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## MRM-APPLICABLE MEASURES FOR THE POWER FUNCTION OF THE SECOND ORDER

IZUMI KUBO, HUI-HSIUNG KUO, AND SUAT NAMLI

ABSTRACT. The authors have studied multiplicative renormalization method (MRM) for generating functions of orthogonal polynomials. They determined all MRM-applicable measures for renormalizing functions  $h(x) = \exp[x]$ ,  $h(x) = (1-x)^{-\kappa}$ ,  $\kappa = 1/2$  and 1. For the cases of  $h(x) = \exp[x]$  MRM-applicable measures are Meixner class and very rich. For the case  $h(x) = (1-x)^{-\kappa}$ ,  $\kappa = 1$ , MRM-applicable measures are rather rich. For the case  $\kappa = 1/2$ , MRM-applicable measures are only uniform measures on intervals. In this paper, they determine all MRM-applicable measures for  $h(x) = (1-x)^{-2}$ , whose typical cases are three special beta distributions.

### 1. Multiplicative Renormalization

A probability measure  $\mu$  on  $\mathbb{R}$  with density  $f_\mu(x)$  is said to be applicable to *the multiplicative renormalization method for  $h(x)$*  if there exists a suitable analytic function  $\rho(t)$  around  $t = 0$  with  $\rho(0) = 0$ ,  $\alpha = \rho'(0) \neq 0$  such that

$$\psi(t, x) = \frac{h(\rho(t)x)}{\theta(\rho(t))} \quad \text{with} \quad \theta(t) = \int h(tx) d\mu(x) \quad (1.1)$$

is a generating function of the orthogonal polynomials  $\{P_n(x)\}$  in  $L^2(\mu)$  with the leading coefficient equal 1. We say that  $\mu$  is *MRM-applicable for  $h(x)$  with  $\rho(t)$* , simply. Then there exist Jacobi-Szegő parameters  $\{\alpha_n, \omega_n\}$  satisfying the recursive relation

$$P_{n+1}(x) = (x - \alpha_n)P_n(x) - \omega_{n-1}P_{n-1}(x) \quad \text{for} \quad n \geq 0 \quad (1.2)$$

with conventions  $\omega_{-1} = 1$ ,  $P_{-1}(x) = 0$ . It is known that

$$\|P_n\|^2 = \lambda_n = \omega_{-1}\omega_0\omega_1 \cdots \omega_{n-1} \quad \text{for} \quad n \geq 0. \quad (1.3)$$

The following proposition is shown in [3].

**Proposition 1.1.** *The function  $\psi(t, x) = \frac{h(\rho(t)x)}{\theta(\rho(t))}$  is a generating function of orthogonal polynomials if and only if*

$$\Theta_\rho(t, s) = \frac{\tilde{\theta}(\rho(t), \rho(s))}{\theta(\rho(t))\theta(\rho(s))} \quad (1.4)$$

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is a function  $J(ts)$  depending only on  $ts$ , where

$$\tilde{\theta}(t, s) = \int h(tx)h(sx) d\mu(x). \tag{1.5}$$

In [3], we showed that Gaussian measure, Poisson measure, Gamma distribution and negative binomial are MRM-applicable for  $h(x) = e^x$  and that beta distribution  $\tilde{B}(\kappa + 1/2, \kappa + 1/2)$ , which is defined in (4.10), is MRM-applicable for  $h^\kappa(x) = 1/(1 - x)^\kappa$ . Especially, uniform measure is MRM-applicable for  $h^{1/2}(x)$  and the semi-circle distribution is for  $h^1(x)$ .

The problem to determine all MRM-measures applicable for  $h(x) = e^x$  is solved in [6], [12]. The measures coincide with Meixner class. In [8], we showed that typical measures MRM-applicable for  $h^1(x)$  are rich and typical cases are given by

$$f_\mu(x) = \frac{b\sqrt{1-x^2}}{\pi(a^2 + b^2 - 2a(1-b)x + (1-2b)x^2)} \quad \text{for } |x| \leq 1,$$

where  $b > 0$  and  $|a| \leq 1 - b$ . In [9], we showed that only uniform measures are MRM-applicable for  $h^{1/2}(x)$ .

Our purpose is to determine all MRM-applicable measures for  $h^k(x)$ . In a forth coming paper, we will discuss for general  $h^k(x)$ ,  $\kappa \neq 1/2, 1$ . The cases of  $\kappa = 1/2$  and  $\kappa = 1$  are extremal and must be discussed separately. The case of  $\kappa = 2$  can be included in the general case, but it is so special that we can obtain  $\mu$  from  $\theta(s)$  directly applying 'the inverse Hilbert transform' discussed in [8]. In this paper we assume that  $h(x) = \frac{1}{(1-x)^2}$  and  $\mu$  is MRM-applicable for  $h(x)$ . Then

$$\psi(t, x) = \frac{1}{\theta(\rho(t))(1-\rho(t)x)^2} \quad \text{with} \quad \theta(t) = \int \frac{f_\mu(x)}{(1-tx)^2} dx. \tag{1.1}'$$

Put  $\varphi(t) = \theta(\rho(t))$  and

$$\begin{aligned} \alpha &= \rho'(0), & c_1 &= \alpha^{-2}\rho''(0), & c_4 &= \alpha^{-3}\rho'''(0), \\ a &= \alpha^{-1}\varphi'(0), & c_2 &= \alpha^{-2}\varphi''(0), & c_5 &= \alpha^{-3}\varphi'''(0), \\ b &= \alpha^{-1}J'(0), & c_3 &= \alpha^{-2}J''(0), & c_6 &= \alpha^{-3}J'''(0). \end{aligned} \tag{1.6}$$

Those parameters are not free. Here, we will see some of constrains. From  $h(x) = \sum_{n=0}^\infty (n+1)x^n$ , we get the expansion

$$\psi(t, x) = \sum_{n=0}^\infty (n+1)\alpha^n P_n(x)t^n. \tag{1.7}$$

By  $\theta'(0) = \varphi'(0)/\rho'(0) = a$ , we see that

$$\alpha_0 = \int x f_\mu(x) dx = \frac{1}{2}\theta'(0) = \frac{a}{2}.$$

Since  $\varphi''(0) = \rho'(0)^2\theta''(0) + \rho''(0)\theta'(0)$ ,

$$\omega_0 = \|P_1\|^2 = \int (x - \alpha_0)^2 f_\mu(x) dx = \frac{1}{6}\theta''(0) - \alpha_0^2 = \frac{2c_2 - 2ac_1 - 3a^2}{12} > 0$$

by (1.3). Thus we have

$$\alpha_0 = \frac{a}{2} \quad \text{and} \quad \omega_0 = \frac{2c_2 - 2ac_1 - 3a^2}{12} > 0. \tag{1.8}$$

Since  $J(t) = \sum_{n=0}^{\infty} (n+1)^2 \alpha^{2n} \|P_n\|^2 t^n$  holds by the orthogonality and (1.7),

$$\alpha b = 4\alpha^2 \|P_1\|^2 > 0, \quad \alpha^2 c_3 = 18\alpha^4 \|P_2\|^2 > 0, \quad \alpha^3 c_6 = 96\alpha^6 \|P_3\|^2 > 0. \tag{1.9}$$

By (1.8) and (1.9), we have an equality  $\alpha b = 4\alpha^2(2c_2 - 2ac_1 - 3a^2)/12$ . Hence

$$\alpha = \frac{3b}{2c_2 - 2ac_1 - 3a^2}. \tag{1.10}$$

For the convenience of discussion we introduce

$$\theta_0(t) = \int \frac{1}{1-tx} d\mu(x), \quad \varphi(t) = \theta(\rho(t)), \quad \varphi_0(t) = \theta_0(\rho(t)). \tag{1.11}$$

Since  $\psi(t, x)$  is analytic around  $t = 0$ , the support of  $\mu$  must be bounded. Moreover,  $\theta(t)$  cannot be analytic at  $t$  with  $1/t \in S_\mu$ , where  $S_\mu$  is the support of  $\mu$ .

Define

$$\begin{aligned} F_1(X, Y, t) &= \frac{bX + 2(a + bt)Y}{2 - (a + bt)X}, \\ F_2(X, Y, t) &= \frac{-3b + 2(ab + c_3t)X + 2(c_2 + 2abt + c_3t^2)Y}{2c_1 + 3a + 3bt - (c_2 + 2abt + c_3t^2)X}, \\ F_3(X, Y, t) &= \frac{A(X, Y, t)}{B(X, Y, t)}, \end{aligned} \tag{1.12}$$

where

$$\begin{aligned} A(X, Y, t) &= -8ab - 5bc_1 - 8c_3t + 3(bc_2 + 2ac_3t + c_6t^2)X \\ &\quad + 2(c_5 + 3bc_2t + 3ac_3 + c_6t^2)Y, \\ B(X, Y, t) &= 5ac_1 + 4c_2 + 2c_4 + (8ab + 5c_1)t + 4c_3t^2 \\ &\quad - (c_5 + 3bc_2t + 3ac_3t^2 + c_6t^3)X. \end{aligned} \tag{1.13}$$

The following proposition can be proved similarly to Lemma 3.3, 3.4 and 3.5 of [9].

**Proposition 1.2.** *Assume that  $\Theta_\rho(t, s)$  is a function of  $ts$ . Then the following conditions are satisfied.*

$$\frac{\varphi'(t)}{\varphi(t)} = F_1(\rho(t), \rho'(t), t) = F_2(\rho(t), \rho'(t), t) = F_3(\rho(t), \rho'(t), t). \tag{1.14}$$

It is useful to remark that the equations (1.14) at  $t = 0$  is considered as

$$\alpha = \varphi'(0) = \lim_{t \rightarrow 0} F_1(\rho(t), \rho'(t), t) = \lim_{t \rightarrow 0} F_2(\rho(t), \rho'(t), t) = \lim_{t \rightarrow 0} F_3(\rho(t), \rho'(t), t) \tag{1.15}$$

and  $\rho(t)$  satisfies

$$\rho(t) = \alpha t + \frac{\alpha^2 c_1}{2} t^2 + \frac{\alpha^3 c_4}{6} t^3 + O(t^4),$$

if necessary.

**2. Actions of the Motion Group**

We now observe the motion group. Let  $f_\mu(x)$  be the density of  $\mu$  ( $d\mu(x) = f_\mu(x) dx$ ). Define

$$\tau_{p,q}x = px + q, \quad d\tau_{p,q}\mu(x) = |p|f_\mu(px + q) dx \tag{2.1}$$

and the scale modification  $rt$  of  $t$  by  $\lambda_r$ . Then we can see how functions and parameters are deformed by the actions similarly to §2 in [9].

