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Luigi Accardi
Università di Roma Tor Vergata, Roma I-00133, Italy, accardi@volterra.uniroma2.it

Irina Ya. Aref'eva
Steklov Mathematical Institute, Gubkin St.8, GSP-1, 17966, Moscow, Russia, arefeva@mi.ras.ru

Yungang Lu
Università di Bari, n.4, Via E. Orabona, 70125 Bari, Italy, yungang.lu@uniba.it

Igor' Vasil'evich Volovich
Steklov Mathematical Institute, Gubkin St.8, GSP-1, 117966, Moscow, Russia, volovich@mi.ras.ru

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TWO NON–$*$–ISOMORPHIC $*$–LIE ALGEBRA STRUCTURES ON $sl(2,\mathbb{R})$ AND THEIR PHYSICAL ORIGINS

L. ACCARDI*, I.YA. AREF'EVA, Y.G. LU, AND I.V.VOLOVICH

Abstract. We introduce two, Lie algebra isomorphic, real forms of $sl(2,\mathbb{C})$, i.e. two real $*$–Lie algebras, denoted respectively $sl_F(2,\mathbb{R})$ and $sl_B(2,\mathbb{R})$, such that their complexifications ($sl_F(2,\mathbb{C})$ and $sl_B(2,\mathbb{C})$) are both isomorphic to $sl(2,\mathbb{C})$ as Lie algebras. Then we prove that $sl_B(2,\mathbb{C})$ cannot contain a real $*$–Lie sub–algebra $*$–isomorphic to $sl_F(2,\mathbb{R})$ and the same is true exchanging the indexes F and B. The meaning of the indexes B and F is explained in the last section where we show how $sl_B(2,\mathbb{R})$ (resp. $sl_F(2,\mathbb{R})$) can be realized in terms of Bosons (resp. Fermion) operators. These realizations are known in the literature.

1. Introduction

The present paper was motivated by the results in [1] where the $*$–Lie algebra $sl_F(2,\mathbb{C})$ (see section 2 below) was introduced and the vacuum distributions of its Hermitian generators in the Fock representation was computed. In the same paper the same problem was solved for the $*$–Lie algebra $sl_B(2,\mathbb{C})$ using a technique different than that used in the previous paper [2] where the result was first obtained. The resulting classes of probability measures on $\mathbb{R}$ turns out to be quite different in the two cases: in the former case the class is reduced to Bernoulli distributions, in the latter it coincides with the class of non–standard Meixner measures: Gamma, negative binomial and proper Meixner. This fact was somewhat surprising because $sl_F(2,\mathbb{C})$ and $sl_B(2,\mathbb{C})$ **are both isomorphic to** $sl(2,\mathbb{C})$ **as Lie algebras** (see Proposition 2.3 below), so one would naively expect that they have the same unitary representations.

The problem to explain this difference naturally led to a deeper analysis of the structure of these $*$–Lie algebras and the difficulties met in extending the Lie algebra isomorphism to a $*$–Lie algebra isomorphism led to the conjecture that such a $*$–isomorphism cannot exist. In the present paper we prove that this is indeed the case.

By **involution** on a (real or complex) Lie algebra $\mathcal{G}$ we mean a map denoted $*$, linear in the real case and anti–linear in the complex case, satisfying

$$(x^*)^* = x \text{ and } [x,y]^* = [y^*,x^*] \ ; \qquad x,y \in \mathcal{G} \tag{1.1}$$

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* Corresponding author.

A $*$–**Lie algebra** (real or complex) is a pair $(\mathcal{G}, *)$, where $\mathcal{G}$ is a Lie algebra and $*$ an involution on it. A $*$–homomorphism between two $*$–Lie algebras is a Lie algebra homomorphism intertwining the corresponding involutions.

A warning on terminology: there is a large literature on involutions on Lie algebras and groups, starting from Elie Cartan who established a correspondence between involutions on complex Lie groups or algebras and real forms of complex Lie groups or algebras.

However in almost all this literature the term *involution* is understood in a weaker sense than (1.1) namely the second condition in (1.1) (on commutators) is usually not included in the definition. Definition (1.1) was used in [6].

Involutions on reductive algebraic groups over a field k have been classified by Helmink in the framework of symmetric k–varieties [9] and a full classification of involutions of $SL(n, k)$, $(n > 2)$ has been obtained in [10] for fields k more general then the real or complex numbers.

In the present note we work at Lie algebra, rather than Lie group, level and we do not study the problem in general. Our main result is the construction of two real $*$–Lie algebras, denoted $sl_B(2, \mathbb{R})$ and $sl_F(2, \mathbb{R})$, both isomorphic to $sl(2, \mathbb{R})$ in the Lie algebra sense, but such that **the complexification of each of them cannot contain a $*$–isomorphic copy of the other one** (Corollary 2.9).

The fact that, on $sl(2, \mathbb{C})$ one can define a natural 1–parameter family of involutions given by $J_z(A) = zA^*$, where $A \in sl(2, \mathbb{C})$ and $z \in \mathbb{C}$, $|z| = 1$, suggests the result proved in the present note is related to Example 8.15 in [9] (even if the non–isomorphism there is true in the real, but not in the complex case). However this possibility will not be discussed here.

The structure of the present paper is the following. In section 2, we introduce the $*$–Lie algebras $sl_F(2, \mathbb{R})$ and $sl_B(2, \mathbb{R})$ and we prove their Lie algebra isomorphism. In section 2.1 we prove that no $*$–Lie algebra isomorphism can exist between the complexifications of them. More precisely, Proposition 2.7 proves that the existence of such a $*$–Lie algebra isomorphism is equivalent to the existence of non–zero solutions of a system of 5 quadratic equations in 5 complex variables. Then Theorem 2.8 shows that this system admits only the zero solution.

This conclusion is supported by symbolic calculations with MAPLE that give the same result.

In section 3, we briefly recall the physical origins of the two $*$–Lie algebras.

