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Lower bounds for two-level additive schwarz preconditioners with small overlap

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Lower Bounds for Two-Level Additive Schwarz Preconditioners with Small Overlap *

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Summary. Lower bounds for the condition numbers of the preconditioned systems are obtained for two-level additive Schwarz preconditioners for both second order and fourth order problems. They show that the known upper bounds are sharp in the case of a small overlap.

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1. Introduction

Let $\Omega = (0,1) \times (0,1)$, $V = H_0^1(\Omega)$ for the second order model problem and $H_0^2(\Omega)$ for the fourth order model problem, and the variational form $a(\cdot,\cdot)$ be defined by either

$$a(v_1, v_2) = \int_\Omega \nabla v_1 \cdot \nabla v_2 \, dx \qquad \forall \, v_1, v_2 \in H_0^1(\Omega) \tag{1.1}$$

for the second order case, or

$$a(v_1, v_2) = \int_\Omega \sum_{i,j=1,2} (v_1)_{x_i x_j} (v_2)_{x_i x_j} \, dx \qquad \forall \, v_1, v_2 \in H_0^2(\Omega) \tag{1.2}$$

for the fourth order case.

Consider the following variational problem:
Find $u \in V$ such that

$$a(u, v) = \int_\Omega f v \, dx \qquad \forall \, v \in V \, , \tag{1.3}$$

where $f \in L_2(\Omega)$.

The variational problem (1.3) can be discretized using the P_1 conforming finite element (cf. Figure 1) in the second order case and the Hsieh-Clough-Tocher macro element (cf. Figure 2 and [11]) in the fourth order case. The nodal variables of these elements are depicted in Figure 1 and Figure 2 according to the conventions in [10] and [6].

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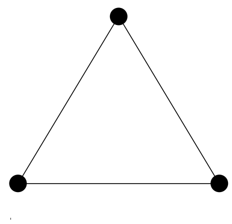

Figure 1.

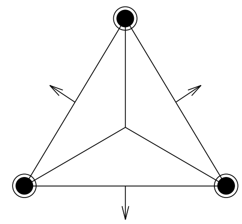

Figure 2.

Anticipating the use of two-level domain decomposition preconditioners, we construct a triangulation of Ω in the following way. Let Ω be divided into $J = 2^{2k}$ nonoverlapping squares $\widehat{\Omega}_1, \ldots, \widehat{\Omega}_J$ (cf. Figure 3 where k=2). By adding a diagonal to each $\widehat{\Omega}_j$ we obtain a triangulation $\mathcal{T}_H$ of Ω (cf. Figure 4). Then we perform a dyadic subdivision of $\mathcal{T}_H$ to obtain the triangulation $\mathcal{T}_h$ (cf. Figure 5). Here H and h are the lengths of the horizontal edges in $\mathcal{T}_H$ and $\mathcal{T}_h$ respectively.

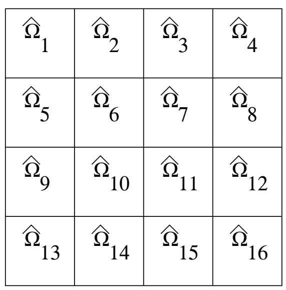

Figure 3.

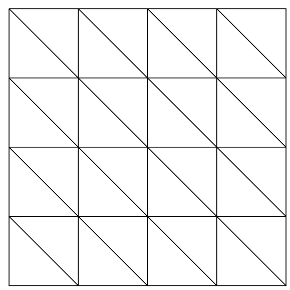

Figure 4.

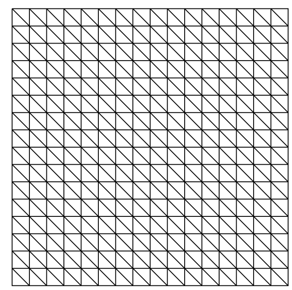

Figure 5.

