

$$\frac{1}{\mathrm{M}_p(a,b)} := c_p \int_0^\infty \frac{dx}{[(x^p+a^p)(x^p+b^p)]^{1/p}}, \quad 0 < p < \infty, \tag{1.5}$$

where the constant c_p, depending on p, will be chosen to have

$$\mathrm{M}_p(a,a) = a.$$

Thus

$$\frac{1}{c_p} = a\int_0^\infty \frac{dx}{(x^p+a^p)^{2/p}} = \int_0^\infty \frac{dy}{(y^p+1)^{2/p}}. \tag{1.6}$$

The means M_0 and M_∞ are defined by taking limits:

$$\mathrm{M}_0(a,b) := \lim_{p\to 0^+} \mathrm{M}_p(a,b), \quad \mathrm{M}_\infty(a,b) := \lim_{p\to\infty} \mathrm{M}_p(a,b). \tag{1.7}$$

A *binary symmetric mean* $M(a,b)$ of positive numbers a and b is a function that satisfies the following properties:

(i) $\min\{a,b\} \le M(a,b) \le \max\{a,b\}$ (In particular, $M(a,a) = a$);
(ii) $M(a,b) = M(b,a)$;
(iii) $M(\alpha a, \alpha b) = \alpha\, M(a,b)$ for all $\alpha > 0$;
(iv) $M(a,b)$ is non-decreasing in a and b.

It is obvious from the definition that the mean M_p satisfies the properties (ii) – (iv). We will give different expressions for M_p from which other properties, including (i) above, become apparent. In particular, we will show that

$$\mathrm{M}_0(a,b) = \sqrt{ab}, \tag{1.8}$$

$$\mathrm{M}_\infty(a,b) = \frac{2\,\max\{a,b\}}{2+\ln\frac{\max\{a,b\}}{\min\{a,b\}}} \le \frac{a+b}{2}. \tag{1.9}$$

We conjecture that *for fixed a and b, $\mathrm{M}_p(a,b)$ is an increasing function of p.* At this time, we can prove that

$$\mathrm{M}_2(a,b) \le \mathrm{M}_\infty(a,b). \tag{1.10}$$

Inequalities already known then lead to the chain

$$\mathrm{M}_0(a,b) \le \mathrm{M}_1(a,b) \le \mathrm{M}_2(a,b) \le \mathrm{M}_\infty(a,b). \tag{1.11}$$

The first of these inequalities is well-known and easy to prove; the second has been given different proofs in [9, 10, 19].

The rest of this paper is organized as follows. Section 2 investigates various properties of M_p in detail, including different expressions for M_p and the inequality (1.10). Section 3 gives a relation between M_p and the *power difference mean* K_p. In section 4, we evaluate the norm of the integral operator induced on the space $L_2(\mathbb{R}_+)$ by the kernel $1/\mathrm{M}_p(x, y)$. This gives an extension of the famous Hilbert inequality. In section 5, we discuss positive definiteness of certain matrices as implications of some relations between M_p and K_p for which another conjecture is also proposed.

2. Mean M_p

Expressions (1.8) for M_0 and (1.9) for M_∞ will be proved after a detailed investigation on M_p for $0 < p < \infty$ is completed. Then we will prove the inequality (1.10).

2.1. $0 < p < \infty$.

Theorem 2.1. $\min\{a, b\} \le \sqrt{ab} \le \mathrm{M}_p(a, b) \le \left(\frac{a^p+b^p}{2}\right)^{1/p} \le \max\{a, b\}$.

Proof. The first inequality is easy to see, and the last one is easy to see too by replacing both a and b in $\left(\frac{a^p+b^p}{2}\right)^{1/p}$ with $\max\{a, b\}$. We now prove the second and the third inequalities. Since $(x^p + a^p)(x^p + b^p) = (x^p)^2 + (a^p + b^p)x^p + a^p b^p$, we have

$$[x^p + (\sqrt{ab})^p]^2 \le (x^p + a^p)(x^p + b^p) \le \left[x^p + \frac{a^p + b^p}{2}\right]^2.$$

Therefore by (1.5),

$$\frac{1}{\mathrm{M}_p\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}, \left[\frac{a^p+b^p}{2}\right]^{1/p}\right)} \le \frac{1}{\mathrm{M}_p(a, b)} \le \frac{1}{\mathrm{M}_p(\sqrt{ab}, \sqrt{ab})}$$

which, together with the condition $\mathrm{M}_p(z, z) = z$ for any $z > 0$, leads to the desired inequalities. □

In the last integral in (1.6), substitute $t = (y^p + 1)^{-1}$ to get

$$y^p = \frac{1}{t} - 1 = \frac{1-t}{t}, \tag{2.1}$$

$$py^{p-1}\, dy = -\frac{1}{t^2}\, dt, \tag{2.2}$$

$$\begin{aligned} dy &= -\frac{1}{p}\left(\frac{t}{1-t}\right)^{\frac{p-1}{p}} \frac{1}{t^2}\, dt \\ &= -\frac{1}{p} t^{-1-1/p}(1-t)^{1/p-1}\, dt, \end{aligned} \tag{2.3}$$

$$\begin{aligned} \frac{1}{c_p} &= \frac{1}{p}\int_0^1 t^{1/p-1}(1-t)^{1/p-1}\, dt & (2.4) \\ &= \frac{B(\frac{1}{p}, \frac{1}{p})}{p}, & (2.5) \end{aligned}$$

where $B(\cdot,\cdot)$ is the Beta-function [1]. In the integral in (1.5), substitute $x^p + a^p = a^p t^{-1}$ to get

$$\begin{aligned}
x^p &= a^p\left(\frac{1}{t}-1\right) = a^p\,\frac{1-t}{t},\\
px^{p-1}\,dx &= -a^p\frac{1}{t^2}\,dt,\\
dx &= -\frac{a}{p}\left(\frac{t}{1-t}\right)^{\frac{p-1}{p}}\frac{1}{t^2}\,dt\\
&= -\frac{a}{p}t^{-1-1/p}(1-t)^{1/p-1}\,dt,\\
\frac{1}{\mathrm{M}_p(a,b)} &= c_p\frac{a}{p}\int_0^1\frac{t^{-1/p-1}(1-t)^{1/p-1}}{[(a^pt^{-1})(a^p\frac{1-t}{t}+b^p)]^{1/p}}\,dt\\
&= \frac{c_p}{p}\int_0^1\frac{t^{1/p-1}(1-t)^{1/p-1}}{[a^p(1-t)+b^pt]^{1/p}}\,dt. \qquad (2.6)
\end{aligned}$$