Let $sl_B(2, \mathbb{R})$ be a **Bosonic realization of** $sl(2; \mathbb{R})$ (see section 3.1 below), that is known since a long time and has been used in multiple different contexts (two relatively recent examples are [2] and [4]).

Similarly, let $sl_F(2, \mathbb{R})$ be a **Fermionic realization of** $sl(2; \mathbb{R})$ (see section 3.2 below). Such a realization too has been considered in the literature, in fact it can be identified to a Lie–sub–algebra of the associative algebra **CA**, introduced in [8], with 15 generators (that are also closed under anti–commutators).

However the problem of the non–$*$–isomorphism of these two Lie–isomorphic algebras, in the strong sense mentioned above, has not been considered in the literature to our knowledge.

The non–∗–isomorphism theorem can be interpreted physically as the fact that the bosonized fields, obtained by even products of Fermi fields, keep in their mathematical structure some peculiarity that **remembers their Fermionic origins**. Such a phenomenon first emerged in the stochastic limit of quantum theory with the emergence of the **super–parity operator** (see section 11.10 of [3]). In this case the situation is different because bosonization is not algebraic, but asymptotic, however the phenomenon of **memory of Fermionic origins** of a bosonized structure is exactly the same. This suggests that, on the basis of their vacuum distributions in Fock representation, **one can distinguish, among Bosons**, those which are originally such and those obtained by coalescence of an even number of Fermions.

Our result deals with the important special case of $sl(2,\mathbb{R})$, but it is natural to conjecture that in higher dimensions the class of inequivalent ∗–Lie algebra structures will be larger and that **at least some of them, may have a physical interpretation**.

This result, combined with those in [1], also suggests that this multiplicity of inequivalent ∗–Lie algebra structures will be reflected in **a multiplicity of inequivalent unitary representations** and in the emergence of inequivalent classes of probability measures on $\mathbb{R}$ associated to special vectors in the corresponding representation spaces (typical example being the Fock representation, when it exists, but many different situations can arise).

2. $sl_F(2,\mathbb{R})$, $sl_B(2,\mathbb{R})$ and Their Lie Algebra Isomorphism

Let $sl(2,\mathbb{R})$ (respectively, $sl(2,\mathbb{C})$) be the Lie algebra of real (respectively, complex) 2×2 matrices with trace zero. A traditional realization of $sl(2,\mathbb{R})$ is obtained considering the linear span of the 2×2 matrices:

$$B_2^- := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \qquad B_2^+ := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ , \qquad M_2 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.1}$$

(see [11], Chapter X, where the + and − signs are exchanged with respect to our notations). Here the index 2 refers to the realization in terms of 2×2 matrices. One then proves that the elements $\{B_2^{\pm}, M_2\}$ satisfy the commutation relations:

$$[B_2^-, B_2^+] = M_2 \ ; \qquad [M_2, B_2^+] = -2B_2^+ \ ; \qquad [M_2, B_2^-] = 2B_2^- \tag{2.2}$$

and this suggests to define abstractly $sl(2,\mathbb{R})$ as the unique real Lie algebra with generators denoted $\{B^{\pm}, M\}$ and Lie brackets uniquely determined by (2.2) (with $\{B, B^+, M\}$ replacing $\{B_2, B_2^+, M_2\}$). This abstract definition is also widely used in the literature.

Under the involution ∗ given by matrix transposition, the generators (2.1) satisfy

$$(B_2^+)^* = B_2^- \ ; \qquad M_2^* = M_2 \tag{2.3}$$

Definition 2.1. The complex ∗–Lie algebra with generators $\{B_F, B_F^+, M_F\}$, anti–linear involution given by (2.3) and commutation relations by (2.2) (with $\{B_F, B_F^+, M_F\}$ replacing $\{B_2, B_2^+, M_2\}$), will be denoted $sl_F(2;\mathbb{C})$. We denote

$$sl_{F,c}(2;\mathbb{R}) := \{X \in sl_F(2;\mathbb{C}) \ : \ X^* = -X\}$$

$sl_{F,c}(2;\mathbb{R})$ is a real $*$–Lie algebra.

Definition 2.2. The complex $*$–Lie algebra with generators $\{B, B^+, M\}$, involution given by (2.3) (with $\{B, B^+, M\}$ replacing $\{B_2, B_2^+, M_2\}$) and commutation relations

$$[B, B^+] = M \ ; \qquad [M, B^+] = 2B^+ \ ; \qquad [M, B] = -2B$$

will be denoted $sl_B(2;\mathbb{C})$. We denote

$$sl_{B,c}(2;\mathbb{R}) := \{X \in sl_B(2;\mathbb{C}) \ : \ X^* = -X\}$$

$sl_{B,c}(2;\mathbb{R})$ is a real $*$–Lie algebra.

Proposition 2.3. *The unique linear extension of the map $u_F : sl_F(2,\mathbb{C}) \longmapsto sl_B(2,\mathbb{C})$ defined by*

$$u_F(B_F) := iB \ ; \qquad u_F(B_F^+) := iB^+ \ ; \qquad u_F(M_F) := -M$$

is a Lie algebra isomorphism from $sl_F(2,\mathbb{C})$ to $sl_B(2,\mathbb{C})$. Moreover the restriction of u_F on the real linear span of $\{B_F, B_F^+, M_F\}$, i.e. $sl_F(2;\mathbb{R})$, is a real Lie algebra isomorphism onto the real linear span of $\{iB, iB^+, -M\}$.