Let $V_h \subseteq V$ be the finite element space associated with $\mathcal{T}_h$. The discretization of (1.3) is:

Find $u_h \in V_h$ such that

$$a(u_h, v) = \int_\Omega f v \, dx \qquad \forall \, v \in V_h \, . \tag{1.4}$$

Let $A_h : V_h \longrightarrow V_h'$ be the linear operator from V_h to its dual space defined by

$$\langle A_h v_1, v_2 \rangle = a(v_1, v_2) \qquad \forall \, v_1, v_2 \in V_h \, , \tag{1.5}$$

where $\langle \cdot, \cdot \rangle$ is the canonical bilinear form between a vector space and its dual. The operator A_h is symmetric positive definite (SPD) in the sense that $\langle A_h v_1, v_2 \rangle = \langle A_h v_2, v_1 \rangle$ for all $v_1, v_2 \in V_h$ and $\langle A_h v, v \rangle > 0$ for $0 \neq v \in V_h$. Note that if $f_h \in V_h'$ is defined by $\langle f_h, v \rangle = \int_\Omega f v \, dx$ for all $v \in V_h$, then (1.4) can be written as $A_h u_h = f_h$.

The two-level additive Schwarz preconditioner (cf. [8], [21] and the references therein) for A_h is constructed as follows. Let $\widehat{\Omega}_j$ be enlarged in all directions by the amount $\delta = \ell h$ ($\ell \in \mathbb{N}$) and Ω_j be the intersection of this enlarged square with Ω (cf. Figure 6).

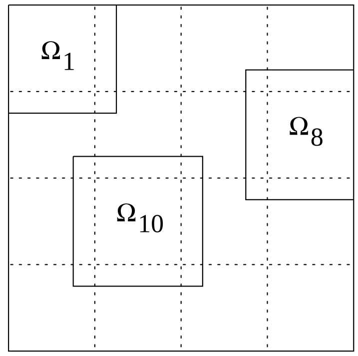

Figure 6.

We define $V_H \subseteq V$ to be the finite element space associated with $\mathcal{T}_H$, and V_j to be the subspace of V_h whose members vanish identically outside Ω_j, for $1 \leq j \leq J$. The SPD operators $A_H : V_H \longrightarrow V_H'$ and $A_j : V_j \longrightarrow V_j'$ are defined by

$$\langle A_H v_1, v_2 \rangle = a(v_1, v_2) \qquad \forall\, v_1, v_2 \in V_H\,, \tag{1.6}$$

$$\langle A_j v_1, v_2 \rangle = a(v_1, v_2) \qquad \forall\, v_1, v_2 \in V_j\,. \tag{1.7}$$

The operators $I_H : V_H \longrightarrow V_h$ and $I_j : V_j \longrightarrow V_h$ are just natural injections, and we denote by $I_H^t : V_h' \longrightarrow V_H'$ and $I_j^t : V_h' \longrightarrow V_j'$ their transposes with respect to the canonical bilinear forms, i.e.,

$$\langle I_H^t \alpha, v \rangle = \langle \alpha, I_H v \rangle \qquad \forall\, \alpha \in V_h', v \in V_H\,, \tag{1.8}$$

$$\langle I_j^t \alpha, v \rangle = \langle \alpha, I_j v \rangle \qquad \forall\, \alpha \in V_h', v \in V_j\,. \tag{1.9}$$

The two-level additive Schwarz preconditioner $B : V_h' \longrightarrow V_h$ is defined by

$$B = I_H A_H^{-1} I_H^t + \sum_{j=1}^{J} I_j A_j^{-1} I_j^t\,. \tag{1.10}$$

It is easy to check that $BA_h : V_h \longrightarrow V_h$ is SPD with respect to the bilinear form $\langle A\cdot, \cdot \rangle = a(\cdot, \cdot)$. It is known (cf. [13], [23]) that for second order problems

$$\kappa(BA_h) \leq C\left(1 + \frac{H}{\delta}\right)\,, \tag{1.11}$$

and for fourth order problems (cf. [4], [3])

$$\kappa(BA_h) \leq C\left(1 + \frac{H}{\delta}\right)^3\,, \tag{1.12}$$

where the (generic) constant C in (1.11) and (1.12) is independent of h, H, J and δ.