Combine (2.4) and (2.6) to get

$$\frac{1}{\mathrm{M}_p(a,b)} = \frac{\displaystyle\int_0^1\frac{t^{1/p-1}(1-t)^{1/p-1}}{[a^p(1-t)+b^pt]^{1/p}}\,dt}{\displaystyle\int_0^1 t^{1/p-1}(1-t)^{1/p-1}\,dt}. \qquad (2.7)$$

Theorem 2.2. *Given $a,b>0$ and $0<p<\infty$, we have*

$$\frac{1}{\mathrm{M}_p(a,b)} = (\max\{a,b\})^{-1}\sum_{k=0}^{\infty}\prod_{i=0}^{k-1}\frac{\left(\frac{1}{p}+i\right)^2}{\frac{2}{p}+i}\,\frac{1}{k!}\left[1-\left(\frac{\min\{a,b\}}{\max\{a,b\}}\right)^p\right]^k, \qquad (2.8)$$

where, by convention, $\prod_{i=0}^{-1}(\cdots)\equiv 1$ and $0!=1$.

Proof. Both sides of (2.8) are equal to a if $a=b$. Assume without loss of generality that $a>b>0$. Let $\alpha = 1-(b/a)^p$ and then $0<\alpha<1$. We have

$$\begin{aligned}
a^p(1-t)+b^pt &= a^p[1-t+(b/a)^pt] = a^p(1-\alpha t),\\
[a^p(1-t)+b^pt]^{-1/p} &= a^{-1}(1-\alpha t)^{-1/p}\\
&= a^{-1}\sum_{k=0}^{\infty}\frac{\prod_{i=0}^{k-1}\left(\frac{1}{p}+i\right)}{k!}\alpha^k t^k. \qquad (2.9)
\end{aligned}$$

The series in (2.9) converges for $\alpha<1$ which justifies the term-by-term integration below. Equation (2.9), together with (2.7), yields

$$\frac{1}{\mathrm{M}_p(a,b)} = \frac{a^{-1}\displaystyle\int_0^1\sum_{k=0}^{\infty}\frac{\prod_{i=0}^{k-1}\left(\frac{1}{p}+i\right)}{k!}\alpha^k t^{k+1/p-1}(1-t)^{1/p-1}\,dt}{\displaystyle\int_0^1 t^{1/p-1}(1-t)^{1/p-1}}$$

$$= \frac{a^{-1}\sum_{k=0}^{\infty}\int_0^1 \frac{\prod_{i=0}^{k-1}\left(\frac{1}{p}+i\right)}{k!}\alpha^k t^{k+1/p-1}(1-t)^{1/p-1}\,dt}{\int_0^1 t^{1/p-1}(1-t)^{1/p-1}}$$

$$= a^{-1}\sum_{k=0}^{\infty}\frac{\prod_{i=0}^{k-1}\left(\frac{1}{p}+i\right)}{k!}\alpha^k\cdot\frac{B(k+\frac{1}{p},\frac{1}{p})}{B(\frac{1}{p},\frac{1}{p})}. \tag{2.10}$$

Using the well-known properties of the Beta and Gamma functions [1]

$$B(s,t)=\frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)},\quad \Gamma(z)=(z-1)\Gamma(z-1),$$

we have

$$\begin{aligned}\frac{B(k+\frac{1}{p},\frac{1}{p})}{B(\frac{1}{p},\frac{1}{p})} &= \frac{\Gamma(k+\frac{1}{p})\Gamma(\frac{1}{p})}{\Gamma(k+\frac{2}{p})}\frac{\Gamma(\frac{2}{p})}{\Gamma(\frac{1}{p})\Gamma(\frac{1}{p})}\\ &= \frac{\Gamma(k+\frac{1}{p})}{\Gamma(\frac{1}{p})}\frac{\Gamma(\frac{2}{p})}{\Gamma(k+\frac{2}{p})}\\ &= \prod_{i=0}^{k-1}\frac{\frac{1}{p}+i}{\frac{2}{p}+i}.\end{aligned}$$

Substituting this into (2.10) gives (2.8). □

Theorem 2.3. *Given $a,b>0$ and $0<p<\infty$, we have*

$$\frac{1}{\mathrm{M}_p(a,b)}=\left(\frac{a^p+b^p}{2}\right)^{-1/p}\sum_{k=0}^{\infty}\frac{1}{k!}\left[\prod_{i=0}^{k-1}\left(\frac{1}{p}+i\right)\right]\left[\prod_{i=0}^{2k-1}\frac{\frac{1}{p}+i}{\frac{2}{p}+i}\right]\left(\frac{a^p-b^p}{a^p+b^p}\right)^{2k}. \tag{2.11}$$

Proof. We have

$$\begin{aligned}(x^p+a^p)(x^p+b^p) &= \left(x^p+\frac{a^p+b^p}{2}\right)^2-\left(\frac{a^p-b^p}{2}\right)^2\\ &= \left(x^p+\frac{a^p+b^p}{2}\right)^2\left(1-r^2\right),\end{aligned}$$

where $r=\frac{a^p-b^p}{2}/\left(x^p+\frac{a^p+b^p}{2}\right)$. Therefore $|r|<1$ and

$$\begin{aligned}[(x^p+a^p)(x^p+b^p)]^{-1/p} &= \left(x^p+\frac{a^p+b^p}{2}\right)^{-2/p}\left(1-r^2\right)^{-1/p},\\ &= \left(x^p+\frac{a^p+b^p}{2}\right)^{-2/p}\sum_{k=0}^{\infty}\left[\prod_{i=0}^{k-1}\left(\frac{1}{p}+i\right)\right]\frac{r^{2k}}{k!},\end{aligned}$$