**Proposition 2.1.** *For a given probability measure  $\mu$ , let  $\tau_{p,q}\mu$  be given by (2.1). If  $\mu$  is MRM-applicable for  $h(x)$  with  $\rho(t)$  and  $\varphi(t)$ , then  $\tau_{p,q}\mu$  is so for  $h(x)$  with  $\lambda_r\tau_{p,q}\rho(t)$  and  $\lambda_r\tau_{p,q}\varphi(t)$ , where*

$$\begin{aligned} \tau_{p,q}\theta(t) &= \left(\frac{p}{p+qt}\right)^2 \theta\left(\frac{t}{p+qt}\right), \\ \tau_{p,q}\tilde{\theta}(t,s) &= \left(\frac{p}{p+qt}\right)^2 \left(\frac{p}{p+qs}\right)^2 \tilde{\theta}\left(\frac{t}{p+qt}, \frac{s}{p+qs}\right), \\ \lambda_r\tau_{p,q}\rho(t) &= \frac{p\rho(rt)}{1-q\rho(rt)}, \\ \lambda_r\tau_{p,q}\varphi(t) &= (1-q\rho(rt))^2 \varphi(rt) \end{aligned} \tag{2.2}$$

for  $p \neq 0, r \neq 0$ .

**Proposition 2.2.** *Parameters are deformed by  $\tau_{p,q}$  and  $\lambda_r$  as*

$$\begin{aligned} \hat{\alpha} &= pr\alpha, \quad \hat{a} = \frac{a-2q}{p}, \quad \hat{b} = \frac{rb}{p}, \quad \hat{c}_1 = \frac{c_1+2q}{p}, \\ \hat{c}_2 &= \frac{c_2-4qa-2qc_1+2q^2}{p^2}, \quad \hat{c}_3 = \frac{r^2c_3}{p^2}, \quad \hat{c}_4 = \frac{c_4+6qc_1+6q^2}{p^2}, \\ \hat{c}_5 &= \frac{c_5-6qac_1-6qc_2-2qc_4+6q^2a+6q^2c_1}{p^3}, \quad \hat{c}_6 = \frac{r^3c_6}{p^3}. \end{aligned} \tag{2.3}$$

Replacing the parameters  $a, b, c_1, c_2, c_3, c_4, c_5$  and  $c_6$  in  $F_1(X, Y, t), F_2(X, Y, t), F_3(X, Y, t), A(X, Y, t)$  and  $B(X, Y, t)$  by  $\hat{a}, \hat{b}, \hat{c}_1, \hat{c}_2, \hat{c}_3, \hat{c}_4, \hat{c}_5$  and  $\hat{c}_6$ , denote them by  $\hat{F}_1(X, Y, t), \hat{F}_2(X, Y, t), \hat{F}_3(X, Y, t), \hat{A}(X, Y, t)$  and  $\hat{B}(X, Y, t)$ , respectively. Then  $\rho_{pqr}(t) = \lambda_r\tau_{p,q}\rho(t)$  and  $\varphi_{p,q}(t) = \tau_{p,q}\varphi(t)$  satisfy

$$\frac{\varphi'_{p,q}(t)}{\varphi_{p,q}(t)} = \hat{F}_1(\rho_{pqr}(t), \rho'_{pqr}(t), t) = \hat{F}_2(\rho_{pqr}(t), \rho'_{pqr}(t), t) = \hat{F}_3(\rho_{pqr}(t), \rho'_{pqr}(t), t). \tag{2.4}$$

**Corollary 2.3.** *Let us take  $q = \frac{a}{2}$ . Then*

$$\begin{aligned} \hat{\alpha} &= pr\alpha, \quad \hat{a} = 0, \quad \hat{b} = \frac{rb}{p}, \quad \hat{c}_1 = \frac{c_1+a}{p}, \\ \hat{c}_2 &= \frac{2c_2-2ac_1-3a^2}{2p^2}, \quad \hat{c}_3 = \frac{r^2c_3}{p^2}, \quad \hat{c}_4 = \frac{2c_4+6ac_1+3a^2}{2p^2}, \\ \hat{c}_5 &= \frac{2c_5-3a^2c_1-6ac_2-2ac_4+3a^3}{2p^3}, \quad \hat{c}_6 = \frac{r^3c_6}{p^3}. \end{aligned} \tag{2.5}$$

**3. Formulas of  $\rho(t)$  and  $\varphi(t)$**

By Proposition 1.2,  $\rho(t)$  satisfies the equations

$$\frac{\varphi'(t)}{\varphi(t)} = F_1(\rho(t), \rho'(t), t) = F_2(\rho(t), \rho'(t), t) = F_3(\rho(t), \rho'(t), t). \tag{1.14}$$

The parameters must satisfy

$$\begin{aligned} \alpha &= \frac{3b}{2c_2 - 2ac_1 - 3a^2}, \\ \frac{\alpha}{b} &= \frac{\frac{3}{2}(c_2 - ac_1) - 3a^2}{2(c_2 - ac_1) - 3a^2} > 0, \quad \frac{c_3}{b^2} > 0, \quad \frac{c_6}{b^3} > 0 \end{aligned} \tag{3.1}$$

by (1.8), (1.9) and (1.10). Since we can show that the case  $5ac_1 + 4c_2 + 2c_4 = 0$  is impossible, we have an equation

$$\alpha a = \lim_{t \rightarrow 0} F_3(\rho(t), \rho'(t), t) = \frac{-8ab - 5bc_1 + 2\alpha c_5}{5ac_1 + 4c_2 + 2c_4},$$

by (1.13), (1.14) and (1.15). Solving the equation in  $c_5$ , we have

$$c_5 = -\frac{1}{3} \left( 12a^3 - 14ac_2 + 8a^2c_1 + 5ac_1^2 - 5c_1c_2 - 3ac_4 \right). \tag{3.2}$$

**Theorem 3.1.** *For parameters satisfying (3.1), suppose that  $\rho(t)$  and  $\varphi(t)$  satisfy (1.14). Then there are only three possible cases as follows:*

(i)  $b \neq 0, a + c_1 = 0, c_3 = \frac{45}{32}b^2$

*In this case, the parameters and functions satisfy*

$$c_1 = -a, \quad c_2 > \frac{1}{2}a^2, \quad c_3 = \frac{45}{32}b^2 > 0,$$

$$c_4 = \frac{3}{4}(3a^2 - 2c_2), \quad c_5 = \frac{3}{4}(2c_2 - a^2)a, \quad c_6 = \frac{81}{32}b^3$$

and

$$\begin{aligned} \rho(t) &= \frac{24bt}{8a^2(2c_2 - a^2) + 12abt + 9b^2t^2}, \\ \varphi(t) &= \left( \frac{8(2c_2 - a^2) + 12abt + 9b^2t^2}{8(2c_2 - a^2)} \right)^2. \end{aligned}$$

(ii)  $b \neq 0, a + c_1 = 0, c_3 = \frac{9}{8}b^2$

*In this case, the parameters and functions satisfy*

$$c_1 = -a, \quad c_2 > \frac{1}{2}a^2, \quad c_3 = \frac{9}{8}b^2,$$

$$c_4 = 2a^2 - c_2, \quad c_5 = a(2c_2 - a^2), \quad c_6 = \frac{3}{2}b^3$$

and

$$\rho(t) = \frac{12bt}{4(2c_2 - a^2) + 6abt + 3b^2t^2},$$

$$\varphi(t) = \frac{4(2c_2 - a^2)}{4(2c_2 - a^2) - 3b^2t^2} \left( \frac{4(2c_2 - a^2) + 6abt + 3b^2t^2}{4(2c_2 - a^2)} \right)^2.$$

$$(iii) \quad b \neq 0, \quad a + c_1 \neq 0, \quad c_3 = \frac{4}{3}b^2$$

In this case, the parameters and functions satisfy

$$c_2 = \frac{1}{2}(12a^2 + 20ac_1 + 9c_1^2), \quad c_3 = \frac{4}{3}b^2, \quad c_4 = -\frac{3}{2}(2a + c_1)(2a + 3c_1),$$

$$c_5 = \frac{1}{2}(36a^3 + 36a^2c_1 - 3ac_1^2 - c_1^3 + 48a^2c_1 + 66ac_1^2 + 16c_1^3), \quad c_6 = \frac{20}{9}b^3$$

and

$$\begin{aligned} \rho(t) &= \frac{6bt}{18(a + c_1)^2 - 3bc_1t + 2b^2t^2}, \\ \varphi(t) &= \frac{(18(c_1 + a)^2 - 3bc_1t + 2b^2t^2)^2}{108(c_1 + a)^3(3(c_1 + a) - bt)}. \end{aligned}$$

*Proof.* Remark that  $a + c_1$  is transformed by  $\tau_{p,q}$  as  $\widehat{a} + \widehat{c}_1 = (a + c_1)/p$  because of (2.3). Hence the condition  $a + c_1 = 0$  is invariant under the motion  $\tau_{p,q}$ . Since  $\widehat{a}\widehat{b} = \alpha b$ ,  $\widehat{\alpha}\widehat{c}_6 = \alpha c_6/p^2$ ,  $\widehat{c}_3 = c_3/p^2$  hold by (2.3), the signs of  $\alpha b$ ,  $c_3$ ,  $\alpha c_6$  are invariant also. By (2.5), we can deform the measure  $\mu$  by the shift  $\tau_{1,a/2}$  as  $\widehat{a} = 0$ . Hence it is sufficient to prove the following lemma for the case  $a = 0$ .  $\square$

**Lemma 3.2.** For parameters satisfying

$$\begin{aligned} a &= 0, \quad c_5 = 10c_1c_2, \quad \alpha = \frac{3b}{2c_2} \\ c_2 &> 0, \quad \frac{c_3}{b^2} > 0, \quad \frac{c_6}{b^3} > 0, \end{aligned} \tag{3.1}'$$

suppose that  $\rho(t)$  and  $\varphi(t)$  satisfy (1.14). Then there are only three possible cases in the following:

$$(i) \quad a = 0, \quad a + c_1 = 0, \quad c_3 = \frac{45}{32}b^2$$

In this case,  $c_2 > 0$ ,

$$a = 0, \quad b \neq 0, \quad c_1 = 0, \quad c_3 = \frac{45}{32}b^2, \quad c_4 = -\frac{1}{2}c_2, \quad c_5 = 0, \quad c_6 = \frac{81}{32}b^3,$$

$$\rho(t) = \frac{24bt}{16c_2 + 9b^2t^2} \quad \text{and} \quad \varphi(t) = \left( \frac{16c_2 + 9b^2t^2}{16c_2} \right)^2.$$

$$(ii) \quad a = 0, \quad a + c_1 = 0, \quad c_3 = \frac{9}{8}b^2$$

In this case,  $c_2 > 0$ ,

$$a = 0, \quad b \neq 0, \quad c_1 = 0, \quad c_3 = \frac{9}{8}b^2,$$

$$c_4 = -\frac{1}{3}c_2, \quad c_5 = 0, \quad c_6 = \frac{3}{2}b^3,$$

$$\rho(t) = \frac{12bt}{8c_2 + 3b^2t^2} \quad \text{and} \quad \varphi(t) = \frac{8c_2}{8c_2 - 3b^2t^2} \left( \frac{8c_2 + 3b^2t^2}{8c_2} \right)^2.$$