Proof. For the first statement, it is sufficient to prove that the commutation relations are preserved. This follows from:

$$\begin{aligned}
[u_F(B_F), u_F(B_F^+)] &= [iB, iB^+] = -[B, B^+] = -M \\
&= u_F(M_F) = u_F([B_F, B_F^+]) \\
[u_F(M_F), u_F(B_F^+)] &= [-M, iB^+] = -i[M, B^+] = -2iB^+ \\
&= u_F(-2B_F^+) = u_F([M_F, B_F^+]) \\
[u_F(M_F), u_F(B_F)] &= [-M, iB] = -i[M, B] = 2(iB) \\
&= u_F(2B_F) = u_F([M_F, B_F])
\end{aligned}$$

The second statement follows from the fact that multiplication by i is a real linear map. $\square$

Remark 2.4. In addition to the one given in Proposition 2.3, there are infinitely many Lie algebra isomorphisms between $sl_B(2,\mathbb{C})$ and $sl_F(2,\mathbb{C})$. In fact, for any $a \in \mathbb{C}$ and $b \in \mathbb{C} \setminus \{0\}$, the application $u : sl_B(2,\mathbb{C}) \longmapsto sl_F(2,\mathbb{C})$ given by

$$u(M) := M_F + 2abB_F^- \ ; \qquad u(B) := aM_F + a^2bB_F - \frac{1}{b}B_F^+ \ ; \qquad u(B^+) := bB_F$$

is a Lie algebra isomorphism. Notice that all these isomorphisms **are not** $*$–Lie algebra isomorphisms.

Remark 2.5. Notice that the Lie algebra isomorphism u_F in Proposition 2.3, **does not map** $sl_{F,c}(2,\mathbb{R})$ into $sl_{B,c}(2,\mathbb{R})$. In fact the typical skew–adjoint element of $sl_{F,c}(2,\mathbb{R})$ has the form $i(aB_F^+ + \overline{a}B_F + cM_F)$ and its image under u_F is $-(aB^+ + \overline{a}B + icM)$ which is not skew–adjoint.

2.1. Non existence of a ∗–isomorphism between $sl_B(2,\mathbb{C})$ and $sl_F(2,\mathbb{C})$. It is natural, in view of Proposition 2.3, to ask oneself if there exists a ∗–Lie algebra isomorphism between $sl_B(2,\mathbb{C})$ and $sl_F(2,\mathbb{C})$. In this section we prove that this is not the case. To this goal, one introduces the notations

$$B^{(1)} = B^- \ ; \qquad B^{(2)} = B^+ \ ; \qquad B^{(0)} = M$$

for the fixed blinear basis of $sl_B(2,\mathbb{C})$ and

$$B_F^{(1)} = B_F^- \ ; \qquad B_F^{(2)} = B_F^+ \ ; \qquad B_F^{(0)} = M_F$$

for the fixed linear basis of $sl_F(2,\mathbb{C})$. Any linear map

$$\lambda : sl_B(2,\mathbb{C}) \longmapsto sl_F(2,\mathbb{C})$$

is uniquely determined by its (complex) matrix $\lambda \equiv (\lambda_{ij})$ in these bases:

$$\lambda(B^{(i)}) = \sum_{j\in\{1,2,0\}} \lambda_{ij} B_F^{(j)} \quad ; \qquad \forall i \in \{1,2,0\} \equiv \{-,+,0\} \tag{2.4}$$

In the following we fix a linear map λ given by (2.4).

Proposition 2.6. *λ preserves the involution, i.e.*

$$\lambda(x)^* = \lambda(x^*), \qquad \forall\, x \in sl_B(2,\mathbb{C}) \tag{2.5}$$

if and only if

$$\lambda_{0,0} = \overline{\lambda}_{0,0}\ ,\ \lambda_{0,1} = \overline{\lambda}_{0,2}\ ,\quad \lambda_{1,0} = \overline{\lambda}_{2,0}\ ,\quad \lambda_{1,1} = \overline{\lambda}_{2,2}\ ,\quad \lambda_{1,2} = \overline{\lambda}_{2,1} \tag{2.6}$$

Proof. Taking $x = B^{(0)} = M$ in (2.5), one gets

$$\sum_j \lambda_{0,j} B_F^{(j)} = \lambda(M) = \lambda(M^*) = \lambda(M)^* = \sum_j \overline{\lambda}_{0,j} B_F^{(j)*} \tag{2.7}$$

Linear independence of the generators and comparison of their coefficients imply that (2.7) is equivalent to the first two equalities in (2.6). Taking $x = B^{(1)}$ in (2.5), one deduces that

$$\sum_{j\in\{1,2,0\}} \bar{\lambda}_{1j} B_F^{(j)*} = \lambda(B^{(1)})^* = \lambda(B^{(1)*}) = \lambda(B^{(2)}) = \sum_{j\in\{1,2,0\}} \lambda_{2j} B_F^{(j)}$$

and the equivalence of this identity to the last three equalities in (2.6) follows again by linear independence of the generators and comparison of the coefficients. □

(2.6) implies that the $\lambda_{k,j}$'s are uniquely determined by $\{\lambda_{0,0}, \lambda_{0,1}, \lambda_{1,0}, \lambda_{1,1}, \lambda_{2,1}\}$ and moreover $\lambda_{0,0} \in \mathbb{R}$.

Proposition 2.7. *λ is a ∗–homomorphism if and only if its coefficients $\{\lambda_{0,0}, \lambda_{0,1}, \lambda_{1,0}, \lambda_{1,1}, \lambda_{2,1}\}$* ***verify the following system of equations***

$$\lambda_{0,0} = |\lambda_{1,1}|^2 - |\lambda_{2,1}|^2 \tag{2.8}$$

$$\lambda_{1,0}\lambda_{2,1} - \overline{\lambda}_{1,0}\lambda_{1,1} = \frac{1}{2}\lambda_{0,1} \tag{2.9}$$

$$\lambda_{0,1}\overline{\lambda}_{1,1} - \overline{\lambda}_{0,1}\lambda_{2,1} = 2\overline{\lambda}_{1,0} \tag{2.10}$$

$$\lambda_{0,0}\lambda_{2,1} - \lambda_{0,1}\overline{\lambda}_{1,0} = \lambda_{2,1} \tag{2.11}$$

$$\overline{\lambda}_{0,1}\overline{\lambda}_{1,0} - \lambda_{0,0}\overline{\lambda}_{1,1} = \overline{\lambda}_{1,1} \tag{2.12}$$