$$\frac{1}{\mathrm{M}_p(a,b)}=c_p\int_0^{\infty}\frac{1}{\left(x^p+\frac{a^p+b^p}{2}\right)^{2/p}}\sum_{k=0}^{\infty}\left[\prod_{i=0}^{k-1}\left(\frac{1}{p}+i\right)\right]\frac{r^{2k}}{k!}\,dx$$

$$
\begin{aligned}
&= c_p \int_0^\infty \frac{dx}{\left(x^p + \frac{a^p+b^p}{2}\right)^{2/p}} \\
&\quad + c_p \sum_{k=1}^\infty \frac{1}{k!}\left[\prod_{i=0}^{k-1}\left(\frac{1}{p}+i\right)\right]\int_0^\infty \frac{r^{2k}}{\left(x^p + \frac{a^p+b^p}{2}\right)^{2/p}}\,dx \\
&= \left(\frac{a^p+b^p}{2}\right)^{-1/p} \\
&\quad + \sum_{k=1}^\infty \frac{1}{k!}\left[\prod_{i=0}^{k-1}\left(\frac{1}{p}+i\right)\right] c_p \int_0^\infty \frac{\left(\frac{a^p-b^p}{2}\right)^{2k}}{\left(x^p + \frac{a^p+b^p}{2}\right)^{2k+2/p}}\,dx.
\end{aligned}
\tag{2.12}
$$

Substitute $x = \left(\frac{a^p+b^p}{2}\right)^{1/p} y$ and $y^p + 1 = t^{-1}$ as in (2.1) – (2.3) to get

$$
\begin{aligned}
\int_0^\infty \frac{dx}{\left(x^p + \frac{a^p+b^p}{2}\right)^{2k+2/p}} &= \left(\frac{a^p+b^p}{2}\right)^{-2k-1/p} \int_0^\infty \frac{dy}{(y^p+1)^{2k+2/p}} \\
&= \left(\frac{a^p+b^p}{2}\right)^{-2k-1/p} \int_0^1 \frac{1}{p} t^{2k+1/p-1}(1-t)^{1/p-1}\,dt \\
&= \left(\frac{a^p+b^p}{2}\right)^{-2k-1/p} \frac{B(2k+\frac{1}{p},\frac{1}{p})}{p}.
\end{aligned}
$$

This together with (2.5) leads to

$$
\begin{aligned}
c_p \int_0^\infty \frac{dx}{\left(x^p + \frac{a^p+b^p}{2}\right)^{2k+2/p}} &= \left(\frac{a^p+b^p}{2}\right)^{-2k-1/p} \frac{B(2k+\frac{1}{p},\frac{1}{p})}{B(\frac{1}{p},\frac{1}{p})} \\
&= \left(\frac{a^p+b^p}{2}\right)^{-2k-1/p} \prod_{i=0}^{2k-1} \frac{\frac{1}{p}+i}{\frac{2}{p}+i}.
\end{aligned}
\tag{2.13}
$$

Now (2.11) is a consequence of (2.12) and (2.13). □

2.2. $p = 0$.

Theorem 2.4. *Given $a, b > 0$, we have*

$$\mathrm{M}_0(a,b) = \sqrt{ab}. \tag{2.14}$$

Proof. It can be verified that $\lim_{p\to 0^+}\left(\frac{a^p+b^p}{2}\right)^{1/p} = \sqrt{ab}$. The equality $\mathrm{M}_0(a,b) = \sqrt{ab}$ is then a consequence of Theorem 2.1. □

2.3. $p = \infty$.

Theorem 2.5. *Given $a, b > 0$, we have*

$$\mathrm{M}_\infty(a,b) = \frac{2\max\{a,b\}}{2 + \ln\frac{\max\{a,b\}}{\min\{a,b\}}}, \tag{2.15}$$

and

$$\mathrm{M}_2(a,b) \le \mathrm{M}_\infty(a,b) \le (a+b)/2. \tag{2.16}$$

Proof. Both (2.15) and (2.16) are obvious if $a = b$. Assume that $a > b > 0$. Then

$$\lim_{p\to\infty}\int_0^\infty \frac{dy}{(y^p+1)^{2/p}} = \int_0^1 dy + \int_1^\infty \frac{dy}{y^2} = 2,$$

$$\begin{aligned}\lim_{p\to\infty}\int_0^\infty \frac{dx}{[(x^p+a^p)(x^p+b^p)]^{1/p}} &= \int_0^b \frac{dx}{ab} + \int_b^a \frac{dx}{xa} + \int_a^\infty \frac{dx}{x^2}\\ &= \frac{1}{a} + \frac{\ln a - \ln b}{a} + \frac{1}{a}\\ &= \frac{2 + (\ln a - \ln b)}{a}.\end{aligned}$$

Therefore $c_\infty = 1/2$, and (2.15) holds by definition. The change of order of taking limits and the integrals above is justified by Lebesgue's Dominated Convergence Theorem [21, p.76] because

$$\frac{1}{(y^p+1)^{2/p}} \le \begin{cases} 1 & \text{for } 0 \le y \le 1,\\ y^{-2} & \text{for } 1 < y,\end{cases}$$

$$\frac{1}{[(x^p+a^p)(x^p+b^p)]^{1/p}} \le \begin{cases} (ab)^{-1} & \text{for } 0 \le x \le a,\\ x^{-2} & \text{for } a < x.\end{cases}$$

The second inequality in (2.16) is relatively easy to show. It goes as follows. Since

$$\mathrm{M}_1(a,b) = \frac{a-b}{\ln a - \ln b} \le \frac{a+b}{2},$$

we have successively

$$\begin{aligned}2(a-b) &\le (a+b)(\ln a - \ln b),\\ 2a &\le 2b + (a+b)(\ln a - \ln b),\\ 4a &= 2(a+b) + (a+b)(\ln a - \ln b),\\ \frac{2a}{2+\ln a - \ln b} &\le \frac{a+b}{2},\end{aligned}$$

as expected.