(iii)  $a = 0, a + c_1 \neq 0, c_3 = \frac{4}{3}b^2$

In this case,

$$a = 0, \quad b \neq 0, \quad c_1 \neq 0, \quad c_2 = \frac{9}{2}c_1^2, \quad c_3 = \frac{4}{3}b^2,$$

$$c_4 = -\frac{9}{2}c_1^2, \quad c_5 = \frac{15}{2}c_1^3, \quad c_6 = \frac{20}{9}b^3,$$

$$\rho(t) = \frac{6bt}{18c_1^2 - 3bc_1t + 2b^2t^2} \quad \text{and} \quad \varphi(t) = \frac{(18c_1^2 - 3bc_1t + 2b^2t^2)^2}{108c_1^3(3c_1 - bt)}.$$

*Proof.* Case (1) Now we assume that  $a + c_1 = 0$  with (3.1)' and prove (i), (ii) in the lemma. Here conditions on the parameters are

$$\begin{aligned} a = 0, \quad c_1 = 0, \quad c_5 = 0, \quad \alpha = \frac{3b}{2c_2}, \\ c_2 > 0, \quad \frac{c_3}{b^2} > 0, \quad \frac{c_6}{b^3} > 0 \end{aligned} \tag{3.3}$$

by (3.1)'. The equations (1.12) are

$$\begin{aligned} F_1(X, Y, t) &= \frac{b(X + 2tY)}{(2 - btX)}, \\ F_2(X, Y, t) &= \frac{-3b + 2c_3tX + 2(c_2 + c_3t^2)Y}{3bt - (c_2 + c_3t^2)X}, \\ F_3(X, Y, t) &= \frac{-8c_3t + 3(bc_2 + c_6t^2)X + 2t(3bc_2 + c_6t^2)Y}{4c_2 + 2c_4 + 4c_3t^2 - t(3bc_2 + c_6t^2)X}. \end{aligned}$$

Solve the equations

$$F_1(X, Y, t) = F_2(X, Y, t) \quad \text{and} \quad F_1(X, Y, t) = F_3(X, Y, t)$$

in  $Y$ , and denote their solutions  $Y_2(X, t)$  and  $Y_3(X, t)$ , which are

$$\begin{aligned} Y_2(X, t) &= \frac{6b - 4c_3tX - b(c_2 - c_3t^2)X^2}{2(2c_2 - (3b^2 - 2c_3)t^2)}, \\ Y_3(X, t) &= \frac{8c_3t - (b(c_2 - c_4) - (2bc_3 + 3c_6)t^2)X + bc_6t^3X^2}{2t(b(c_2 - c_4) - (2bc_3 - c_6)t^2)}, \end{aligned}$$

respectively. Now, a candidate of  $\rho(t)$  satisfying  $\rho(0) = 0$  is given by solving the equation  $Y_2(X, t) = Y_3(X, t)$  in  $X$  as

$$\rho(t) = \frac{t \times R_{num}(t)}{R_{den}(t)}, \tag{3.4}$$

where

$$\begin{aligned} R_{num}(t) &= 3b^2(c_2 - c_4) - 8c_2c_3 + (6b^2c_3 - 8c_3^2 + 3bc_6)t^2, \\ R_{den}(t) &= -bc_2(c_2 - c_4) + (3bc_2c_3 - bc_3c_4 - 3c_2c_6)t^2 - (2bc_3^2 - 3b^2c_6 + c_3c_6)t^4. \end{aligned}$$



If  $c_4 = c_2$ , then  $R_{num}(0) \neq 0$  is shown and hence  $\rho'(0)$  does not exist; that is,  $c_4 \neq c_2$ . We see

$$\alpha = \rho'(0) = \lim_{t \rightarrow 0} \frac{R_{num}(t)}{R_{den}(t)} = \frac{3b^2(c_2 - c_4) - 8c_2c_3}{-bc_2(c_2 - c_4)}.$$

By (3.3), we have an equation

$$\frac{3b^2(c_2 - c_4) - 8c_2c_3}{-bc_2(c_2 - c_4)} = \frac{3b}{2c_2}.$$

Solving it in  $c_4$ , we have

$$c_4 = \frac{c_2(9b^2 - 16c_3)}{9b^2}.$$

Then we can see the numerator of  $\rho'(t) - Y_2(\rho(t), t)$  is expressed in the form

$$(K_2t^2 + K_4t^4 + K_6t^6), \quad (3.5)$$

where

$$\begin{aligned} K_2 &= 16c_2^2c_3(72b^2c_3 - 224c_3^2 + 135bc_6), \\ K_4 &= -3c_2(108b^4c_3^2 + 324b^3c_3c_6 - 405b^2c_6^2 - 480b^2c_3^3 + 912bc_3^2c_6 - 512c_3^4), \\ K_6 &= 27b^2c_3(-36bc_3c_6 - 36b^2c_3^2 + 36b^3c_6 + 3c_6^2 + 32c_3^2). \end{aligned}$$

Since the form (3.5) must be equal to 0 with  $b \neq 0$ , we get

$$c_6 = \frac{8c_3(-9b^2 + 28c_3)}{135b} \quad (3.6)$$

solving the equation  $K_2 = 0$  in  $c_6$ . Replacing  $c_6$  in  $K_4$  by the right hand side of (3.6), we have expressions of  $K_4$  and  $K_6$  as

$$\begin{aligned} K_4 &= \frac{4}{3}c_2c_3^2(45b^2 - 32c_3)(9b^2 - 8c_3), \\ K_6 &= -\frac{4}{225}c_3^2(72b^2 - 49c_3)(45b^2 - 32c_3)(9b^2 - 8c_3). \end{aligned}$$

The condition for  $K_4 = 0$  and  $K_6 = 0$  is given by

$$c_3 = \frac{45}{32}b^2 \quad \text{or} \quad c_3 = \frac{9}{8}b^2.$$

Case (1-1) :  $c_3 = \frac{45}{32}b^2$

In this case, parameters are

$$a = 0, \quad b \neq 0, \quad c_1 = 0, \quad c_2 > 0, \quad c_3 = \frac{45}{32}b^2,$$

$$c_4 = -\frac{3}{2}c_2, \quad c_5 = 0, \quad c_6 = \frac{81}{32}b^3$$

and  $\rho(t)$  is

$$\rho(t) = \frac{24bt}{16c_2 + 9b^2t^2}, \quad \rho'(0) = \alpha = \frac{3b}{2c_2}$$

by (3.4). Then

$$\frac{\varphi'(t)}{\varphi(t)} = F_1(\rho(t), \rho'(t), t) = \frac{36b^2t}{16c_2 + 9b^2t^2}, \quad \varphi(0) = 1.$$

Therefore, we have

$$\log \varphi(t) = \int_0^t \frac{36b^2t}{16c_2 + 9b^2t^2} dt = 2 \log \frac{16c_2 + 9b^2t^2}{16c_2}.$$

Thus we proved (i).  $\square$

Case (1-2):  $c_3 = \frac{9}{8}b^2$

In this case, parameters are

$$a = 0, \quad b \neq 0, \quad c_1 = 0, \quad c_2 > 0, \quad c_3 = \frac{9}{8}b^2,$$

$$c_4 = -c_2, \quad c_5 = 0, \quad c_6 = \frac{3}{2}b^3$$

and  $\rho(t)$  is

$$\rho(t) = \frac{12bt}{8c_2 + 3b^2t^2}, \quad \rho'(0) = \alpha = \frac{3b}{2c_2}$$

by (3.4). Then

$$\frac{\varphi'(t)}{\varphi(t)} = F_1(\rho(t), \rho'(t), t) = \frac{18b^2t(8c_2 - b^2t^2)}{(8c_2 - 3b^2t^2)(8c_2 + 3b^2t^2)}.$$

Therefore, we have

$$\begin{aligned} \log \varphi(t) &= \int_0^t \frac{18b^2t(8c_2 - b^2t^2)}{(8c_2 + 3b^2t^2)(8c_2 - 3b^2t^2)} dt \\ &= 2 \log \frac{8c_2 + 3b^2t^2}{8c_2} - \log \frac{8c_2 - 3b^2t^2}{8c_2}. \end{aligned}$$

Thus we proved (ii).  $\square$

Case (2) Assume (3.1)' and  $a + c_1 \neq 0$ . Then

$$a = 0, \quad b \neq 0, \quad c_1 \neq 0, \quad c_2 > 0, \quad c_3 > 0, \quad c_5 = \frac{5}{3}c_1c_2, \quad \frac{c_6}{b^3} > 0 \quad (3.7)$$

by (3.1)'. The equations (1.12) are

$$\begin{aligned} F_1(X, Y, t) &= \frac{b(X + 2tY)}{(2 - btX)}, \\ F_2(X, Y, t) &= \frac{-3b + 2c_3tX + 2(c_2 + c_3t^2)Y}{2c_1 + 3bt - (c_2 + c_3t^2)X}, \\ F_3(X, Y, t) &= \frac{-15bc_1 - 24c_3t + 9(bc_2 + c_6t^2)X + 2(5c_1c_2 + 9bc_2t + 3c_6t^3)Y}{6(2c_2 + c_4) + 15bc_1t + 12c_3t^2 - (5c_1c_2 + 9bc_2t + 3c_6t^3)X}. \end{aligned}$$

Solve equations

$$F_1(X, Y, t) = F_2(X, Y, t) \quad \text{and} \quad F_1(X, Y, t) = F_3(X, Y, t)$$

in  $Y$ , and denote their solutions  $Y_2(X, t)$  and  $Y_3(X, t)$ , which are

$$Y_2(X, t) = \frac{6b + 2(bc_1 - 2c_3t)X - b(c_2 - c_3t^2)X^2}{2(2c_2 - 2bc_1t - (3b^2 - 2c_3)t^2)},$$

$$Y_3(X, t) = \frac{30bc_1 + 48c_3t - 6(b(c_2 - c_4) + (2bc_3 + 3c_6)t^2)X - b(5c_1c_2 - 6c_6t^3)X^2}{2(10c_1c_2 + 6b(c_2 - c_4)t - 15b^2c_1t^2 - 6(2bc_3 - c_6)t^3)}.$$

Now, the candidate of  $\rho(t)$  satisfying  $\rho(0) = 0$  is given by solving the equation  $Y_2(X, t) = Y_3(X, t)$  in  $X$  as

$$\rho(t) = \frac{t \times R_{num}(t)}{R_{den}(t)}, \quad (3.8)$$

where

$$R_{num}(t) = 6(3b^2(c_2 - c_4) + 5b^2c_1^2 - 8c_2c_3 + 3bc_1c_3t + (6b^2c_3 - 8c_3^2 + 3bc_6)t^2),$$

$$R_{den}(t) = -2bc_2(3(c_2 - c_4) + 5c_1^2) + 20c_1c_2c_3t + 6(3bc_2c_3 - bc_3c_4 - 3c_2c_6)t^2 - 3bc_1(5bc_3 - 4c_6)t^3 + 6(3b^2c_6 - 2bc_3^2 - c_3c_6)t^4.$$