In this paper we will show that for $\delta = h$ (*minimal* overlap) the following estimate holds for the second order model problem

$$\kappa(BA_h) \geq c\left(\frac{H}{h}\right), \tag{1.13}$$

while the estimate

$$\kappa(BA_h) \geq c\left(\frac{H}{h}\right)^3 \tag{1.14}$$

holds for the fourth order model problem, where the (generic) positive constant c is independent of h, H and J.

Hence, the known upper bounds are sharp in the case of a small overlap for both second and fourth order problems. We note that the sharpness of (1.11) has already been remarked upon in [13].

The rest of the paper is organized as follows. Section 2 contains some lemmas that are needed in the subsequent sections. We prove the lower bound (1.13) for the second order model problem in Section 3 and the lower bound (1.14) for the fourth order model problem in Section 4.

2. Some Lemmas

First we state an abstract result for additive Schwarz preconditioners. Let $\mathscr{V}$ and $\mathscr{W}_j$, $0 \leq j \leq J$, be finite dimensional vector spaces, and $\mathscr{A} : \mathscr{V} \longrightarrow \mathscr{V}'$ and $\mathscr{B}_j : \mathscr{W}_j \longrightarrow \mathscr{W}'$ be linear SPD operators. Let the vectors spaces be connected by the linear operators $\mathscr{I}_j : \mathscr{W}_j \longrightarrow \mathscr{V}$. Then the additive Schwarz preconditioner $\mathscr{B} : \mathscr{V}' \longrightarrow \mathscr{V}$ is defined by

$$\mathscr{B} = \sum_{j=0}^{J} \mathscr{I}_j \mathscr{B}_j^{-1} \mathscr{I}_j^t ,$$

where $\mathscr{I}_j^t : \mathscr{V}' \longrightarrow \mathscr{W}'$ is the transpose of $\mathscr{I}_j$ with respect to the canonical bilinear forms. We have the following lemma (cf. [17], [19], [20], [12], [24], [14]) on the eigenvalues of $\mathscr{B}\mathscr{A}$.

Lemma 2.1. The operator $\mathscr{B}\mathscr{A}$ is symmetric positive semi-definite with respect to $\langle \mathscr{A}\cdot, \cdot\rangle$. The minimum eigenvalue $\lambda_{\min}(\mathscr{B}\mathscr{A})$ and the maximum eigenvalue $\lambda_{\max}(\mathscr{B}\mathscr{A})$ of $\mathscr{B}\mathscr{A}$ have the following characterizations:

$$\text{(i)} \qquad \lambda_{\min}(\mathscr{B}\mathscr{A}) = \min_{\substack{v \in \mathscr{V} \\ v \neq 0}} \frac{\langle \mathscr{A} v, v\rangle}{\displaystyle\min_{\substack{v=\sum_{j=0}^{J} \mathscr{I}_j w_j \\ w_j \in \mathscr{W}_j}} \sum_{j=0}^{J} \langle \mathscr{B}_j w_j, w_j\rangle},$$

$$\text{(ii)} \qquad \lambda_{\max}(\mathscr{B}\mathscr{A}) = \max_{\substack{v \in \mathscr{V} \\ v \neq 0}} \frac{\langle \mathscr{A} v, v\rangle}{\displaystyle\min_{\substack{v=\sum_{j=0}^{J} \mathscr{I}_j w_j \\ w_j \in \mathscr{W}_j}} \sum_{j=0}^{J} \langle \mathscr{B}_j w_j, w_j\rangle}.$$

Next we state three lemmas concerning discrete norms and semi-norms for finite element spaces. They can all be easily proved by straight-forward calculations and standard scaling arguments. The Sobolev semi-norms in these lemmas are defined by

$$|v|_{H^\ell(G)} = \Big(\int_G \sum_{|\alpha|=\ell} (\partial_x^\alpha v)^2 \, dx \Big)^{1/2} ,$$

where G is an open subset of $\mathbb{R}^n$, $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_n}^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