Let us focus on the first inequality in (2.16) now. As in the proof by John Todd for *Problem 19-17* in [10], let $r = (a-b)/(a+b)$. Then $0 < r < 1$ and $a = \frac{a+b}{2}(1+r)$ and $b = \frac{a+b}{2}(1-r)$. It suffices to show that $\mathrm{M}_2(1+r,1-r) \le \mathrm{M}_\infty(1+r,1-r)$. It was shown by Gauss [12] (see also [8, p.7]) that for $|r| < 1$

$$\frac{1}{\mathrm{M}_2(1+r,1-r)} = 1 + \frac{1}{4}r^2 + \frac{9}{64}r^4 + \cdots + \left(\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}\right)^2 r^{2n} + \cdots.$$

On the other hand, by (1.9),

$$\begin{aligned}\frac{1}{\mathrm{M}_\infty(1+r,1-r)} &= \frac{2+\ln\frac{1+r}{1-r}}{2(1+r)}\\ &= \frac{2 + 2r\left(1 + \frac{1}{3}r^2 + \frac{1}{5}r^4 + \cdots + \frac{1}{2n+1}r^{2n} + \cdots\right)}{2(1+r)}\end{aligned}$$

$$= \frac{1 + r\left(1 + \frac{1}{3}r^2 + \frac{1}{5}r^4 + \cdots + \frac{1}{2n+1}r^{2n} + \cdots\right)}{1+r}.$$

So for $\mathrm{M}_2(1+r, 1-r) \le \mathrm{M}_\infty(1+r, 1-r)$ to hold, it suffices to have

$$(1+r)\left[1 + \frac{1}{4}r^2 + \frac{9}{64}r^4 + \cdots + \left(\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}\right)^2 r^{2n} + \cdots\right] \ge 1 + r\left(1 + \frac{1}{3}r^2 + \frac{1}{5}r^4 + \cdots + \frac{1}{2n+1}r^{2n} + \cdots\right), \quad (2.17)$$

or, equivalently,

$$(1+r)\left[\frac{1}{4}r^2 + \frac{9}{64}r^4 + \cdots + \left(\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}\right)^2 r^{2n} + \cdots\right] \ge r\left(\frac{1}{3}r^2 + \frac{1}{5}r^4 + \cdots + \frac{1}{2n+1}r^{2n} + \cdots\right). \quad (2.18)$$

Since $1 + r > 2r$, (2.18) holds if

$$2r\left[\frac{1}{4}r^2 + \frac{9}{64}r^4 + \cdots + \left(\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}\right)^2 r^{2n} + \cdots\right] \ge r\left(\frac{1}{3}r^2 + \frac{1}{5}r^4 + \cdots + \frac{1}{2n+1}r^{2n} + \cdots\right) \quad (2.19)$$

which is guaranteed if the corresponding coefficients of r^{2n+1} from both sides satisfy

$$2\left(\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}\right)^2 \ge \frac{1}{2n+1} \quad (2.20)$$

for $n \ge 1$. This is what we shall prove now. To this end, we shall use the following estimate for factorial $n!$ [18, 20]

$$\sqrt{2\pi}n^{n+1/2}e^{-n+1/(12n+1)} < n! < \sqrt{2\pi}n^{n+1/2}e^{-n+1/(12n)}. \quad (2.21)$$

We have

$$\begin{aligned}
\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)} &= \frac{(2n)!}{2^{2n}(n!)^2} \\
&> \frac{\sqrt{2\pi}(2n)^{2n+1/2}e^{-2n+1/(24n+1)}}{2^{2n}[\sqrt{2\pi}n^{n+1/2}e^{-n+1/(12n)}]^2} \\
&= \frac{e^{1/(24n+1)}}{\sqrt{\pi n}\,e^{1/(6n)}}, \\
2(2n+1)\left(\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}\right)^2 &> 4n\left(\frac{e^{1/(24n+1)}}{\sqrt{\pi n}\,e^{1/(6n)}}\right)^2 \\
&= \frac{4}{\pi}\,e^{2/(24n+1)-1/(3n)} \\
&= \frac{4}{\pi}\,e^{-(18n+1)/[3n(24n+1)]}
\end{aligned}$$

$$\geq 1.12 \quad \text{for } n \geq 2.$$

This proves that (2.19) holds for $n \geq 2$. It can be verified that (2.19) holds for $n = 1$ also. The proof is completed. □

Remark 2.6. One can use (2.21) to also show that

$$\left(\frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)}\right)^2 < \frac{1}{\pi n} < \frac{1}{2n+1}.$$

This is used by John Todd [10] to show $\mathrm{M}_1(1+r, 1-r) \leq \mathrm{M}_2(1+r, 1-r)$.

3. Relation to the Power Difference Mean

The *power difference mean* $\mathrm{K}_p(a,b)$ is defined for any p and a, $b > 0$ as follows [14, 6].

$$\mathrm{K}_p(a,b) := \frac{p-1}{p} \frac{a^p - b^p}{a^{p-1} - b^{p-1}}, \tag{3.1}$$

where it is understood that

$$\mathrm{K}_p(a,a) := a, \quad \mathrm{K}_1(a,b) := \lim_{p \to 1} \mathrm{K}_p(a,b) = \mathrm{L}(a,b). \tag{3.2}$$

Alternatively, $\mathrm{K}_p(a,b)$ admits the following integral expression:

$$\frac{1}{\mathrm{K}_p(a,b)} = \int_0^1 \frac{dt}{[(1-t)a^p + tb^p]^{1/p}}. \tag{3.3}$$

By (1.1), (1.3), and (3.2), we have $\mathrm{M}_1(a,b) = \mathrm{L}(a,b) = \mathrm{K}_1(a,b)$. It makes us wonder what kind of relations are between $\mathrm{M}_p(a,b)$ and $\mathrm{K}_p(a,b)$ for $p \neq 1$. Theorem 3.2 below provides an answer. But first we establish an expansion formula for $\mathrm{K}_p(a,b)$.