If  $3c_4 - 5c_1^2 - 3c_2 = 0$ , then we have

$$0 = \rho(0) = \lim_{t \rightarrow 0} \frac{tR_{num}(t)}{R_{den}(t)} = -\frac{12}{5c_1},$$

which is a contradiction. Therefore, we see that  $3c_4 - 5c_1^2 - 3c_2 \neq 0$  and

$$\rho'(0) = \lim_{t \rightarrow 0} \frac{R_{num}(t)}{R_{den}(t)} = \frac{3(3b^2c_2 + 5b^2c_1^2 - 8c_2c_3 - 3b^2c_4)}{-bc_2(3(c_2 - c_4) + 5c_1^2)}. \quad (3.9)$$

By (3.1)', (3.7) and (3.9) with  $\rho'(0) = \alpha$ , we have an equation

$$\frac{3(3b^2c_2 + 5b^2c_1^2 - 8c_2c_3 - 3b^2c_4)}{-bc_2(3(c_2 - c_4) + 5c_1^2)} = \frac{3b}{2c_2}.$$

Solving it in  $c_4$ , we have

$$c_4 = \frac{9b^2c_2 + 15b^2c_1^2 - 16c_2c_3}{9b^2}.$$

Then we can see the numerator of  $\rho'(t) - Y_2(\rho(t), t)$  is expressed in the form

$$-3b(K_2t^2 + K_3t^3 + K_4t^4 + K_5t^5 + K_6t^6), \quad (3.10)$$

where

$$K_2 = 16c_2c_3(144b^2c_2c_3 + 15b^2c_1^2c_3 - 448c_2c_3^2 + 270bc_2c_6),$$

$$K_3 = -12bc_1c_3(168b^2c_2c_3 + 15b^2c_1^2c_3 - 736c_2c_3^2 + 486bc_2c_6),$$

$$K_4 = 3(32c_2c_3^2(32c_3^2 - 57bc_6) + 30b^2(-32c_1^2c_3^3 + 32c_2c_3^3 + 27c_2c_6^2) + 18b^3c_3c_6(41c_1^2 - 36c_2) + 9b^4c_3^2(5c_1^2 - 24c_2)),$$

$$K_5 = 324b^2c_1(-10bc_3^3 + 9b^2c_3c_6 + 3c_3^2c_6 - 3bc_6^2),$$

$$K_6 = 54b^2c_3(32c_3^3 + 3c_6^2 - 36bc_3c_6 - 36b^2c_3^2 + 36b^3c_6).$$

Since the form (3.10) must be equal to 0, we get

$$c_6 = -\frac{c_3}{270bc_2} \left( 15b^2c_1^2 + 144b^2c_2 - 448c_2c_3 \right)$$

solving  $K_2 = 0$ , because of  $b \neq 0, c_2 > 0, c_3 > 0$ . Replacing  $c_6$  in  $K_3$  by the above value and solving  $K_3 = 0$  in  $c_3$ , we have

$$c_3 = \frac{3b^2(38c_2 + 5c_1^2)}{88c_2}. \tag{3.11}$$

For these values of  $c_3$  and  $c_6$ ,  $K_4$  is

$$K_4 = \frac{135b^8}{1874048c_2^3} \left( -2c_2 + 9c_1^2 \right) \left( 38c_2 + 5c_1^2 \right)^2 \left( 104c_2 + 27c_1^2 \right).$$

Since  $c_1 \neq 0, c_2 > 0, K_4 = 0$  implies

$$c_2 = \frac{9}{2}c_1^2.$$

Then the parameters are

$$\begin{aligned} a = 0, \quad b \neq 0, \quad c_1 \neq 0, \quad c_2 = \frac{9}{2}c_1^2, \quad c_3 = \frac{4}{3}b^2, \\ c_4 = -\frac{9}{2}c_1^2, \quad c_5 = \frac{15}{2}c_1^3, \quad c_6 = \frac{20}{9}b^3 \end{aligned}$$

and  $\rho(t)$  is

$$\rho(t) = \frac{6bt}{18c_1^2 - 3bc_1t + 2b^2t^2}$$

by (3.8). Then

$$\frac{\varphi'(t)}{\varphi(t)} = F_1(\rho(t), \rho'(t), t) = \frac{3b^2t(9c_1 - 2bt)}{(3c_1 - bt)(18c_1^2 - 3bc_1t + 2b^2t^2)}.$$

Integrating it from 0 to  $t$ , we have

$$\log \varphi(t) = 2 \log \frac{18c_1^2 - 3bc_1t + 2b^2t^2}{18c_1^2} - \log \frac{3c_1 - bt}{3c_1}.$$

Thus we conclude (iii) and all statements in the lemma. □

#### 4. MRM-applicable Measures for $h(x) = \frac{1}{(1-x)^2}$

The case  $\kappa = 2$  is a very special case for which we can calculate  $\Theta_\rho(t, s)$  and 'the Hilbert transform' of the measure  $\mu$

$$\tilde{\theta}_0(s) = \text{p.v.} \int \frac{f_\mu(x)}{1-sx} dx \quad \text{for } s \in \mathbb{R}. \tag{4.1}$$

Obviously,

$$\theta_0(s) = \tilde{\theta}_0(s) \quad \text{for } \frac{1}{s} \notin S_\mu$$

holds. Then we can obtain measures  $\mu$  explicitly by 'the inverse Hilbert transform' by applying the method in [8].

It is easy to see that

$$\theta(t) = \theta_0(t) + t\theta'_0(t) = \{t\theta_0(t)\}'. \tag{4.2}$$

and

$$\begin{aligned} \tilde{\theta}(t, s) &= \int \frac{1}{(1-tx)^2(1-sx)^2} d\mu(x) \\ &= \frac{1}{(t-s)^3} (t^2(t-s)\theta(t) - 2t^2s\theta_0(t) + 2ts^2\theta_0(s) + s^2(t-s)\theta(s)) \end{aligned} \tag{4.3}$$

by (1.5). Therefore, we have

$$\tilde{j}(t, s) = \frac{\tilde{\theta}(t, s)}{\theta(t)\theta(s)} = \frac{1}{(t-s)^3} \left( \frac{t^2(t-s)}{\theta(s)} - \frac{2t^2s\theta_0(t)}{\theta(t)\theta(s)} + \frac{2ts^2\theta_0(s)}{\theta(t)\theta(s)} + \frac{s^2(t-s)}{\theta(t)} \right) \tag{4.4}$$

and hence

$$\left. \frac{\partial \tilde{j}(t, s)}{\partial s} \right|_{s=0} = \frac{2}{t} + \frac{1}{t^3} \left( -\theta'(0)t^2 - \frac{2t^2\theta_0(t)}{\theta(t)} \right) = \frac{2}{t} - \theta'(0) - \frac{2\theta_0(t)}{t\theta(t)}. \tag{4.5}$$

Suppose that  $\mu$  is MRM-applicable for  $h(x) = 1/(1-x)^2$  with  $\rho(t)$ . Then  $\Theta_\rho(t, s) = \tilde{j}(\rho(t), \rho(s))$  depends only on  $ts$ , say  $J(ts) = \Theta_\rho(t, s)$ . By (4.5), we have

$$\frac{tJ'(0)}{\rho'(0)} = \frac{1}{\rho'(0)} \left. \frac{\partial \tilde{j}(\rho(t), \rho(s))}{\partial s} \rho'(s) \right|_{s=0} = \frac{2}{\rho(t)} - \theta'(0) - \frac{2\theta_0(\rho(t))}{\rho(t)\theta(\rho(t))} \tag{4.6}$$

and equivalently

$$\frac{J'(0)}{\rho'(0)} \rho^{-1}(t) = \frac{2}{t} - \theta'(0) - \frac{2\theta_0(t)}{t\theta(t)}. \tag{4.6}'$$

Hence

$$\theta_0(s) = \frac{\theta(s)}{2} (2 - (a + b\rho^{-1}(s))s) \tag{4.7}$$

with  $\alpha = \rho'(0)$ ,  $a = \theta'(0)$ ,  $b = J'(0)/\rho'(0)$ . When  $\rho^{-1}(s)$  and  $\theta(s)$  are given, we can calculate  $\theta_0(s)$  and  $\tilde{j}(t, s)$  by (4.7) and (4.4). The equalities (4.3) and (4.4) can be rewritten as

$$\tilde{\theta}(t, s) = \frac{t^2(t-3s + (a + b\rho^{-1}(t))ts)\theta(t) - s^2(s-3t + (a + b\rho^{-1}(s))ts)\theta(s)}{(t-s)^3}, \tag{4.3}'$$

$$\tilde{j}(t, s) = \frac{1}{(t-s)^3} \left( \frac{t^2(t-3s + (a + b\rho^{-1}(t))ts)}{\theta(s)} - \frac{s^2(s-3t + (a + b\rho^{-1}(s))ts)}{\theta(t)} \right). \tag{4.4}'$$

To calculate  $J(ts) = \Theta_\rho(t, s)$  in (1.4), the following expression is convenient:

$$\begin{aligned} \Theta_\rho(t, s) &= \frac{\tilde{\theta}(\rho(t), \rho(s))}{\theta(\rho(t))\theta(\rho(s))} \\ &= \frac{1}{(\rho(t) - \rho(s))^3} \left( \frac{\rho(t)^2(\rho(t) - \rho(s))}{\varphi(s)} - \frac{2\rho(t)^2\rho(s)\varphi_0(t)}{\varphi(t)\varphi(s)} \right. \\ &\quad \left. + \frac{2\rho(t)\rho(s)^2\varphi_0(s)}{\varphi(t)\varphi(s)} + \frac{\rho(s)^2(\rho(t) - \rho(s))}{\varphi(t)} \right), \end{aligned} \tag{4.8}$$

where  $\varphi_0(t) = \theta_0(\rho(t))$  is calculated by

$$\varphi_0(t) = \frac{\varphi(t)}{2}(2 - (a + bt)\rho(t)) \tag{4.9}$$

using (4.7). Appealing (4.7), (4.8) and (4.9), we have the following theorems by Theorems 4.1, 4.2 and 4.3. Those cases are all possible ones, which give probability measures MRM-applicable.