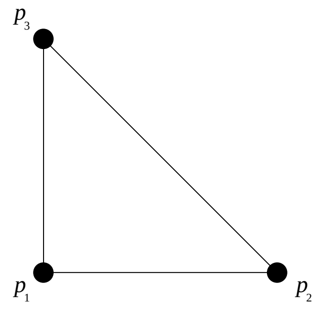

Figure 7.

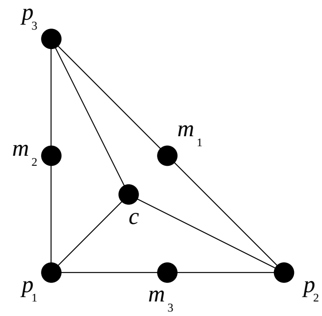

Figure 8.

Lemma 2.2. Let $v(x_1, x_2)$ be a linear polynomial on an isosceles right-angled triangle T with vertices p_1, p_2 and p_3 (cf. Figure 7). Then there exists a positive constant C independent of $\operatorname{diam} T$ and v such that

$$\sum_{j=2,3} \big[v(p_1) - v(p_j) \big]^2 \le C |v|^2_{H^1(T)} .$$

Lemma 2.3. Let $v(x_1, x_2)$ be a C^1 function on an isosceles right-angled triangle T such that v is piecewise cubic with respect to the triangulation formed by the vertices p_i $(1 \le i \le 3)$ and the centroid c of T (cf. Figure 8). Let m_i, $1 \le i \le 3$, be the midpoints of the three sides of T. Then there exists a positive constant C independent of $\operatorname{diam} T$ and v such that

$$\sum_{i=1,2} \sum_{j=2,3} \big[v_{x_i}(p_1) - v_{x_i}(p_j) \big]^2 + \left(\frac{v(p_1) - v(p_2)}{|p_1 p_2|} - \frac{\partial v}{\partial n}(m_2) \right)^2$$
$$+ \left(\frac{v(p_1) - v(p_3)}{|p_1 p_3|} - \frac{\partial v}{\partial n}(m_3) \right)^2 \le C |v|^2_{H^2(T)} ,$$

where $\partial v / \partial n$ denotes the normal derivative of v in the direction of the outer normal.

Lemma 2.4. Let $\mathcal{I}$ be an interval with endpoints p_1 and p_2. Let $P_1(\mathcal{I})$, $P_3(\mathcal{I})$ be respectively the space of linear and cubic polynomials defined on $\mathcal{I}$. Then there exist positive constants C_1 and C_2 independent of $|\mathcal{I}|$ such that

(i) $$\|v\|^2_{L_2(\mathcal{I})} \le C_1 |\mathcal{I}| \sum_{i=1,2} v^2(p_i) \qquad \forall\, v \in P_1(\mathcal{I}) ,$$

(ii) $$\|v\|^2_{L_2(\mathcal{I})} \le C_3 |\mathcal{I}| \sum_{i=1,2} \big[v^2(p_i) + |\mathcal{I}|^2 (v')^2(p_i) \big] \qquad \forall\, v \in P_3(\mathcal{I}) .$$

3. The Second Order Case

In this section we consider the preconditioner B (cf. (1.10)) for the second order model problem, where $V = H_0^1(\Omega)$, $a(\cdot,\cdot)$ is defined by (1.1), and the P_1 conforming finite element is used. The overlap δ is taken to be h, i.e., we consider the case of minimal overlap.

In order to avoid the proliferation of constants, we will henceforth use the notation $A \lesssim B$ (or $B \gtrsim A$) to represent the statement that $A \leq \text{constant} \times B$, where the constant is independent of h, H, J and the variables in A and B. The notation $A \approx B$ means that $A \lesssim B$ and $A \gtrsim B$.