Lemma 3.1. *Given $a, b > 0$, we have*

$$\frac{1}{\mathrm{K}_p(a,b)} = \left(\frac{a^p + b^p}{2}\right)^{-1/p} \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \left[\prod_{i=0}^{2k-1} \left(\frac{1}{p} + i\right)\right] \left(\frac{a^p - b^p}{a^p + b^p}\right)^{2k}. \tag{3.4}$$

Proof. K_p as defined by (3.1) has a removable singularity at $p = 1$. In this case, equation (3.4) can be verified either by using $\mathrm{K}_1(a,b) = \mathrm{L}(a,b)$ or by taking the limit as p goes to 1. In what follows, we shall assume $p \neq 1$. It suffices to show (3.4) for $a = 1$ and $0 < b \neq 1$. It follows from (3.1) that

$$\begin{aligned}
\frac{1}{\mathrm{K}_p(a,b)} \left(\frac{a^p + b^p}{2}\right)^{1/p} &= \frac{p}{p-1} \frac{a^{p-1} - b^{p-1}}{a^p - b^p} \left(\frac{a^p + b^p}{2}\right)^{1/p} \\
&= \frac{p}{p-1} \frac{1 - b^{p-1}}{1 - b^p} \left(\frac{1 + b^p}{2}\right)^{1/p}.
\end{aligned} \tag{3.5}$$

Let $r = (1 - b^p)/(1 + b^p)$. Then $|r| < 1$, and

$$b^p = \frac{1-r}{1+r}, \ \frac{1+b^p}{2} = \frac{1}{1+r}, \ 1 - b^p = \frac{2r}{1+r}, \ b^{p-1} = (b^p)^{(p-1)/p} = \left(\frac{1-r}{1+r}\right)^{1-1/p}.$$

Therefore by (3.5)

$$\frac{1}{\mathrm{K}_p(a,b)}\left(\frac{a^p+b^p}{2}\right)^{1/p} = \frac{p}{p-1}\frac{1-\left(\frac{1-r}{1+r}\right)^{1-1/p}}{\frac{2r}{1+r}}(1+r)^{-1/p}$$
$$= \frac{p}{p-1}\frac{(1+r)^{1-1/p}-(1-r)^{1-1/p}}{2r}. \qquad (3.6)$$

Use the binomial series expansion to get

$$(1+r)^{1-1/p} = \sum_{k=0}^{\infty}\left[\prod_{i=0}^{k-1}\left(1-\frac{1}{p}-i\right)\right]\frac{r^k}{k!}$$
$$= 1+\sum_{k=1}^{\infty}(-1)^{k-1}\left(1-\frac{1}{p}\right)\left[\prod_{i=0}^{k-2}\left(\frac{1}{p}+i\right)\right]\frac{r^k}{k!},$$
$$(1-r)^{1-1/p} = 1+\sum_{k=1}^{\infty}(-1)^{k-1}\left(1-\frac{1}{p}\right)\left[\prod_{i=0}^{k-2}\left(\frac{1}{p}+i\right)\right]\frac{(-r)^k}{k!}$$
$$= 1-\sum_{k=1}^{\infty}\left(1-\frac{1}{p}\right)\left[\prod_{i=0}^{k-2}\left(\frac{1}{p}+i\right)\right]\frac{r^k}{k!},$$
$$(1+r)^{1-1/p}-(1-r)^{1-1/p} = 2\left(1-\frac{1}{p}\right)\sum_{\ell=0}^{\infty}\left[\prod_{i=0}^{2\ell-1}\left(\frac{1}{p}+i\right)\right]\frac{r^{2\ell+1}}{(2\ell+1)!},$$

and from (3.6)

$$\frac{1}{\mathrm{K}_p(a,b)}\left(\frac{a^p+b^p}{2}\right)^{1/p} = \sum_{\ell=0}^{\infty}\left[\prod_{i=0}^{2\ell-1}\left(\frac{1}{p}+i\right)\right]\frac{r^{2\ell}}{(2\ell+1)!},$$

as was to be shown. □

Theorem 3.2. *Given $a, b > 0$ and $a \neq b$, we have*

(1) $\mathrm{M}_p(a,b) > \mathrm{K}_p(a,b)$ *for* $0 \le p < 1$,
(2) $\mathrm{M}_1(a,b) = \mathrm{K}_1(a,b)$,
(3) $\mathrm{M}_p(a,b) < \mathrm{K}_p(a,b)$ *for* $p > 1$.

Proof. We compare the right hand side of (2.11) and that of (3.4). For the purpose here, we may ignore the factor $\left(\frac{a^p+b^p}{2}\right)^{-1/p}$ in both expressions and compare the two series. Let

$$\alpha_k = \frac{1}{k!}\left[\prod_{i=0}^{k-1}\left(\frac{1}{p}+i\right)\right]\left[\prod_{i=0}^{2k-1}\frac{\frac{1}{p}+i}{\frac{2}{p}+i}\right], \quad \beta_k = \frac{1}{(2k+1)!}\left[\prod_{i=0}^{2k-1}\left(\frac{1}{p}+i\right)\right]$$

which are the coefficients of $\left(\frac{a^p-b^p}{a^p+b^p}\right)^{2k}$ in the two series, respectively. Since $\alpha_0 = 1 = \beta_0$, it suffices to show that for $k \ge 1$,

$$\alpha_k < \beta_k \text{ for } 0 < p < 1;\ \alpha_k = \beta_k \text{ for } p = 1;\ \text{and } \alpha_k > \beta_k \text{ for } p > 1.$$