Proof. By definition a $*$–homomorphism is a $*$–map that preserves the Lie brackets:

$$\lambda([x,y]) = [\lambda(x), \lambda(y)]\ , \qquad \forall\ x,y \in sl_B(2,\mathbb{C})$$

By Proposition 2.7 the $*$–map condition is equivalent to (2.6), which shows that in this case the coefficients of λ are uniquely determined by $\{\lambda_{0,0}, \lambda_{0,1}, \lambda_{1,0}, \lambda_{1,1}, \lambda_{2,1}\}$. The homomorphism condition

$$\left[\lambda\Big(B^{(1)}\Big), \lambda\Big(B^{(2)}\Big)\right] = \lambda\Big(B^{(0)}\Big) = \lambda\big(M\big)$$

together with the $*$–property of λ (i.e., (2.5)) is equivalent to

$$\begin{aligned}
&\sum_{k,j} \lambda_{1,k}\lambda_{2,j}\left[B_F^{(k)}, B_F^{(j)}\right] = \lambda\left(M^*\right) = \lambda\left(M\right)^* = \sum_j \overline{\lambda}_{0,j} B_F^{(j)*} \\
\Longleftrightarrow\ &(\lambda_{1,1}\lambda_{2,2} - \lambda_{1,2}\lambda_{2,1})\left[B_F^{(1)}, B_F^{(2)}\right] + (\lambda_{1,0}\lambda_{2,1} - \lambda_{1,1}\lambda_{2,0})\left[B_F^{(0)}, B_F^{(1)}\right] \\
&+ (\lambda_{1,0}\lambda_{2,2} - \lambda_{1,2}\lambda_{2,0})\left[B_F^{(0)}, B_F^{(2)}\right] \\
&= \overline{\lambda}_{0,0} B_F^{(0)*} + \overline{\lambda}_{0,1} B_F^{(1)*} + \overline{\lambda}_{0,2} B_F^{(2)*} = \overline{\lambda}_{0,0} B_F^{(0)} + \overline{\lambda}_{0,1} B_F^{(2)} + \overline{\lambda}_{0,2} B_F^{(1)} \\
\Longleftrightarrow\ &\overline{\lambda}_{0,0} B_F^{(0)} + \overline{\lambda}_{0,1} B_F^{(2)} + \overline{\lambda}_{0,2} B_F^{(1)} \\
&= (\lambda_{1,1}\lambda_{2,2} - \lambda_{1,2}\lambda_{2,1}) B_F^{(0)} + 2\left(\lambda_{1,0}\lambda_{2,1} - \lambda_{1,1}\lambda_{2,0}\right) B_F^{(1)} \\
&- 2\left(\lambda_{1,0}\lambda_{2,2} - \lambda_{1,2}\lambda_{2,0}\right) B_F^{(2)}
\end{aligned} \tag{2.13}$$

Identifying coefficients and using (2.6), one sees that (2.13) is equivalent to:

$$\begin{aligned}
\lambda_{0,0} &= \overline{\lambda}_{0,0} = (\lambda_{1,1}\lambda_{2,2} - \lambda_{1,2}\lambda_{2,1}) = |\lambda_{1,1}|^2 - |\lambda_{2,1}|^2 \\
\lambda_{0,1} &= \overline{\lambda}_{0,2} = 2\left(\lambda_{1,0}\lambda_{2,1} - \lambda_{1,1}\lambda_{2,0}\right) = 2\left(\lambda_{1,0}\lambda_{2,1} - \lambda_{1,1}\overline{\lambda}_{1,0}\right)
\end{aligned}$$

which are equivalent to (2.8) and (2.9) respectively.

The homomorphism condition

$$\left[\lambda\left(B^{(0)}\right), \lambda\left(B^{(2)}\right)\right] = \lambda\left(\left[B^{(0)}, B^{(2)}\right]\right) = 2\lambda\left(B^{(2)}\right)$$

is equivalent to

$$\sum_{k,j} \lambda_{0,k}\lambda_{2,j}\left[B_F^{(k)}, B_F^{(j)}\right] = 2\sum_j \lambda_{2,j} B_F^{(j)} \tag{2.14}$$

In other words

$$\begin{aligned}
&(\lambda_{0,1}\lambda_{2,2} - \lambda_{0,2}\lambda_{2,1})\left[B_F^{(1)}, B_F^{(2)}\right] + (\lambda_{0,0}\lambda_{2,1} - \lambda_{0,1}\lambda_{2,0})\left[B_F^{(0)}, B_F^{(1)}\right] \\
&+ (\lambda_{0,0}\lambda_{2,2} - \lambda_{0,2}\lambda_{2,0})\left[B_F^{(0)}, B_F^{(2)}\right] \\
=&\,(\lambda_{0,1}\lambda_{2,2} - \lambda_{0,2}\lambda_{2,1}) B_F^{(0)} + 2\left(\lambda_{0,0}\lambda_{2,1} - \lambda_{0,1}\lambda_{2,0}\right) B_F^{(1)} \\
&- 2\left(\lambda_{0,0}\lambda_{2,2} - \lambda_{0,2}\lambda_{2,0}\right) B_F^{(2)} \\
=&\,2\lambda_{2,0} B_F^{(0)} + 2\lambda_{2,1} B_F^{(1)} + 2\lambda_{2,2} B_F^{(2)}
\end{aligned} \tag{2.15}$$

Equality of the $B_F^{(0)}$–coefficients is equivalent to:

$$2\overline{\lambda}_{1,0} = 2\lambda_{2,0} = \lambda_{0,1}\lambda_{2,2} - \lambda_{0,2}\lambda_{2,1} = \lambda_{0,1}\overline{\lambda}_{1,1} - \overline{\lambda}_{0,1}\lambda_{2,1}$$

In other words

$$2\lambda_{1,0} = \overline{\lambda}_{0,1}\lambda_{1,1} - \lambda_{0,1}\overline{\lambda}_{2,1}$$

and which is equivalent to (2.10). Equality of the $B_F^{(1)}$–coefficients is equivalent to:

$$\lambda_{2,1} = \lambda_{0,0}\lambda_{2,1} - \lambda_{0,1}\lambda_{2,0} = \lambda_{0,0}\lambda_{2,1} - \lambda_{0,1}\overline{\lambda}_{1,0}$$

which is equivalent to (2.11).