Now we see three typical cases and corresponding three general cases. To state assertions, we introduce *beta distributions*  $\tilde{B}(p, q)$  over  $[-1, 1]$  by

$$d\tilde{\mu}_{p,q}(x) = \tilde{f}_{p,q}(x) dx, \tag{4.10}$$

$$\tilde{f}_{p,q}(x) = \begin{cases} \frac{\Gamma(p+q)}{2^{p+q-1}\Gamma(p)\Gamma(q)}(1+x)^{p-1}(1-x)^{q-1} & \text{if } -1 < x < 1, \\ 0 & \text{otherwise} \end{cases}$$

for  $p, q > 0$ . Let us denote the usual beta distribution over  $[0, 1]$  by  $B(p, q)$ ;

$$d\mu_{p,q}(x) = f_{p,q}(x) dx, \quad f_{p,q}(x) = \begin{cases} \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}x^{p-1}(1-x)^{q-1} & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

**Example 4.1.** Put  $a = 0, b = \frac{4}{3}$  in Theorem 3.1 (i). Then parameters are  $c_1 = 0, c_2 = 1, c_3 = \frac{5}{2}, c_4 = -\frac{3}{2}, c_5 = 0, c_6 = 6$  and functions are

$$\begin{aligned} \rho(t) &= \frac{2t}{1+t^2}, \quad \varphi(t) = (1+t^2)^2, \quad \varphi_0(t) = \frac{1}{3}(3-t^2)(1+t^2), \\ \rho^{-1}(s) &= \frac{s}{1+\sqrt{1-s^2}}, \quad \theta(s) = \frac{4}{(1+\sqrt{1-s^2})^2}, \\ \theta_0(s) &= \frac{4(3-2s^2+3\sqrt{1-s^2})}{3(1+\sqrt{1-s^2})^3}, \quad J(ts) = \Theta_\rho(t, s) = \frac{3-ts}{3(1-ts)^3}, \\ \tilde{\theta}(t, s) &= \frac{4}{3(t-s)^3} \left( \frac{s^2(-3s+5t+4t\sqrt{1-s^2})}{(1+\sqrt{1-s^2})^2} + \frac{t^2(-5s+3t-4s\sqrt{1-t^2})}{(1+\sqrt{1-t^2})^2} \right), \\ f_\mu(x) &= \frac{8}{3\pi}(1+x)^{\frac{3}{2}}(1-x)^{\frac{3}{2}} \quad (\text{beta distribution } \tilde{B}(\frac{5}{2}, \frac{5}{2}) \text{ over } [-1, 1]). \end{aligned}$$

*Proof.* Applying Lemma 3.2 (i), we see  $\rho(t) = \frac{2t}{1+t^2}$  and  $\varphi(t) = (1+t^2)^2$ . Solving  $\rho(t) = s$  in  $t$ , we see  $\rho^{-1}(s) = \frac{s}{1+\sqrt{1-s^2}}$ . Hence

$$\theta(s) = \varphi(\rho^{-1}(s)) = (1+\rho^{-1}(s)^2)^2 = \left( \frac{2}{1+\sqrt{1-s^2}} \right)^2,$$

since

$$1 + \rho^{-1}(s)^2 = 1 + \left( \frac{s}{1+\sqrt{1-s^2}} \right)^2 = \frac{2-s^2+2\sqrt{1-s^2}+s^2}{(1+\sqrt{1-s^2})^2} = \frac{2}{1+\sqrt{1-s^2}}.$$

By (4.7), we have

$$\theta_0(s) = \frac{1}{2} \left( \frac{2}{1 + \sqrt{1 - s^2}} \right)^2 \left( 2 - \frac{4}{3} \frac{s^2}{1 + \sqrt{1 - s^2}} \right) = \frac{4(3 - 2s^2 + 3\sqrt{1 - s^2})}{3(1 + \sqrt{1 - s^2})^3},$$

which satisfies (4.2).  $\varphi_0(t)$  is obtained by (4.9) as

$$\varphi_0(t) = \frac{1}{2}(1 + t^2)^2 \left( 2 - \frac{4t}{3} \frac{2t}{1 + t^2} \right) = \frac{1}{3}(1 + t^2)(3 - t^2).$$

Replacing as  $\rho(t) = 2t/(1 + t^2)$  and  $\varphi(t) = (1 + t^2)^2$  in Equation (4.8), we have

$$\begin{aligned} \Theta_\rho(t, s) &= \frac{1}{3(s-t)^3(1-ts)^3} \left( 4s^2(6s(1+t^2) - 18t(1+s^2) + 16s^2t) \right. \\ &\quad \left. - 4t^2(6t(1+s^2) - 18s(1+t^2) + 16t^2t) \right) \\ &= \frac{8(s-t)^3(3-ts)}{3 \times 8(s-t)^3(1-ts)^3} = \frac{(3-ts)}{3(1-ts)^3}, \end{aligned}$$

since

$$\begin{aligned} \rho(s)^2\varphi(s) &= 4s^2, \\ 3\rho(s) - 9\rho(t) + 4s\rho(s)\rho(t) &= \frac{2(s-3t)(3-ts)}{(1+t^2)(1+s^2)}, \\ (\rho(s) - \rho(t))^3\varphi(t)\varphi(s) &= \frac{(2s(1+t^2) - 2t(1+s^2))^3}{(1+t^2)(1+s^2)} = \frac{8(s-t)^3(1-ts)^3}{(1+t^2)(1+s^2)}. \end{aligned}$$

By (4.3)',  $\theta(s) = 4/(1 + \sqrt{1 - s^2})^2$ ,  $\rho^{-1}(s) = s/(1 + \sqrt{1 - s^2})$  and  $1/(1 + \sqrt{1 - t^2}) = (1 - \sqrt{1 - t^2})/t^2$ ,

$$\begin{aligned} \tilde{\theta}(t, s) &= \frac{1}{3(t-s)^3} \left( t^2 \left( 3t - 9s + \frac{4t^2s}{1 + \sqrt{1 - t^2}} \right) \theta(t) \right. \\ &\quad \left. - s^2 \left( 3s - 9t + \frac{4t^2s}{1 + \sqrt{1 - s^2}} \right) \theta(s) \right) \\ &= \frac{1}{3(t-s)^3} \left( \frac{t^2(3t - 5s - 4s\sqrt{1 - t^2})}{(1 + \sqrt{1 - t^2})^2} - \frac{s^2(3s - 5t - 4t\sqrt{1 - s^2})}{(1 + \sqrt{1 - s^2})^2} \right). \end{aligned}$$

The corresponding measure  $\mu$  can be obtained by 'the inverse Hilbert transform' of  $\theta_0(s)$ . However, here we get it more simply by observing the result in Section 4.7 of [4]. There we showed that  $\tilde{B}(\beta + \frac{1}{2}, \beta + \frac{1}{2})$  is MRM-applicable for  $h_\beta(x) = \frac{1}{(1-x)^\beta}$

with  $\theta(s) = \left( \frac{2}{1 + \sqrt{1 - s^2}} \right)^\beta$ . Hence we see that  $\mu$  is  $\tilde{B}(\frac{5}{2}, \frac{5}{2})$ .  $\square$

**Theorem 4.2.** For  $c_1 = -a$ ,  $c_2 > \frac{1}{2}a^2$ ,  $c_3 = \frac{45}{32}b^2$ ,  $c_4 = -\frac{3}{4}(2c_2 - a^2)$ ,  $c_5 = \frac{3}{4}a(2c_2 - a^2)$ ,  $c_6 = \frac{81}{32}b^3$ , corresponding functions are given as

$$\begin{aligned} \rho(t) &= \frac{24bt}{8(2c_2 - a^2) + 12abt + 9b^2t^2}, \\ \varphi(t) &= \frac{(8(2c_2 - a^2) + 12abt + 9b^2t^2)^2}{64(2c_2 - a^2)^2}, \\ \varphi_0(t) &= \frac{(8(2c_2 - a^2) + 12abt + 9b^2t^2)(8(2c_2 - a^2) - 3b^2t^2)}{64(2c_2 - a^2)^2}, \\ J(ts) &= \Theta_\rho(t, s) = \frac{64(2c_2 - a^2)^2(8(2c_2 - a^2) - 3b^2st)}{(8(2c_2 - a^2) - 9b^2st)^3}, \\ \rho^{-1}(s) &= \frac{4(2c_2 - a^2)s}{3b(2 - as + \sqrt{4 - 4as + (3a^2 - 4c_2)s^2})^2}, \\ \theta(s) &= \frac{16}{3b(2 - as + \sqrt{4 - 4as - (4c_2 - 3a^2)s^2})^2}, \\ \theta_0(s) &= \frac{8(2 - as + 2\sqrt{4 - 4as - (4c_2 - 3a^2)s^2})}{3(2 - as + \sqrt{4 - 4as - (4c_2 - 3a^2)s^2})^2}, \\ \tilde{\theta}(t, s) &= \frac{16}{3(t - s)^3} \left( \frac{s^2(-3s + 5t - ast + 2\sqrt{4 - 4as - (4c_2 - 3a^2)s^2}t)}{(2 - as + \sqrt{4 - 4as - (4c_2 - 3a^2)s^2})^2} \right. \\ &\quad \left. + \frac{t^2(-5s + 3t + ast - 2s\sqrt{4 - 4at - (4c_2 - 3a^2)t^2})}{(2 - at + \sqrt{4 - 4at - (4c_2 - 3a^2)t^2})^2} \right), \\ f_\mu(x) &= \frac{4}{3\pi(2c_2 - a^2)^2} (2(2c_2 - a^2) - (2x - a)^2)^{\frac{3}{2}}. \end{aligned}$$

*Proof.* Let us apply Proposition 2.1 to Example 4.1 with parameters

$$p = \frac{2}{\sqrt{2}\sqrt{2c_2 - a^2}}, \quad q = -\frac{a}{\sqrt{2}\sqrt{2c_2 - a^2}}, \quad r = \frac{3b}{\sqrt{2}\sqrt{2c_2 - a^2}}.$$

Then we have the conclusion. □

**Example 4.3.** Put  $a = 0$ ,  $b = 2$  in Theorem 3.1 (ii). Then parameters are  $c_1 = 0$ ,  $c_2 = \frac{3}{2}$ ,  $c_3 = \frac{9}{2}$ ,  $c_4 = -\frac{3}{2}$ ,  $c_5 = 0$ ,  $c_6 = 12$  and functions are

$$\begin{aligned} \rho(t) &= \frac{2t}{1 + t^2}, \quad \varphi(t) = \frac{(1 + t^2)^2}{1 - t^2}, \quad \varphi_0(t) = 1 + t^2, \\ \rho^{-1}(s) &= \frac{s}{1 + \sqrt{1 - s^2}}, \quad \theta(s) = \frac{2}{\sqrt{1 - s^2}(1 + \sqrt{1 - s^2})}, \quad \theta_0(s) = \frac{2}{1 + \sqrt{1 - s^2}}, \\ J(ts) &= \Theta_\rho(t, s) = \frac{1 + ts}{(1 - ts)^3}, \\ \tilde{\theta}(t, s) &= \frac{2}{(t - s)^3} \left( \frac{t + s - 2t^2s}{\sqrt{1 - t^2}} - \frac{t + s - 2ts^2}{\sqrt{1 - s^2}} \right), \\ f_\mu(x) &= \frac{2}{\pi} \sqrt{1 - x^2} \quad (\text{semi-circle law}). \end{aligned}$$