Comparing α_k and β_k, after canceling the common factor $\prod_{i=0}^{2k-1}\left(\frac{1}{p}+i\right)$ in α_k and β_k, is equivalent to comparing the two quantities

$$\frac{\prod_{i=0}^{k-1}\left(\frac{1}{p}+i\right)}{\prod_{i=0}^{2k-1}\left(\frac{2}{p}+i\right)} = \frac{p^k\prod_{i=0}^{k-1}(1+pi)}{\prod_{i=0}^{2k-1}(2+pi)} = \frac{p^k}{2^k\prod_{i=1}^{k}[2+p(2i-1)]}$$

and $k!/(2k+1)!$. We have

$$\begin{aligned}&\frac{p^k}{2^k\prod_{i=1}^{k}[2+p(2i-1)]} - \frac{k!}{(2k+1)!}\\ &\qquad\qquad = \frac{p^k\,(2k+1)! - k!\,2^k(2+p)(2+3p)\cdots[2+(2k-1)p]}{2^k\,(2k+1)!\,\prod_{i=1}^{k}[2+p(2i-1)]}\end{aligned}$$

whose numerator denoted by $g(p)$ is a polynomial of degree k in p with the leading coefficient (of p^k)

$$\begin{aligned}(2k+1)! - k!\,2^k\,(2k-1)!! &= (2k+1)! - k!\,2^k\,\frac{(2k+1)!}{(2k)!!\,(2k+1)}\\ &= (2k+1)!\left(1-\frac{1}{2k+1}\right) > 0,\end{aligned}$$

and the rest of the coefficients (of p^i for $i<k$) are all negative, and

$$g(1) = (2k+1)! - k!\,2^k\,(2k+1)!! = 0.$$

Therefore $g(p) < g(1)p^k = 0$ for $0<p<1$, and $g(p) > g(1)p^k = 0$ for $p>1$. This completes the proof. □

4. Integral Operators Induced by M_p

Let $\phi^{[0]}(x) \geq 0$ be any function on $\mathbb{R}_+ = \{x : x > 0\}$. This introduces a function on $\mathbb{R}_+\times\mathbb{R}_+$:

$$\phi^{[1]}(x,y) = \int_0^\infty \phi^{[0]}(tx)\phi^{[0]}(ty)\,dt, \tag{4.1}$$

and, in turn, another function

$$\phi^{[2]}(x,y) = \int_0^\infty \phi^{[1]}(x,t)\phi^{[1]}(y,t)\,dt. \tag{4.2}$$

For $0<p<\infty$, let

$$\phi_p^{[0]}(x) = e^{-x^p}. \tag{4.3}$$

Then for $x,\,y>0$

$$\begin{aligned}\phi_p^{[1]}(x,y) &= \int_0^\infty e^{-t^p(x^p+y^p)}\,dt \qquad\qquad (\text{substitute } s = t^p(x^p+y^p))\\ &= \int_0^\infty e^{-s}\,\frac{1}{p}\,\frac{s^{1/p-1}}{(x^p+y^p)^{1/p}}\,ds\\ &= \frac{\Gamma(\frac{1}{p})}{p}\,\frac{1}{(x^p+y^p)^{1/p}},\end{aligned} \tag{4.4}$$

$$\phi_p^{[2]}(x,y) = \frac{\Gamma(\frac{1}{p})^2}{p^2}\int_0^\infty \frac{dt}{[(x^p+t^p)(y^p+t^p)]^{1/p}} \qquad \text{(use (2.5))}$$
$$= \frac{\Gamma(\frac{1}{p})^2}{p^2}\,\frac{B(\frac{1}{p},\frac{1}{p})}{p}\,\frac{1}{\mathrm{M}_p(x,y)}. \tag{4.5}$$

Remark 4.1. Instead of (4.3), we could have started with

$$\phi_p^{[0]}(x) = \alpha_p\, e^{-x^p}, \quad \alpha_p = \frac{\Gamma(\frac{2}{p})^{1/4}\,p^{3/4}}{\Gamma(\frac{1}{p})}. \tag{4.3′}$$

Then we will get

$$\phi_p^{[2]}(x) = \frac{1}{\mathrm{M}_p(x,y)}. \tag{4.5′}$$

This provides another way of looking at the family of means $\mathrm{M}_p(x,y)$.

Note for $p=1$, (4.5) is

$$\phi_1^{[2]}(x,y) = \frac{1}{\mathrm{M}_1(x,y)} = \frac{1}{\mathrm{L}(x,y)}, \tag{4.6}$$

and for $p=2$, it is

$$\phi_2^{[2]}(x,y) = \frac{\pi^2}{8}\,\frac{1}{\mathrm{M}_2(x,y)} = \frac{\pi^2}{8}\,\frac{1}{\mathrm{AG}(x,y)}. \tag{4.7}$$

We obtain the values of the norms of the integral operators with kernel $1/\mathrm{M}_p(x,y)$. These results are extensions of the famous Hilbert inequality. We use a familiar technique from Hardy, Littlewood, and Pólya [13].

Theorem 4.2 ([13, Theorem 319, page 229])**.** *Let $\phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be homogeneous of order -1, i.e., $\phi(\lambda x, \lambda y) \equiv \lambda^{-1}\phi(x,y)$ for $\lambda > 0$, and that*

$$\int_0^\infty \frac{\phi(x,1)}{\sqrt{x}}\,dx = \int_0^\infty \frac{\phi(1,y)}{\sqrt{y}}\,dy =: \kappa \tag{4.8}$$

Then the induced operator on $L_2(\mathbb{R}_+)$

$$\Phi f(x) := \int_0^\infty \phi(x,y) f(y)\,dy$$

has norm $\|\Phi\|_{L_2} \le \kappa$. If $\phi(1,y)$ is uniformly bounded in $y \in \mathbb{R}_+$, then[1] $\|\Phi\|_{L_2} = \kappa$.

For the kernel (4.4), $\phi_p^{[1]}(1,y)$ is uniformly bounded in y and satisfies (4.8). Let $\Phi_p^{[1]}$ be the integral operator with $\phi_p^{[1]}(x,y)$ in (4.4) as its kernel. Apply Theorem 4.2 to get

$$\|\Phi_p^{[1]}\|_{L_2} = \kappa_p := \int_0^\infty \frac{\phi_p^{[1]}(1,x)}{\sqrt{x}}\,dx$$
$$= \frac{\Gamma(\frac{1}{p})}{p}\int_0^\infty \frac{dx}{(1+x^p)^{1/p}\,x^{1/2}}$$

[1]This is not explicitly asserted in [13], but can be inferred from the discussion there. See, e.g., [15, page 149].

$$= \frac{\Gamma(\frac{1}{p})}{p}\,\frac{1}{p}\,B(\tfrac{1}{2p},\tfrac{1}{2p})$$
$$= \frac{1}{p^2}\,[\Gamma(\tfrac{1}{2p})]^2. \tag{4.9}$$

Since the operator $\Phi_p^{[2]}$ induced by the kernel (4.5) is the square of $\Phi_p^{[1]}$ induced by (4.4) and also $\Phi_p^{[1]}$ is self-adjoint because $\phi_p^{[1]}(x,y) = \phi_p^{[1]}(y,x)$, we have $\|\Phi_p^{[2]}\|_{L_2} = \|\Phi_p^{[1]}\|_{L_2}^2$.