Finally, using the fact that $\lambda_{0,0} \in \mathbb{R}$, one sees that equality of the $B_F^{(2)}$–coefficients is equivalent to:

$$\overline{\lambda}_{1,1} = \lambda_{2,2} = \lambda_{0,2}\lambda_{2,0} - \lambda_{0,0}\lambda_{2,2} = \overline{\lambda}_{0,1}\overline{\lambda}_{1,0} - \lambda_{0,0}\overline{\lambda}_{1,1}$$

In other words,

$$\lambda_{1,1} = \lambda_{0,1}\lambda_{1,0} - \lambda_{0,0}\lambda_{1,1}$$

and which is equivalent to (2.12). Since all the other commutation relations among generators can be deduced from (2.13) and (2.14), this completes the proof of equivalence. □

The system of equations (2.8), (2.9), (2.10), (2.11), (2.12) has the trivial solution $(0, 0, 0, 0, 0)$ which does not correspond to a ∗–isomorphism. The following Theorem 2.9 shows that this is the only solution.

Theorem 2.8. *The system of equations* (2.8)*,* (2.9)*,* (2.10)*,* (2.11)*,* (2.12) *in the unknowns* $\lambda_{0,0}, \lambda_{1,1}, \lambda_{1,0}, \lambda_{2,1}, \lambda_{0,1}$ *(will be called **the system S** in the following) admits only the solution in which all unknowns are zero.*

Proof. First note that, if $\lambda_{1,1} = 0$, the system S becomes

$$\lambda_{0,0} = -|\lambda_{2,1}|^2 \tag{2.16}$$

$$\lambda_{1,0}\lambda_{2,1} = \frac{1}{2}\lambda_{0,1} \tag{2.17}$$

$$-\overline{\lambda}_{0,1}\lambda_{2,1} = 2\overline{\lambda}_{1,0} \tag{2.18}$$

$$\lambda_{0,0}\lambda_{2,1} - \lambda_{0,1}\overline{\lambda}_{1,0} = \lambda_{2,1} \tag{2.19}$$

$$\overline{\lambda}_{0,1}\overline{\lambda}_{1,0} = 0 \tag{2.20}$$

If also $\lambda_{2,1} = 0$, (2.16), (2.17) and (2.18) become respectively

$$\lambda_{0,0} = 0 \ ; \qquad 0 = \frac{1}{2}\lambda_{0,1} \ ; \qquad 0 = 2\overline{\lambda}_{1,0}$$

so that $\lambda_{0,0} = \lambda_{2,1} = \lambda_{0,1} = \lambda_{1,0} = \lambda_{1,1} = 0$. Thus, if $\lambda_{1,1} = \lambda_{2,1} = 0$, the zero solution is the only possible one. If $\lambda_{1,1} = 0$ but $\lambda_{2,1} \neq 0$, (2.20) implies that either $\lambda_{0,1} = 0$ or $\lambda_{1,0} = 0$. Then, by (2.17) and (2.18), $\lambda_{0,1} = \lambda_{0,1} = 0$. But then, by (2.19), $\lambda_{2,1} = 0$ against the assumption. Thus $\lambda_{1,1} = 0$ implies that $\lambda_{2,1} = 0$ and in this case we have already seen that the zero solution is the only possible one. It follows that, for the existence of non–zero solutions, $\lambda_{1,1}$ must be $\neq 0$. Assuming this, if $\lambda_{1,0} = 0$, the system S becomes

$$\lambda_{0,0} = |\lambda_{1,1}|^2 - |\lambda_{2,1}|^2 \tag{2.21}$$

$$0 = \lambda_{0,1} \tag{2.22}$$

$$\lambda_{0,1}\overline{\lambda}_{1,1} - \overline{\lambda}_{0,1}\lambda_{2,1} = 0 \tag{2.23}$$

$$\lambda_{0,0}\lambda_{2,1} = \lambda_{2,1} \tag{2.24}$$

$$-\lambda_{0,0}\overline{\lambda}_{1,1} = \overline{\lambda}_{1,1} \tag{2.25}$$

and (2.23) is identically satisfied due to (2.22). Multiplication of both sides of (2.25) by $\lambda_{1,1}(\neq 0)$ gives $-\lambda_{0,0}|\lambda_{1,1}|^2 = |\lambda_{1,1}|^2$, i.e., $\lambda_{0,0} = -1$. Then by (2.24), $\lambda_{2,1} = 0$ and this, due to (2.21), leads to the contradiction $|\lambda_{1,1}|^2 = -1$. Thus non–zero solutions may exist only if $\lambda_{1,1}\lambda_{1,0} \neq 0$. Assuming this, (2.10) implies that also $\lambda_{0,1} \neq 0$, and consequently $\lambda_{1,1}\lambda_{1,0}\lambda_{0,1} \neq 0$. Then (2.11) implies that also $\lambda_{2,1} \neq 0$. We conclude that non–zero solutions may exist only if

$$\lambda_{1,1}\lambda_{1,0}\lambda_{0,1}\lambda_{2,1} \neq 0 \tag{2.26}$$

Supposing that (2.26) holds and that $\lambda_{0,0} = 0$, the system S becomes

$$|\lambda_{1,1}|^2 = |\lambda_{2,1}|^2 \tag{2.27}$$

$$\lambda_{1,0}\lambda_{2,1} - \overline{\lambda}_{1,0}\lambda_{1,1} = \frac{1}{2}\lambda_{0,1} \tag{2.28}$$

$$\lambda_{0,1}\overline{\lambda}_{1,1} - \overline{\lambda}_{0,1}\lambda_{2,1} = 2\overline{\lambda}_{1,0} \tag{2.29}$$

$$-\lambda_{0,1}\overline{\lambda}_{1,0} = \lambda_{2,1} \tag{2.30}$$

$$\overline{\lambda}_{0,1}\overline{\lambda}_{1,0} = \overline{\lambda}_{1,1} \tag{2.31}$$