*Proof.* Applying Lemma 3.2 (ii), we see  $\rho(t) = \frac{2t}{1+t^2}$  and  $\varphi(t) = \frac{(1+t^2)^2}{1-t^2}$ . We see  $\rho^{-1}(s) = \frac{s}{1+\sqrt{1-s^2}}$  as in Example 4.1. Hence

$$\theta(s) = \frac{(1+\rho^{-1}(s)^2)^2}{1-\rho^{-1}(s)^2} = \left(\frac{2}{1+\sqrt{1-s^2}}\right)^2 \frac{1+\sqrt{1-s^2}}{2\sqrt{1-s^2}} = \frac{2}{\sqrt{1-s^2}(1+\sqrt{1-s^2})},$$

since

$$1+\rho^{-1}(s)^2 = \frac{2}{1+\sqrt{1-s^2}}, \quad 1-\rho^{-1}(s)^2 = \frac{2-s^2+2\sqrt{1-s^2}-s^2}{(1+\sqrt{1-s^2})^2} = \frac{2\sqrt{1-s^2}}{1+\sqrt{1-s^2}}.$$

By (4.7), we obtain

$$\theta_0(s) = \frac{1}{2} \frac{2}{\sqrt{1-s^2}(1+\sqrt{1-s^2})} \left(2 - 2 \frac{s^2}{1+\sqrt{1-s^2}}\right) = \frac{2}{1+\sqrt{1-s^2}},$$

which satisfies (4.2).  $\varphi_0(t)$  is obtained by (4.9) as

$$\varphi_0(t) = \frac{1}{2} \frac{(1+t^2)^2}{1-t^2} \left(2 - 2 \frac{2t^2}{1+t^2}\right) = \frac{(1+t^2)^2(1-t^2)}{(1+t^2)(1-t^2)} = 1+t^2.$$

By (4.8) with  $\rho(t) = 2t/(1+t^2)$  and  $\varphi(t) = (1+t^2)^2/(1-t^2)$ ,

$$\begin{aligned} \Theta_\rho(t, s) &= \frac{(1+t^2)(1+s^2)(1-t^2)(1-s^2)}{8(s-t)^3(1-ts)^3} \\ &\quad \times \left( \frac{8s^2(s-3t+s^2t+st^2)}{(1+t^2)(1+s^2)(1-s^2)} - \frac{8t^2(t-3s+t^2s+ts^2)}{(1+t^2)(1+s^2)(1-t^2)} \right) \\ &= \frac{8(s-t)^3(3-ts)}{3 \times 8(s-t)^3(1-ts)^3} = \frac{(1+ts)}{(1-ts)^3}, \end{aligned}$$

since

$$\begin{aligned} \rho(t)^2\varphi(t) &= \frac{4t^2}{1-t^2}, \\ \rho(s) - 3\rho(t) + 2s\rho(s)\rho(t) &= \frac{2(s-3t+s^2+st^2)}{(1+t^2)(1+s^2)}, \\ (\rho(s) - \rho(t))^3\varphi(t)\varphi(s) &= \frac{8(s-t)^3(1-ts)^3}{(1+t^2)(1+s^2)(1-t^2)(1-s^2)}. \end{aligned}$$

By (4.3)' with  $\rho^{-1}(s) = s/(1+\sqrt{1-s^2})$  and  $\theta(s) = 2/(\sqrt{1-s^2}(1+\sqrt{1-s^2}))$ ,

$$\begin{aligned} \tilde{\theta}(t, s) &= \frac{1}{(t-s)^3} \left( t^2(t-3s + \frac{2t^2s}{1+\sqrt{1-t^2}})\theta(t) \right. \\ &\quad \left. - s^2(s-3t + \frac{2t^2s}{1+\sqrt{1-s^2}})\theta(s) \right) \\ &= \frac{2}{(t-s)^3} \left( \frac{t^2(t-s-2s\sqrt{1-t^2})}{\sqrt{1-t^2}(1+\sqrt{1-t^2})} - \frac{s^2(s-t-2t\sqrt{1-s^2})}{\sqrt{1-s^2}(1+\sqrt{1-s^2})} \right) \\ &= \frac{2}{(t-s)^3} \left( \frac{t+s-2t^2s}{\sqrt{1-t^2}} - \frac{t+s-2ts^2}{\sqrt{1-s^2}} \right). \end{aligned}$$

The corresponding measure  $\mu$  can be obtained by 'the inverse Hilbert transform' of  $\theta_0(s) = \frac{2}{1 + \sqrt{1 - s^2}}$ . We showed that it is the semi-circle distribution in §4.6 of [4].  $\square$

**Theorem 4.4.** For  $c_1 = -a$ ,  $c_2 > \frac{1}{2}a^2$ ,  $c_3 = \frac{9}{8}b^2$ ,  $c_4 = -c_2 + 2a^2$ ,  $c_5 = a(2c_2 - a^2)$ ,  $c_6 = \frac{3}{2}b^3$ , corresponding functions are given as

$$\begin{aligned}\rho(t) &= \frac{12bt}{4(2c_2 - a^2) + 6abt + 3b^2t^2}, \\ \varphi(t) &= \frac{(4(2c_2 - a^2) + 6abt + 3b^2t^2)^2}{4(2c_2 - a^2)(4(2c_2 - a^2) - 3b^2t^2)}, \\ \varphi_0(t) &= \frac{4(2c_2 - a^2) + 6abt + 3b^2t^2}{4(2c_2 - a^2)}, \\ J(ts) &= \Theta_\rho(t, s) = \frac{16(2c_2 - a^2)^2(4(2c_2 - a^2) + 3b^2st)}{(4(2c_2 - a^2) - 3b^2st)^3}, \\ \rho^{-1}(s) &= \frac{4(2c_2 - a^2)s}{b(6 - 3as + \sqrt{3}\sqrt{12 - 12as - (8c_2 - 7a^2)s^2})}, \\ \theta(s) &= \frac{24}{12 - 12as - (8c_2 - 7a^2)s^2 + \sqrt{3}(2 - as)\sqrt{12 - 12as - (8c_2 - 7a^2)s^2}}, \\ \theta_0(s) &= \frac{12}{(3(2 - as) + \sqrt{3}\sqrt{12(1 - as) - (8c_2 - 7a^2)s^2})}, \\ \tilde{\theta}(t, s) &= \frac{2\sqrt{3}}{(2c_2 - a^2)(s - t)^3} \\ &\quad \times \left( \frac{(6s + 6t - 9ast - 8c_2st^2 - 3at^2 + 7a^2st^2)}{\sqrt{12 - 12at - (8c_2 - 7a^2)t^2}} \right. \\ &\quad \left. - \frac{(6s + 6t - 9ast - 8c_2s^2t - 3as^2 + 7a^2ts^2)}{\sqrt{12 - 12as - (8c_2 - 7a^2)s^2}} \right) \\ f_\mu(x) &= \frac{\sqrt{3}}{\pi(2c_2 - a^2)} \sqrt{4(2c_2 - a^2) - 3(2x - a)^2}.\end{aligned}$$

*Proof.* Let us apply Proposition 2.1 to Example 4.3 with parameters

$$p = \frac{\sqrt{3}}{\sqrt{2c_2 - a^2}}, \quad q = -\frac{\sqrt{3}a}{2\sqrt{2c_2 - a^2}}, \quad r = \frac{\sqrt{3}b}{2\sqrt{2c_2 - a^2}}.$$

Then we have the conclusion.  $\square$

Let us observe the last case.

**Example 4.5.** Take  $a = \frac{1}{2}$ ,  $b = \frac{3}{2}$ ,  $c_1 = 0$ ,  $c_2 = \frac{3}{2}$ ,  $c_3 = 3$ ,  $c_4 = -\frac{3}{2}$ ,  $c_5 = \frac{9}{4}$ ,  $c_6 = \frac{15}{2}$ . Then

$$\begin{aligned}\rho(t) &= \frac{2t}{1+t^2}, \quad \varphi(t) = \frac{(1+t^2)^2}{1-t}, \quad \varphi_0(t) = \frac{1}{2}(2+t)(1+t^2)^2, \\ \rho^{-1}(s) &= \frac{s}{1+\sqrt{1-s^2}}, \quad \theta(s) = \frac{4}{2-s-s^2+(2-s)\sqrt{1-s^2}}, \\ \theta_0(s) &= \frac{4+5s}{2+2s-s^2+2(1+s)\sqrt{1-s^2}}, \\ J(ts) &= \Theta_\rho(t, s) = \frac{1}{(1-ts)^3}, \\ \tilde{\theta}(t, s) &= \frac{2(1-s)(2+s-t-2ts)\sqrt{1-t^2} - 2(1-t)(2-s+t-2ts)\sqrt{1-s^2}}{(1-t)(1-s)(t-s)^3}, \\ f_\mu(x) &= \frac{2}{\pi}(1+x)^{\frac{3}{2}}(1-x)^{\frac{1}{2}} \quad (\text{beta distribution } \tilde{B}(\frac{5}{2}, \frac{3}{2}) \text{ over } [-1, 1]).\end{aligned}$$

*Proof.* In Lemma 3.2 (iii), put  $b = \frac{3}{2}$ ,  $c_1 = \frac{1}{2}$ . Then

$$a = 0, \quad b = \frac{3}{2}, \quad c_1 = \frac{1}{2}, \quad c_2 = \frac{9}{8}, \quad c_3 = 3, \quad c_4 = -\frac{9}{8}, \quad c_5 = \frac{15}{16}, \quad c_6 = \frac{15}{2},$$

$$\rho(t) = \frac{4t}{2-t+2t^2}, \quad \text{and} \quad \varphi(t) = \frac{(2-t+2t^2)^2}{4(1-t)}.$$

Deform those parameters and functions by  $\tau_{1, -\frac{1}{4}}$ , and assigned them by " $\hat{\phantom{x}}$ ". Then we have

$$\hat{a} = \frac{1}{2}, \quad \hat{b} = \frac{3}{2}, \quad \hat{c}_1 = 0, \quad \hat{c}_2 = \frac{3}{2}, \quad \hat{c}_3 = 3, \quad \hat{c}_4 = -\frac{3}{2}, \quad \hat{c}_5 = \frac{9}{4}, \quad \hat{c}_6 = \frac{15}{2},$$

$$\hat{\rho}(t) = \tau_{1, -\frac{1}{4}}\rho(t) = \frac{2t}{1+t^2} \quad \text{and} \quad \hat{\varphi}(t) = \tau_{1, -\frac{1}{4}}\varphi(t) = \frac{(1+t^2)^2}{1-t}$$

by Proposition 2.1. Obviously,  $\hat{\rho}^{-1}(s) = \frac{s}{1+\sqrt{1-s^2}}$ . Hence

$$\begin{aligned}\hat{\theta}(s) &= \hat{\varphi}(\hat{\rho}^{-1}(s)) = \left(\frac{2}{1+\sqrt{1-s^2}}\right)^2 \frac{1+\sqrt{1-s^2}}{1-s+\sqrt{1-s^2}} \\ &= \frac{4}{2-s-s^2+(2-s)\sqrt{1-s^2}},\end{aligned}$$

$$\begin{aligned}\hat{\theta}_0(s) &= \frac{2}{(1-s)(2+s)+(2-s)\sqrt{1-s^2}} \left(2 - \left(\frac{1}{2} + \frac{3}{2} \frac{2}{1+\sqrt{1-s^2}}\right)\right) \\ &= \frac{4+5s}{2+2s-s^2+2(1+s)\sqrt{1-s^2}}\end{aligned}$$

by (4.7).  $\hat{\theta}_0(s)$  satisfies  $\hat{\theta}(s) = \hat{\theta}_0(s) + s\hat{\theta}'_0(s)$ . By (4.9), we see

$$\hat{\varphi}_0(t) = \frac{(1+t^2)^2}{2(1-t)} \left(2 - \left(\frac{1}{2} + \frac{3}{2}t\right) \frac{2t}{1+t^2}\right) = \frac{1}{2}(2+t)(1+t^2).$$