First we apply Lemma 2.1 to obtain a lower bound for $\lambda_{\max}(BA_h)$. In this context we have $\mathscr{V} = V_h$, $\mathscr{W}_0 = V_H$, $\mathscr{W}_j = V_j$ for $1 \leq j \leq J$, $\mathscr{A} = A_h$, $\mathscr{B}_0 = A_H$, $\mathscr{B}_j = A_j$ for $1 \leq j \leq J$, $\mathscr{I}_0 = I_H$, and $\mathscr{I}_j = I_j$ for $1 \leq j \leq J$.

Lemma 3.1. The following estimate holds:

$$\lambda_{\max}(BA_h) \geq 1\,. \tag{3.1}$$

Proof. Let $0 \neq v_* \in V_1$. We have a trivial decomposition of v_*: $v_* = v_H + \sum_{j=1}^J v_j$, where $0 = v_H = v_2 = \cdots = v_J$ and $v_1 = v_*$. It follows from (1.5)–(1.7) and (ii) of Lemma 2.1 that

$$\lambda_{\max}(BA_h) \geq a(v_*,v_*)/\Bigg(\min_{\substack{v_*=v_H+\sum_{j=1}^J v_j\\ v_H\in V_H,\, v_j\in V_j}} \Big[a(v_H,v_H)+\sum_{j=1}^J a(v_j,v_j)\Big]\Bigg) \geq \frac{a(v_*,v_*)}{a(v_*,v_*)} = 1\,.$$

□

By (1.5)–(1.7) and (i) of Lemma 2.1, in order to show that $\lambda_{\min}(BA_h) \lesssim (h/H)$, it suffices to find one function $v_\dagger \in V_h$ such that

$$a(v_\dagger,v_\dagger) \lesssim \left(\frac{h}{H}\right) \min_{\substack{v_\dagger=v_H+\sum_{j=1}^J v_j\\ v_H\in V_H,\, v_j\in V_j}} \Big[a(v_H,v_H)+\sum_{j=1}^J a(v_j,v_j)\Big]\,. \tag{3.2}$$

We will construct $v_\dagger$ as one of the *discrete harmonic* functions associated with the nonoverlapping decomposition $\widehat{\Omega}_1,\ldots,\widehat{\Omega}_J$ (cf. Figure 3).

Let $\Gamma = \left(\bigcup_{j=1}^J \partial\widehat{\Omega}_j\right) \setminus \partial\Omega$ be the *skeleton* of the nonoverlapping decomposition. The subspace $V_h(\Omega\setminus\Gamma)$ of V_h is defined by

$$V_h(\Omega\setminus\Gamma) = \{v \in V_h : v \text{ vanishes on } \Gamma\}\,.$$

The subspace $V_h(\Gamma)$ of V_h is the $a(\cdot,\cdot)$-orthogonal complement of $V_h(\Omega\setminus\Gamma)$, i.e.,

$$V_h(\Gamma) = \{v \in V_h : a(v,w) = 0 \quad \forall\, w \in V_h(\Omega\setminus\Gamma)\}\,. \tag{3.3}$$

The functions in $V_h(\Gamma)$ are known as discrete harmonic functions and they are completely determined by their nodal values along Γ. The property of discrete harmonic functions that we will use is stated in the following lemma, the proof of which can be found in [2] and [22].

Lemma 3.2. The following estimate holds:

$$|v|_{H^1(\widehat{\Omega}_j)} \approx |v|_{H^{1/2}(\partial\widehat{\Omega}_j)} \quad \text{for } 1 \le j \le J \quad \text{and} \quad \forall\, v \in V_h(\Gamma)\,.$$

The fractional order Sobolev semi-norm $|\cdot|_{H^{1/2}(\partial\widehat{\Omega}_j)}$ in Lemma 3.2 is defined by

$$|v|^2_{H^{1/2}(\partial\widehat{\Omega}_j)} = \int_{\partial\widehat{\Omega}_j}\int_{\partial\widehat{\Omega}_j} \frac{|v(x)-v(y)|^2}{|x-y|^2}\, ds(x)\, ds(y)\,,$$

where ds denotes the differential of the arc length.