Theorem 4.3. *Let $0 < p < \infty$ and let*

$$\mathscr{M}_p f(x) := \int_0^\infty \frac{1}{\mathrm{M}_p(x,y)}\,f(y)\,dy.$$

Then $\mathscr{M}_p$ is a bounded linear operator on $L_2(\mathbb{R}_+)$ with

$$\|\mathscr{M}_p\|_{L_2} = \frac{\Gamma(\frac{2}{p})\,\Gamma(\frac{1}{2p})^4}{p\,\Gamma(\frac{1}{p})^2}. \tag{4.10}$$

Proof. Note by (4.5)

$$\Phi_p^{[2]} = \frac{\Gamma(\frac{1}{p})^2}{p^2}\,\frac{B(\frac{1}{p},\frac{1}{p})}{p}\,\mathscr{M}_p.$$

By the consideration above,

$$\|\mathscr{M}_p\|_{L_2} = \kappa_p^2\,\frac{p^3\,\Gamma(\frac{2}{p})}{\Gamma(\frac{1}{p})^4} = \frac{\Gamma(\frac{1}{2p})^4}{p^4}\,\frac{p^3\,\Gamma(\frac{2}{p})}{\Gamma(\frac{1}{p})^4} = \frac{\Gamma(\frac{2}{p})\,\Gamma(\frac{1}{2p})^4}{p\,\Gamma(\frac{1}{p})^2},$$

as expected. □

Special case $p = 1$ gives

$$\|\mathscr{M}_1\|_{L_2} = \pi^2. \tag{4.11}$$

This is noted on [13, page 257] (the last statement of §355). Special case $p = 2$ gives

$$\|\mathscr{M}_2\|_{L_2} = \frac{\Gamma(1/4)^2}{2\pi^2} = 8.753758\cdots \tag{4.12}$$

which happens to be $2\pi/\mathrm{AG}(\sqrt{2},1)^2$.

Remark 4.4. Recall the famous Hilbert inequality that says the norm of the operator induced on $L_2(\mathbb{R}_+)$ by the kernel $1/(x+y)$ is π. This is κ_1 in (4.9).

More generally, consider the space $L_r(\mathbb{R}_+)$, where $r > 1$. [13, Theorem 319, page 229] says that if ϕ and Φ are as in Theorem 4.2 and

$$\kappa(r) := \int_0^\infty \frac{\phi(1,x)}{x^{1/r}}\,dx = \int_0^\infty \frac{\phi(1,y)}{y^{1/r'}}\,dx < \infty, \tag{4.13}$$

then Φ is a bounded operator on $L_r(\mathbb{R}_+)$ with norm $\|\Phi\|_{L_r} \le \kappa(r)$.

In our case, for the kernel (4.4), (4.13) gives

$$\kappa_p(r) := \int_0^\infty \frac{dx}{(1+x^p)^{1/p}\,x^{1/r}} \quad \text{(substitute } t = (1+x^p)^{-1}\text{)}$$
$$= \frac{1}{p}\,B(\tfrac{1-1/r}{p},\tfrac{1/r}{p})$$

$$= \frac{1}{p} \frac{\Gamma(\frac{1-1/r}{p})\,\Gamma(\frac{1/r}{p})}{\Gamma(\frac{1}{p})}$$
$$= \frac{1}{p} \frac{\Gamma(\frac{1}{r'p})\,\Gamma(\frac{1}{rp})}{\Gamma(\frac{1}{p})}, \tag{4.14}$$

where $1/r + 1/r' = 1$. So we have

$$\|\mathscr{M}_p\|_{L_r} \le \kappa_p(r)^2 \, \frac{p^3\,\Gamma(\frac{2}{p})}{\Gamma(\frac{1}{p})^4}. \tag{4.15}$$

Special case $p = 1$

$$\kappa_1(r) = \Gamma(\tfrac{1}{r'})\,\Gamma(\tfrac{1}{r}) = \pi \csc(\pi/r)$$

is given in [13, pages 226 and 255].

5. Positive Definiteness of Certain Matrices

An interesting connection between binary means of positive real numbers and positive definite matrices has been developed in the last few years. See [5, Chapters 4 and 5], [7], [14], and references therein.

Let M and $\widetilde{M}$ be two binary means. We say that $M \ll \widetilde{M}$ if for every n and for every choice of positive real numbers $\lambda_1, \lambda_2, \dots, \lambda_n$, the $n \times n$ matrix

$$\left(\frac{M(\lambda_i, \lambda_j)}{\widetilde{M}(\lambda_i, \lambda_j)}\right)_{n\times n}$$

is positive semidefinite. For many interesting means, it has been found that the inequality $M \le \widetilde{M}$ implies the stronger relation $M \ll \widetilde{M}$.

We explore this for the two families K_p and M_p. First we observe that for every $x \ge 0$, the matrix

$$\left(\frac{1}{(x^p + \lambda_i^p)^{1/p}(x^p + \lambda_j^p)^{1/p}}\right)_{n\times n}$$

is positive semidefinite, since it is congruent to the *flat* matrix E (the matrix with all its entries equal to one). It follows from (1.5) that for $0 < p < \infty$, the $n \times n$ matrices with (i, j) entries

$$\frac{1}{\mathrm{M}_p(\lambda_i, \lambda_j)} = c_p \int_0^\infty \frac{dx}{(x^p + \lambda_i^p)^{1/p}(x^p + \lambda_j^p)^{1/p}}$$

are positive semidefinite. By a limiting argument, we see that the matrices $\left(\frac{1}{\mathrm{M}_0(\lambda_i,\lambda_j)}\right)_{n\times n} = \left(\frac{1}{\sqrt{\lambda_i\lambda_j}}\right)_{n\times n}$ and

$$\left(\frac{1}{\mathrm{M}_\infty(\lambda_i, \lambda_j)}\right)_{n\times n} = \left(\frac{2 + \ln\frac{\max\{\lambda_i,\lambda_j\}}{\min\{\lambda_i,\lambda_j\}}}{2\max\{\lambda_i, \lambda_j\}}\right)_{n\times n} \tag{5.1}$$

are also positive semidefinite.