Now, we obtain the corresponding measure  $\widehat{\mu}$ . 'The Hilbert transform'  $\widetilde{\theta}_0(s)$  of  $\widehat{\mu}$  is given in §5 of [8] as follows. Firstly, let us denote the analytic continuation of  $\widehat{\theta}(s)$  for  $s$  with  $s^{-1} \in \mathbb{C} \setminus S_{\widehat{\mu}}$  by the same symbol  $\widehat{\theta}(s)$ .

$$\widetilde{\theta}_0(s) = \lim_{\epsilon \rightarrow +0} \frac{\widehat{\theta}_0(s + i\epsilon) + \widehat{\theta}_0(s - i\epsilon)}{2} \quad \text{for } s \in \mathbb{R}. \tag{4.11}$$

Actually,

$$\widetilde{\theta}_0(s) = \text{p.v.} \int_{-\infty}^{\infty} \frac{f_{\widehat{\mu}}(x)}{1 - xs} dx.$$

holds. Then the density  $f_{\widehat{\mu}}(x)$  is given by 'the inverse Hilbert transform'

$$f_{\widehat{\mu}}(x) = \text{p.v.} \int_{-\infty}^{\infty} \frac{1}{\pi^2} \frac{\widetilde{\theta}_0(s)}{1 - xs} ds. \tag{4.12}$$

By (4.11)

$$\widetilde{\theta}_0(s) = \begin{cases} \frac{4 + 5s}{2 + 2s - s^2 + 2(1 + s)\sqrt{1 - s^2}} & \text{if } |s| < 1, \\ \frac{2 + 2s - s^2}{s^3} & \text{otherwise,} \end{cases}$$

since

$$\begin{aligned} \widehat{\theta}_0(s) &= \frac{4 + 5s}{2 + 2s - s^2 + 2(1 + s)\sqrt{1 - s^2}} \\ &= \frac{(4 + 5s)(2 + 2s - s^2 - 2(1 + s)\sqrt{1 - s^2})}{(2 + 2s - s^2)^2 - 4(1 + s)^2(1 - s^2)} \\ &= \frac{2 + 2s - s^2 - 2(1 + s)\sqrt{1 - s^2}}{s^3}. \end{aligned}$$

Then for  $|x| > 1$ ,

$$\begin{aligned} \text{p.v.} \int_{|s|>1} \frac{\widetilde{\theta}_0(s)}{1 - xs} ds &= \text{p.v.} \int_{|s|>1} \frac{2 + 2s - s^2}{s^3(1 - xs)} ds \\ &= \lim_{A \rightarrow \infty} \int_1^A \left( \frac{4(1 + x)}{s^2} - \frac{x(1 - 2x - 2x^2)}{1 - xs} - \frac{x(1 - 2x - 2x^2)}{1 + xs} \right) ds \\ &= 4(1 + x) + (1 - 2x - 2x^2) \log \frac{|1 + x|}{|1 - x|}. \end{aligned}$$

Similarly, for  $0 \leq x < 1$ ,

$$\begin{aligned} \text{p.v.} \int_{|s|>1} \frac{\widetilde{\theta}_0(s)}{1 - xs} ds &= \text{p.v.} \int_{|s|>1} \frac{2 + 2s - s^2}{s^3(1 - xs)} ds \\ &= \text{p.v.} \int_{|s|>1} \left( \frac{-1 + 2x + 2x^2}{s} + \frac{2(1 + x)}{s^2} + \frac{2}{s^3} - \frac{x(1 - 2x - 2x^2)}{1 - xs} \right) ds \\ &= \lim_{A \rightarrow \infty, \epsilon \rightarrow 0} \left( \int_0^{\frac{1}{x} - \epsilon} + \int_{\frac{1}{x} + \epsilon}^A \right) \\ &\quad \times \left( \frac{4(1 + x)}{s^2} - \frac{x(1 - 2x - 2x^2)}{1 - xs} - \frac{x(1 - 2x - 2x^2)}{1 + xs} \right) ds \\ &= 4(1 + x) + (1 - 2x - 2x^2) \log \frac{|1 + x|}{|1 - x|}. \end{aligned}$$

For  $|x| < 1$ , we have

$$\begin{aligned}
 & \text{p.v.} \int_{|s|<1} \frac{\tilde{\theta}_0(s)}{1-xs} ds \\
 &= \text{p.v.} \int_{|s|<1} \frac{4+5s}{(2+2s-s^2+2(1+s)\sqrt{1-s^2})(1-xs)} ds \\
 &= \text{p.v.} \int_0^\infty \frac{4+5\frac{u^2-1}{u^2+1}}{\left(2+2\frac{u^2-1}{u^2+1}-\frac{(u^2-1)^2}{(u^2+1)^2}+2\left(1+\frac{u^2-1}{u^2+1}\right)\frac{2u}{u^2+1}\right)} \\
 &\quad \times \frac{1}{\left(1-x\frac{u^2-1}{u^2+1}\right)} \frac{4u}{(u^2+1)^2} du \\
 &= \text{p.v.} \int_0^\infty \frac{4u(3u+1)}{(u+1)^3(1+x+(1-x)u^2)} du \\
 &= \lim_{A \rightarrow \infty} \int_0^A \left( -\frac{2(1-2x-2x^2)}{u+1} - \frac{2(2x+3)}{(u+1)^2} + \frac{4}{(u+1)^3} \right. \\
 &\quad \left. + 2(1-x) \frac{(1-2x-2x^2)u+2(1+x)^2}{1+x+(1-x)u^2} \right) du \\
 &= (1-2x-2x^2) \log \frac{1-x}{1+x} - 4(x+1) + 2\pi(1+x)\sqrt{1-x^2},
 \end{aligned}$$

replacing  $u = \sqrt{\frac{1+x}{1-x}}$ ,  $s = \frac{u^2-1}{u^2+1}$ ,  $ds = \frac{4u}{(u^2+1)^2}$ . For  $x > 1$ , we have similarly

$$\begin{aligned}
 & \text{p.v.} \int_{|s|<1} \frac{\tilde{\theta}_0(s)}{1-xs} ds \\
 &= \lim_{A \rightarrow \infty} \int_0^A \left( -\frac{2(1-2x-2x^2)}{u+1} - \frac{2(2x+3)}{(u+1)^2} \right. \\
 &\quad \left. + \frac{4}{(u+1)^3} + (1-x) \frac{2(1-2x-2x^2)u}{1+x+(1-x)u^2} \right) du \\
 &\quad - \lim_{A \rightarrow \infty, \epsilon \rightarrow +0} \left( \int_0^{\sqrt{\frac{x+1}{x-1}-\epsilon}} + \int_{\sqrt{\frac{x+1}{x-1}+\epsilon}}^A \right) \frac{4(1+x)^2(x-1)}{x+1-(x-1)u^2} du \\
 &= (1-2x-2x^2) \log \frac{1-x}{1+x} - 4(x+1).
 \end{aligned}$$

Thus we obtain

$$f_{\hat{\mu}}(x) = \text{p.v.} \int \frac{\tilde{\theta}_0(s)}{\pi^2(1-xs)} ds = \begin{cases} \frac{2}{\pi}(1+x)\sqrt{1-x^2} & \text{if } |x| < 1, \\ 0 & \text{if } |x| > 1 \end{cases}$$

using (4.12). By (4.8) with  $\hat{\rho}(t) = 2t/(1+t^2)$  and  $\hat{\varphi}(t) = (1+t^2)^2/(1-t)$ ,

$$\begin{aligned}
 \hat{\Theta}_{\hat{\rho}}(t, s) &= \frac{(1+t^2)(1+s^2)(1-t)(1-s)}{8(s-t)^3(1-ts)^3} \\
 &\quad \times \left( \frac{8s^2(s-3t+ts+st^2)}{(1+t^2)(1+s^2)(1-s)} - \frac{8t^2(t-3s+ts+ts^2)}{(1+t^2)(1+s^2)(1-t)} \right)
 \end{aligned}$$

$$= \frac{1}{(1-ts)^3},$$

since

$$\begin{aligned}\widehat{\rho}(t)^2\widehat{\varphi}(t) &= \frac{4t^2}{1-t}, \\ 2\widehat{\rho}(s) - 6\widehat{\rho}(t) + (1+3s)\widehat{\rho}(s)\widehat{\rho}(t) &= \frac{4(s-3t+ts+st^2)}{(1+t^2)(1+s^2)}, \\ (\widehat{\rho}(s) - \widehat{\rho}(t))^3\widehat{\varphi}(t)\widehat{\varphi}(s) &= \frac{8(s-t)^3(1-ts)^3}{(1+t^2)(1+s^2)(1-t)(1-s)}.\end{aligned}$$

$\widetilde{\theta}(t, s)$  can be calculated by (4.3)'. □

**Theorem 4.6.** For  $c_1 \neq -a$ ,  $c_2 = \frac{1}{2}(12a^2 + 20ac_1 + 9c_1^2)$ ,  $c_3 = \frac{4}{3}b^2$ ,  $c_4 = -\frac{3}{2}(2a+c_1)(2a+3c_1)$ ,  $c_5 = \frac{3}{2}(a+c_1)(2a+c_1)(6a+5c_1)$ ,  $c_6 = \frac{20}{9}b^3$ , corresponding functions are given as

$$\begin{aligned}\rho(t) &= \frac{6bt}{18(a+c_1)^2 - 3bc_1t + 2b^2t^2}, \\ \varphi(t) &= \frac{(18(a+c_1)^2 - 3bc_1t + 2b^2t^2)^2}{108(a+c_1)^3(3(a+c_1) - bt)}, \\ \varphi_0(t) &= \frac{(6(a+c_1) + bt)(18(a+c_1)^2 - 3bc_1t + 2b^2t^2)}{108(a+c_1)^3}, \\ J(ts) &= \Theta_\rho(t, s) = \frac{729(a+c_1)^6}{(9(a+c_1)^2 - b^2st)^3}, \\ \rho^{-1}(s) &= \frac{12(a+c_1)^2s}{b\left(2+c_1s + \sqrt{(2-(4a+3c_1)s)(2+(4a+5c_1)s)}\right)}, \\ \theta(s) &= \frac{8}{\left((2+(4a+5c_1)s)(2-(4a+3c_1)s) + (2-(2a+c_1)s)\sqrt{(2-(4a+3c_1)s)(2+(4a+5c_1)s)}\right)}, \\ \theta_0(s) &= \frac{4(2+(5a+6c_1)s)}{\left(4+4(2a+3c_1)s - (8a^2+12ac_1+3c_1^2)s^2 + (2+(4a+5c_1)s)\sqrt{(2-(4a+3c_1)s)(2+(4a+5c_1)s)}\right)}, \\ f_\mu(x) &= \frac{\sqrt{(4a+5c_1+2x)^3(4a+3c_1-2x)}}{16\pi|a+c_1|^3}.\end{aligned}$$

*Proof.* Let us apply Proposition 2.1 to Example 4.5 with parameters

$$p = \frac{1}{2(a+c_1)}, \quad q = \frac{c_1}{4(a+c_1)}, \quad r = \frac{b}{3(a+c_1)}.$$

Then we have the conclusion. □

**5. Tables of Typical MRM-applicable Measures**

We have seen that for the renormalizing function  $h(x) = (1 - x)^{-2}$ , MRM-applicable measures are classified into three classes. Their typical cases are beta distributions over  $[-1, 1]$  (or over  $[0, 1]$ ). They are shown in the following Table 1 and Table 2. Here, we say that a measure  $\mu$  is *typical* in a class, if other measures in the class can be obtained by an action of motion group, i.e. an affine transform, of the measure  $\mu$ .