Using (2.30) and (2.31), (2.28) leads to the contradiction as follows:

$$-|\lambda_{1,0}|^2\lambda_{0,1} - |\lambda_{1,0}|^2\lambda_{0,1} = \frac{1}{2}\lambda_{0,1} \iff -2|\lambda_{1,0}|^2\lambda_{0,1} = \frac{1}{2}\lambda_{0,1} \iff -2|\lambda_{1,0}|^2 = \frac{1}{2}$$

We conclude that non–zero solutions may exist only if

$$\lambda_{0,0}\lambda_{1,1}\lambda_{1,0}\lambda_{0,1}\lambda_{2,1} \neq 0 \tag{2.32}$$

Assuming (2.32), one gets

$$(2.12) \iff \lambda_{0,1}\lambda_{1,0} - \lambda_{0,0}\lambda_{1,1} = \lambda_{1,1} \iff \lambda_{0,0} + 1 = \frac{\lambda_{0,1}\lambda_{1,0}}{\lambda_{1,1}} \in \mathbb{R}$$

Introducing polar coordinates for the $\lambda_{i,j} =: |\lambda_{i,j}|e^{-i\theta_{i,j}}$, we obtain

$$\begin{aligned} \frac{\lambda_{0,1}\lambda_{1,0}}{\lambda_{1,1}} \in \mathbb{R} &\iff \frac{|\lambda_{0,1}|\ |\lambda_{1,0}|}{|\lambda_{1,1}|} e^{i\theta_{0,1}} e^{i\theta_{1,0}} e^{-i\theta_{1,1}} \in \mathbb{R} \\ &\iff e^{i\theta_{0,1}} e^{i\theta_{1,0}} e^{-i\theta_{1,1}} \in \{\pm 1\} \\ &\iff e^{i(\theta_{0,1}+\theta_{1,0})} = \varepsilon_+ e^{i\theta_{1,1}} \ ; \qquad \text{with } \varepsilon_+ \in \{\pm 1\} \end{aligned} \tag{2.33}$$

Similarly, thanks to the equivalence,

$$(2.11) \iff \lambda_{0,0} - \frac{\lambda_{0,1}\overline{\lambda}_{1,0}}{\lambda_{2,1}} = 1 \iff \lambda_{0,0} - 1 = \frac{\lambda_{0,1}\overline{\lambda}_{1,0}}{\lambda_{2,1}} \in \mathbb{R}$$

one finds

$$\begin{aligned}
&\frac{\lambda_{0,1}\overline{\lambda}_{1,0}}{\lambda_{2,1}} \in \mathbb{R} \\
\iff &\frac{|\lambda_{0,1}|\ |\lambda_{1,0}|}{|\lambda_{1,1}|} e^{i\theta_{0,1}} e^{-i\theta_{1,0}} e^{-i\theta_{2,1}} \in \mathbb{R} \\
\iff &e^{i\theta_{0,1}} e^{-i\theta_{1,0}} e^{-i\theta_{2,1}} \in \{\pm 1\} \\
\iff &e^{i(\theta_{0,1}-\theta_{1,0})} = \varepsilon_- e^{i\theta_{2,1}} \ ; \qquad \text{with } \varepsilon_- \in \{\pm 1\}
\end{aligned} \tag{2.34}$$

From (2.34) it follows that

$$\begin{aligned}
(2.11) \iff &\lambda_{0,1}\overline{\lambda}_{1,0} = (\lambda_{0,0} - 1)\lambda_{2,1} \\
\iff &|\lambda_{0,1}||\lambda_{1,0}| e^{i(\theta_{0,1}-\theta_{1,0})} = (\lambda_{0,0} - 1)|\lambda_{2,1}| e^{i\theta_{2,1}} \\
\iff &|\lambda_{0,1}||\lambda_{1,0}| = \varepsilon_-(\lambda_{0,0} - 1)|\lambda_{2,1}|
\end{aligned}$$

From (2.33) it follows that

$$\begin{aligned}
(2.12) \iff &\overline{\lambda}_{0,1}\overline{\lambda}_{1,0} = (\lambda_{0,0} + 1)\overline{\lambda}_{1,1} \\
\iff &|\lambda_{0,1}||\lambda_{1,0}| e^{-i(\theta_{0,1}+\theta_{1,0})} = (\lambda_{0,0} + 1)|\lambda_{1,1}e^{-i\theta_{1,1}} \\
\iff &|\lambda_{0,1}||\lambda_{1,0}| = \varepsilon_+(\lambda_{0,0} + 1)|\lambda_{1,1}|
\end{aligned}$$

Combining the above two identities one finds, with $\varepsilon := \varepsilon_+\varepsilon_- \in \{\pm 1\}$,

$$(\lambda_{0,0} - 1)|\lambda_{2,1}| = \varepsilon(\lambda_{0,0} + 1)|\lambda_{1,1}| \tag{2.35}$$

If $\varepsilon = +1$, (2.35) becomes

$$\begin{aligned}
&(\lambda_{0,0} - 1)|\lambda_{2,1}| = (\lambda_{0,0} + 1)|\lambda_{1,1}| \\
\iff &\lambda_{0,0}|\lambda_{2,1}| - |\lambda_{2,1}| = \lambda_{0,0}|\lambda_{1,1}| + |\lambda_{1,1}| \\
\iff &-(|\lambda_{1,1}| + |\lambda_{2,1}|) = \lambda_{0,0}(|\lambda_{1,1}| - |\lambda_{2,1}|)
\end{aligned} \tag{2.36}$$