In [14], Hiai and Kosaki have proved that for $p \le 1/2$, the matrices

$$\left(\mathrm{K}_p(\lambda_i, \lambda_j)\right)_{n\times n}$$

are positive semidefinite. Hence, *the matrix*

$$\left(\frac{\mathrm{K}_p(\lambda_i,\lambda_j)}{\mathrm{M}_p(\lambda_i,\lambda_j)}\right)_{n\times n}$$

is positive semidefinite for $p\le 1/2$, being the Schur product of two such matrices.

The mean $\mathrm{K}_\infty(a,b)$ is equal to $\max\{a,b\}$. Hence we have

$$\frac{\mathrm{M}_\infty(\lambda_i,\lambda_j)}{\mathrm{K}_\infty(\lambda_i,\lambda_j)}=\frac{2}{2+\ln\frac{\max\{\lambda_i,\lambda_j\}}{\min\{\lambda_i,\lambda_j\}}}=\frac{2}{2+|\ln\lambda_i-\ln\lambda_j|}=\frac{1}{1+|\ln\lambda_i^{1/2}-\ln\lambda_j^{1/2}|}.$$

The matrix with this as its (i,j) entry is positive semidefinite, in fact, infinitely divisible [5, p.153].

We have proved that $\mathrm{K}_p\ll\mathrm{M}_p$ *for* $0\le p\le 1/2$*, and that* $\mathrm{M}_\infty\ll\mathrm{K}_\infty$. *We conjecture that*

$$\mathrm{K}_p\ll\mathrm{M}_p\quad \textit{for } 1/2<p<1, \textit{ and}\quad \mathrm{M}_p\ll\mathrm{K}_p\quad \textit{for } 1<p<\infty. \tag{5.2}$$

Remark 5.1. The positive semidefiniteness of the matrices (5.1) can be expressed in another way: the matrix

$$\left(\frac{1+\frac{1}{2}|\ln\lambda_i-\ln\lambda_j|}{\max\{\lambda_i,\lambda_j\}}\right)_{n\times n} \tag{5.3}$$

is always positive semidefinite. It is interesting to note that the matrix

$$\left(\frac{1+|\ln\lambda_i-\ln\lambda_j|}{\max\{\lambda_i,\lambda_j\}}\right)_{n\times n}$$

is not necessarily positive semidefinite, as can be seen from the 2×2 example in which $\lambda_1=1$ and $\lambda_2=e^2$. In fact more can be said. Let r be any real nonnegative number. Then *the matrix*

$$W=\left(\frac{1+r|\ln\lambda_i-\ln\lambda_j|}{\max\{\lambda_i,\lambda_j\}}\right)_{n\times n}$$

is positive semidefinite for $0\le r\le 1/2$ *and not necessarily positive semidefinite for* $r>1/2$. This can be seen as follows. For $0\le r\le 1/2$, we have

$$w_{ij}=\frac{1+|\ln\lambda_i^r-\ln\lambda_j^r|}{\max\{\lambda_i,\lambda_j\}}=\frac{1+|\ln\lambda_i^r-\ln\lambda_j^r|}{\max\{\lambda_i^{2r},\lambda_j^{2r}\}}\cdot\frac{1}{[\max\{\lambda_i,\lambda_j\}]^{1-2r}}=:u_{ij}v_{ij}.$$

The matrix $U=\big(u_{ij}\big)_{n\times n}$ is positive semidefinite (by the case $r=1/2$ already proved), and the matrix $V=\big(v_{ij}\big)_{n\times n}$ is positive semidefinite since the matrix $\big(1/\max\{\lambda_i,\lambda_j\}\big)_{n\times n}$ is infinitely divisible [3, 5]. So $W=(w_{ij})_{n\times n}$ being the Schur product of U and V is positive semidefinite. Now consider the case $r>1/2$. Let $\widetilde{r}$ be any number such that $r>\widetilde{r}>1/2$, and α be the unique positive root of $x=2\widetilde{r}\ln(1+x)$ (such a root exists because at $x=0$, the derivative of x is 1 and the derivative of $2\widetilde{r}\ln(1+x)$ is $2\widetilde{r}>1$). With $\lambda_1=1$ and $\lambda_2=e^{\alpha/r}$, the 2×2 matrix W is

$$W=\begin{pmatrix}1 & (1+\alpha)e^{-\alpha/r}\\ (1+\alpha)e^{-\alpha/r} & e^{-\alpha/r}\end{pmatrix}$$

whose determinant

$$\det W = e^{-\alpha/r} - (1+\alpha)^2 e^{-2\alpha/r} = e^{-2\alpha/r}\left[e^{\alpha/r} - (1+\alpha)^2\right] < 0$$

since $\alpha/r < \alpha/\widetilde{r} = 2\ln(1+\alpha) = \ln(1+\alpha)^2$.

Remark 5.2. Examples of means for which $M \le \widetilde{M}$ but the stronger relation $M \ll \widetilde{M}$ is not true were given in [4], and in [14]. To that list, we add another. We have seen that $\mathrm{M}_\infty(a,b) \le \mathrm{A}(a,b) := (a+b)/2$, where A stands for the arithmetic mean. But the relation $\mathrm{M}_\infty \ll \mathrm{A}$ is not true. For example, with $\lambda_1 = 17/100$, $\lambda_2 = 18/100$, and $\lambda_3 = 72/100$, the 3×3 matrix with its (i,j) entry being $\mathrm{M}_\infty(\lambda_i,\lambda_j)/\mathrm{A}(\lambda_i,\lambda_j)$ has a negative eigenvalue -0.00011509756859 computed by MATLAB.

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