Table 1 shows the examples in §4 together with generating functions, orthogonal polynomials and Jacobi-Szegö parameters. Generating function  $\psi(t, x)$  is easily obtained by using the equality  $\psi(t, x) = h(\rho(t)x)/\varphi(t)$ . From the generating function, we can calculate orthogonal polynomials  $\{P_n(x)\}$  and Jacobi-Szegö parameters applying the method in [3]. In the table,  $\{C_n^\kappa(x)\}$  are Gegenbauer polynomials.

Beta Distributions over $[-1, 1]$ as Typical Examples for $h(x) = \frac{1}{(1-x)^2}$			
	$a = 0, b = \frac{4}{3},$ $c_1 = 0, c_2 = 1,$ $c_3 = \frac{5}{2}, c_4 = -\frac{3}{2},$ $c_5 = 0, c_6 = 6$	$a = 0, b = 2,$ $c_1 = 0, c_2 = \frac{3}{2},$ $c_3 = \frac{9}{2}, c_4 = -\frac{3}{2},$ $c_5 = 0, c_6 = 12$	$a = \frac{1}{2}, b = \frac{3}{2},$ $c_1 = 0, c_2 = \frac{3}{2},$ $c_3 = 3, c_4 = -\frac{3}{2},$ $c_5 = \frac{9}{4}, c_6 = \frac{15}{2}$
$\rho(t)$	$\frac{2t}{1+t^2}$	$\frac{2t}{1+t^2}$	$\frac{2t}{1+t^2}$
$\varphi(t)$	$(1+t^2)^2$	$\frac{(1+t^2)^2}{1-t^2}$	$\frac{(1+t^2)^2}{1-t}$
$\varphi_0(t)$	$\frac{1}{3}(3-t^2)(1+t^2)$	$1+t^2$	$\frac{1}{2}(2+t)(1+t^2)$
$\psi(t, x)$	$\frac{1}{(1-2xt+t^2)^2}$	$\frac{1-t^2}{(1-2xt+t^2)^2}$	$\frac{1-t}{(1-2xt+t^2)^2}$
$\Theta_\rho(t, s)$	$\frac{3-ts}{3(1-ts)^3}$	$\frac{1+ts}{(1-ts)^3}$	$\frac{1}{(1-ts)^3}$
$\rho^{-1}(s)$	$\frac{s}{1+\sqrt{1-s^2}}$	$\frac{s}{1+\sqrt{1-s^2}}$	$\frac{s}{1+\sqrt{1-s^2}}$
$\theta(s)$	$\frac{4}{(1+\sqrt{1-s^2})^2}$	$\frac{2}{\sqrt{1-s^2}(1+\sqrt{1-s^2})}$	$\frac{4}{(2-s-s^2+(2-s)\sqrt{1-s^2})}$
$\theta_0(t)$	$\frac{4(1+2\sqrt{1-s^2})}{3(1+\sqrt{1-s^2})^2}$	$\frac{2}{1+\sqrt{1-s^2}}$	$\frac{4+5s}{(2+2s-s^2+2(1+s)\sqrt{1-s^2})}$
$\mu$	$\tilde{B}(\frac{5}{2}, \frac{5}{2})$	$\tilde{B}(\frac{3}{2}, \frac{3}{2})$	$\tilde{B}(\frac{5}{2}, \frac{3}{2})$
$f_\mu(x)$	$\frac{8}{3\pi}(1+x)^{\frac{3}{2}}(1-x)^{\frac{3}{2}}$	$\frac{2}{\pi}\sqrt{1-x^2}$	$\frac{2}{\pi}(1+x)^{\frac{3}{2}}(1-x)^{\frac{1}{2}}$
$P_n(x)$	$\frac{1}{2^n(n+1)}C_n^2(x)$	$\frac{1}{2^n}C_n^1(x)$	$\frac{1}{2^n(n+1)}(C_n^2(x) - C_{n-1}^1(x))$
$\alpha_n$	0	0	$\frac{1}{2(n+1)(n+2)}$
$\omega_n$	$\frac{(n+1)(n+4)}{4(n+2)(n+3)}$	$\frac{1}{4}$	$\frac{(n+1)(n+3)}{4(n+2)^2}$

(The numbering of  $\{\omega_n\}$  is different from some of our papers (e.g. [4]).

Table 1

Beta Distributions over [0, 1] as Typical Examples for $h(x) = \frac{1}{(1-x)^2}$			
	$a = 1, b = \frac{2}{3},$ $c_1 = -1, c_2 = \frac{3}{4}, c_3 = \frac{5}{8},$ $c_4 = \frac{9}{8}, c_5 = \frac{3}{8}, c_6 = \frac{3}{4}$	$a = 1, b = 1,$ $c_1 = -1, c_2 = \frac{7}{8}, c_3 = \frac{9}{8},$ $c_4 = \frac{9}{8}, c_5 = \frac{3}{4}, c_6 = \frac{1}{2}$	$a = \frac{5}{4}, b = \frac{3}{4},$ $c_1 = -1, c_2 = \frac{11}{8}, c_3 = \frac{3}{4},$ $c_4 = \frac{9}{8}, c_5 = \frac{45}{32}, c_6 = \frac{15}{16}$
$\rho(t)$	$\frac{4t}{(1+t)^2}$	$\frac{4t}{(1+t)^2}$	$\frac{4t}{(1+t)^2}$
$\varphi(t)$	$(1+t)^4$	$\frac{(1+t)^4}{1-t^2}$	$\frac{(1+t)^4}{1-t}$
$\varphi_0(t)$	$\frac{1}{3}(1+t)^2(3-t^2)$	$(1+t)^2$	$\frac{1}{2}(2+t)(1+t)^2$
$\psi(t, x)$	$\frac{1}{(1+2(1-2x)t+t^2)^2}$	$\frac{1-t^2}{(1+2(1-2x)t+t^2)^2}$	$\frac{1-t}{(1+2(1-2x)t+t^2)^2}$
$\Theta_\rho(t, s)$	$\frac{3-ts}{3(1-ts)^3}$	$\frac{1+ts}{(1-ts)^3}$	$\frac{1}{(1-ts)^3}$
$\rho^{-1}(s)$	$\frac{s}{(1+\sqrt{1-s})^2}$	$\frac{s}{(1+\sqrt{1-s})^2}$	$\frac{s}{(1+\sqrt{1-s})^2}$
$\theta(s)$	$\frac{16}{(1+\sqrt{1-s})^4}$	$\frac{4}{\sqrt{1-s}(1+\sqrt{1-s})^2}$	$\frac{8}{\sqrt{1-s}(1+\sqrt{1-s})^3}$
$\theta_0(t)$	$\frac{8(2-s+4\sqrt{1-s})}{3(1+\sqrt{1-s})^4}$	$\frac{4}{(1+\sqrt{1-s})^2}$	$\frac{2(3+\sqrt{1-s})}{(1+\sqrt{1-s})^3}$
$\mu$	$B(\frac{5}{2}, \frac{5}{2})$	$B(\frac{3}{2}, \frac{3}{2})$	$B(\frac{5}{2}, \frac{3}{2})$
$f_\mu(x)$	$\frac{128}{3\pi} x^{\frac{3}{2}}(1-x)^{\frac{3}{2}}$	$\frac{8}{\pi} \sqrt{x(1-x)}$	$\frac{16}{\pi} x^{\frac{3}{2}}(1-x)^{\frac{1}{2}}$

Table 2

The following Table 3 shows all MRM-applicable measures for the general case, which will be proved separately. A similar discussion is given by Demni [5].

Beta Distributions over [-1, 1] as Typical Examples for $h(x) = \frac{1}{(1-x)^\kappa}$ ( $\kappa \neq 0, 1$ )			
$\kappa$	$\kappa > -\frac{1}{2}$	$\kappa > \frac{1}{2}$	$\kappa > \frac{1}{2}$
$\rho(t)$	$\frac{2t}{1+t^2}$	$\frac{2t}{1+t^2}$	$\frac{2t}{1+t^2}$
$\varphi(t)$	$(1+t^2)^\kappa$	$\frac{(1+t^2)^\kappa}{1-t^2}$	$\frac{(1+t^2)^\kappa}{1-t}$
$\psi(t, x)$	$\frac{1}{(1-2xt+t^2)^\kappa}$	$\frac{1-t^2}{(1-2xt+t^2)^\kappa}$	$\frac{1-t}{(1-2xt+t^2)^\kappa}$
$\rho^{-1}(s)$	$\frac{s}{1+\sqrt{1-s^2}}$	$\frac{s}{1+\sqrt{1-s^2}}$	$\frac{s}{1+\sqrt{1-s^2}}$
$\theta(s)$	$\frac{2^\kappa}{(1+\sqrt{1-s^2})^\kappa}$	$\frac{1}{\sqrt{1-s^2}} \frac{2^{\kappa-1}}{(1+\sqrt{1-s^2})^{\kappa-1}}$	$\frac{1-s+\sqrt{1-s^2}}{2^{\kappa-1}} \times \frac{1}{(1+\sqrt{1-s^2})^{\kappa-1}}$
$\mu$	$\tilde{B}(\kappa + \frac{1}{2}, \kappa + \frac{1}{2})$	$\tilde{B}(\kappa - \frac{1}{2}, \kappa - \frac{1}{2})$	$\tilde{B}(\kappa + \frac{1}{2}, \kappa - \frac{1}{2})$

Table 3



Obviously, the case  $\kappa = 0$  is not appropriate for multiplicative renormalization method. The case  $\kappa = 1/2$  is described in the table also. Actually  $\tilde{B}(1, 1)$  is the uniform measure on  $[-1, 1]$ . The table of typical measures on  $[0, 1]$  obtained by the motion group similarly to Table 2 is omitted here. One mode interacting Fock spaces related to MRM-applicable measures and the case of finitely supported measures will be studied too.

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