We know from (2.32) that $|\lambda_{1,1}| + |\lambda_{2,1}| \neq 0$. Therefore $(|\lambda_{1,1}| - |\lambda_{2,1}|) \neq 0$, otherwise

$$(|\lambda_{1,1}| - |\lambda_{2,1}|) = 0 \iff 0 = (|\lambda_{1,1}|^2 - |\lambda_{2,1}|^2) = \lambda_{0,0}$$

contradicting (2.32). Therefore (2.36) is equivalent to

$$\begin{aligned}
&-(|\lambda_{1,1}| + |\lambda_{2,1}|)^2 = \lambda_{0,0}(|\lambda_{1,1}| - |\lambda_{2,1}|)(|\lambda_{1,1}| + |\lambda_{2,1}|) \\
\iff &-(|\lambda_{1,1}| + |\lambda_{2,1}|)^2 = \lambda_{0,0}(|\lambda_{1,1}|^2 - |\lambda_{2,1}|^2)
\end{aligned}$$

and in virtue of (2.8), this is equivalent to

$$-(|\lambda_{1,1}| + |\lambda_{2,1}|)^2 = \lambda_{0,0}^2$$

which is impossible because the left hand side is < 0.

If $\varepsilon = -1$, (2.35) becomes

$$\begin{aligned}
&(\lambda_{0,0} - 1)|\lambda_{2,1}| = -(\lambda_{0,0} + 1)|\lambda_{1,1}| \\
\iff &\lambda_{0,0}|\lambda_{2,1}| - |\lambda_{2,1}| = -\lambda_{0,0}|\lambda_{1,1}| - |\lambda_{1,1}| \\
\iff &|\lambda_{1,1}| - |\lambda_{2,1}| = -\lambda_{0,0}(|\lambda_{1,1}| + |\lambda_{2,1}|) \\
\iff &|\lambda_{1,1}|^2 - |\lambda_{2,1}|^2 = -\lambda_{0,0}(|\lambda_{1,1}| + |\lambda_{2,1}|)^2
\end{aligned}$$

and (2.8) and (2.32) imply that this leads to the contradiction

$$\lambda_{0,0} = -\lambda_{0,0}(|\lambda_{1,1}| + |\lambda_{2,1}|)^2 \iff 1 = -(|\lambda_{1,1}| + |\lambda_{2,1}|)^2$$

In conclusion: non–zero solutions may exist only if (2.32) is satisfied, but in this case no solution exists. □

Theorem 2.8 asserts the non–existence of a $*$–isomorphism between $sl(2,\mathbb{C})$ is to $sl_F(2,\mathbb{C})$. The following result strengthens this statement.

Corollary 2.9. *There is no real $*$–Lie sub–algebra of $sl_B(2,\mathbb{C})$ $*$–isomorphic to $sl_F(2,\mathbb{R})$.*

Proof. Suppose that there exist a real Lie $*$–sub–algebra $\mathcal{L}_\mathbb{R}$ of $sl_B(2,\mathbb{C})$ and a $*$–isomorphism of real Lie $*$–algebras

$$\lambda : sl_F(2,\mathbb{R}) \to \mathcal{L}_\mathbb{R} \subset sl_B(2,\mathbb{C})$$

If we prove that

$$\mathcal{L}_\mathbb{R} \cap i\mathcal{L}_\mathbb{R} = \{0\} \tag{2.37}$$

the thesis will follow because for dimensional reasons (2.37) implies that

$$\mathcal{L}_\mathbb{R} \dot{+} i\mathcal{L}_\mathbb{R} = sl_B(2,\mathbb{C})$$

Therefore the complexification $\lambda \dot{+} i\lambda$ of λ is a $*$–Lie algebra isomorphism from $sl_F(2,\mathbb{C})$ to $sl_B(2,\mathbb{C})$. Since the existence of such an isomorphism is forbidden by Theorem 2.9, it follows that a λ with the above properties cannot exist.

To prove (2.37), note that since the set $\{B_F, B_F^+, M_F\}$ is a linear basis of $sl_F(2,\mathbb{R})$, for any element in $\mathcal{L}_\mathbb{R} \cap i\mathcal{L}_\mathbb{R}$, there will exist $a, b, c, a', b', c' \in \mathbb{R}$ such that:

$$a\lambda(B_F^+) + b\lambda(B_F) + c\lambda(M_F) = ia'\lambda(B_F^+) + ib'\lambda(B_F) + ic'\lambda(M_F)$$

or equivalently,

$$(a - ia')\lambda(B_F^+) + (b - ib)\lambda(B_F) + (c - ic')\lambda(M_F) = 0$$

Taking commutators of both sides with $\lambda(B_F)$ leads to

$$\begin{aligned} &(a - ia')[\lambda(B_F), \lambda(B_F^+)] + (c - ic')[\lambda(B_F), \lambda(M_F)] \\ =&(a - ia')\lambda(M_F) - 2(c - ic')\lambda(B_F) = 0 \end{aligned}$$

Taking commutators of this with $\lambda(M_F)$ gives

$$-2(c - ic')[\lambda(M_F), \lambda(B_F)] = 4(c - ic')\lambda(B_F) = 0$$

Since $\lambda(B_F) \neq 0$ because λ is an isomorphism, it follows that $c = ic'$. But, since both c and c' are real, this is possible if and only if $c = c' = 0$. With similar arguments one proves that $a = b = a' = b' = 0$.

Therefore $\mathcal{L}_\mathbb{R} \cap i\mathcal{L}_\mathbb{R} = \{0\}$ and for the first part of the argument, this implies the thesis. □

3. Physical Origins of $sl_F(2,\mathbb{R})$ and $sl_B(2,\mathbb{R})$

3.1. The 1–mode quadratic Bose algebra. Let $Heis(1,\mathbb{R})$ be the 1–mode real Heisenberg algebra with generators $\{a^+, a, 1\}$ and with relations and involution given respectively by

$$\big[a, a^+\big] = 1 \ ; \qquad (a^+)^* = a \ ; \qquad 1^* = 1$$

all other commutators being zero. Denote $Ue(Heis(1,\mathbb{R}))$ its universally enveloping algebra. The operators

$$\{a^2 \ , \ a^{+2} \ , \ a^+a \ , \ 1\} \subset Ue\big(Heis(1,\mathbb{R})\big)$$

are a linear basis of a ∗–Lie algebra with central element 1 and relations:

$$[a^2, a^{+2}] = 4a^+a + 2\cdot 1 \ ; \qquad [a^+a, a^2] = -2a^2 \ ; \qquad [a^+a, a^{2+}] = 2a^{2+}$$
$$(a^2)^* = a^{+2} \ ; \qquad (a^{+2})^* = a^2 \ ; \qquad (a^+a)^* = a^+a \ ; \qquad 1^* = 1 \tag{3.1}$$

Defining

$$B := a^2/2 \ ; \qquad B^+ := a^{+2}/2 \ ; \qquad M := a^+a + 1/2$$

one verifies that the real linear span of the set $\{B, B^+, M\}$ is a ∗–isomorphic copy of the ∗–Lie algebra $sl_B(2;\mathbb{R})$.

3.2. The quadratic Fermi algebra. A possible candidate for the role of quadratic Fermi algebra is constructed as follows. Consider the 2–mode real CAR algebra, denoted $CAR(2)$, i.e. the associative real ∗–algebra with identity 1_F, generators: $\{a_j \ , \ a_j^+ \ , \ 1 : \ j = 1,2\}$ and relations

$$1^* = 1 \ \text{ and } \ a_j = (a_j^+)^* \ , \quad \forall j = 1,2$$

$$\{a_i, a_j\} = \{a_i^+, a_j^+\} = 0 \ ; \quad \{a_i, a_j^+\} := a_ia_j^+ + a_j^+a_i = \delta_{i,j}\cdot 1_F \ , \ \forall i,j = 1,2 \tag{3.2}$$

Lemma 3.1. *Defining*

$$B_F := a_1a_2 \ ; \qquad B_F^+ := a_2^+a_1^+ \ ; \qquad M_F := 1 - (a_2^+a_2 + a_1^+a_1) \tag{3.3}$$

the real linear span of the set $\{B_F, B_F^+, M_F\}$ is a ∗–isomorphic copy of the ∗–Lie algebra $sl_F(2;\mathbb{R})$.

Proof. We have to prove that the involution relations (2.3) and the commutation relations (2.2) hold with $\{B_F, B_F^+, M_F\}$ replacing $\{B_2, B_2^+, M_2\}$.

That $(B_F)^*B_F^+$ and $M_F = M_F^*$ follows by inspection from (3.3). So (2.3) holds.

Using the anti–commutation relations, one finds:

$$\begin{aligned}
[B_F, B_F^+] &= [a_1a_2, a_2^+a_1^+] \\
&= a_1a_2a_2^+a_1^+ - a_2^+a_1^+a_1a_2 = -a_1a_2a_1^+a_2^+ - a_2^+a_1^+a_1a_2 \\
&= -a_1(a_2a_1^+ + a_1^+a_2)a_2^+ + a_1a_1^+a_2a_2^+ - a_2^+a_1^+a_1a_2 \\
&= a_1a_1^+a_2a_2^+ - a_1^+a_1a_2^+a_2 = (1_F - a_1^+a_1)(1_F - a_2^+a_2) - a_1^+a_1a_2^+a_2 \\
&= 1 - a_2^+a_2 - a_1^+a_1 + a_1^+a_1a_2^+a_2 - a_1^+a_1a_2^+a_2 \\
&= 1_F - a_2^+a_2 - a_1^+a_1 = M_F
\end{aligned}$$

which is the first identity in (2.2). By definition of M_F

$$[M_F, B_F^+] = -[a_2^+ a_2 + a_1^+ a_1, a_2^+ a_1^+]$$

and

$$\begin{aligned}
&[a_2^+ a_2 + a_1^+ a_1, a_2^+ a_1^+] = [a_2^+ a_2, a_2^+ a_1^+] + [a_1^+ a_1, a_2^+ a_1^+] \\
=&a_2^+ a_2 a_2^+ a_1^+ - a_2^+ a_1^+ a_2^+ a_2 + a_1^+ a_1 a_2^+ a_1^+ - a_2^+ a_1^+ a_1^+ a_1 \\
=&a_2^+ (1 - a_2^+ a_2) a_1^+ + a_1^+ a_2^+ a_2^+ a_2 + (1 - a_1 a_1^+) a_2^+ a_1^+ \\
=&a_2^+ a_1^+ + a_2^+ a_1^+ - a_1 a_1^+ a_2^+ a_1^+ = 2a_2^+ a_1^+ + a_1 a_2^+ a_1^+ a_1^+ \\
=&2a_2^+ a_1^+ = 2B_F^+
\end{aligned}$$

Therefore

$$[M_F, B_F^+] = -2B_F^+$$

which is the second identity in (2.2). Taking the adjoint of this and using (2.3) one finds the third identity in (2.2). □

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L. Accardi: Centro Vito Volterra, Università di Roma “Tor Vergata”, Roma I-00133, Italy
Email address: accardi@volterra.uniroma2.it

I.Ya. Aref'eva: Steklov Mathematical Institute, Gubkin St.8, GSP-1, 117966, Moscow, Russia
Email address: arefeva@mi.ras.ru

Y.G. Lu: Dipartimento di Matematica, Univerversità di Bari “Aldo Moro”, Via E. Orabona 4, 70125, Bari, Italy
Email address: yungang.lu@uniba.it

I.V.Volovich: Steklov Mathematical Institute, Gubkin St.8, GSP-1, 117966, Moscow, Russia
Email address: volovich@mi.ras.ru