Let $\mathrm{P}_1\mathrm{P}_2$ be the common boundary of two subdomains $\widehat{\Omega}_{j_1}$ and $\widehat{\Omega}_{j_2}$ which is parallel to the x_1-axis, and Q_1, Q_2 be two points on $\mathrm{P}_1\mathrm{P}_2$ such that $|\mathrm{P}_1\mathrm{Q}_1| = |\mathrm{P}_2\mathrm{Q}_2| = H/4$ (cf. Figure 9).

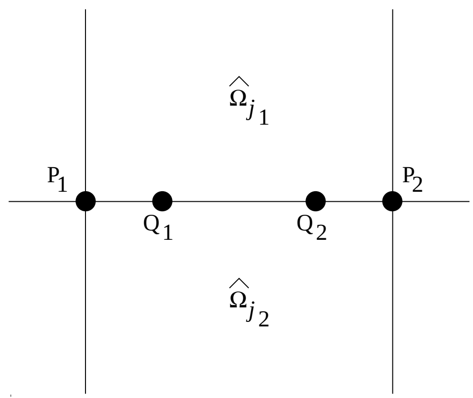

Figure 9.

The restriction to Γ of the function $v_\dagger \in V_h(\Gamma)$ that we are going to construct will vanish outside the line segment $\mathrm{Q}_1\mathrm{Q}_2$. Lemma 3.2 and a simple calculation shows that for such functions the following lemma holds.

Lemma 3.3. Suppose that $v \in V_h(\Gamma)$ and $v\big|_\Gamma$ vanishes outside $\mathrm{Q}_1\mathrm{Q}_2$. Then we have

$$|v|_{H^1(\Omega)} \approx |v|_{H^{1/2}(\mathrm{P}_1\mathrm{P}_2)}\,,$$

where

$$|v|^2_{H^{1/2}(\mathrm{P}_1\mathrm{P}_2)} = \int_{\mathrm{P}_1\mathrm{P}_2}\int_{\mathrm{P}_1\mathrm{P}_2} \frac{|v(x)-v(y)|^2}{|x-y|^2}\, dx_1\, dy_1\,.$$

In view of Lemma 3.3, we can focus our construction to the reference interval $I = [0,1]$. Let $\mathcal{T}_\rho$ be a dyadic subdivision of I with mesh size ρ and $\mathcal{L}_\rho(I)$ be the space of continuous piecewise linear functions on I associated with $\mathcal{T}_\rho$.

Since the dimension of the subspace $\{w \in \mathcal{L}_{1/8}(I) : w = 0 \text{ outside } (1/4, 3/4)\}$ of $\mathcal{L}_{1/8}(I)$ is three, there exists a nontrivial function $\hat{g}$ with the following properties:

(i) $\hat{g} \in \mathcal{L}_{1/8}(I)$,

(ii) $\hat{g} = 0$ outside $(1/4, 3/4)$,

(iii) $\displaystyle\int_{1/4}^{3/4} \hat{g}(x)\,dx = \int_{1/4}^{3/4} x\hat{g}(x)\,dx = 0.$

We denote by α the constant $(|\hat{g}|^2_{H^{1/2}(I)}/|\hat{g}\|^2_{L_2(I)})$, which is of course independent of h, H and J.

The next lemma follows from the construction on I above and a scaling argument.

Lemma 3.4. There exists a continuous function g defined on the line segment $\mathrm{P}_1\mathrm{P}_2$ (cf. Figure 9) which is piecewise linear with respect to the dyadic subdivision induced by $\mathcal{T}_h$, for any $h \leq (H/8)$, and which has the following properties:

$$g \quad \text{vanishes outside the line segment } \mathrm{Q}_1\mathrm{Q}_2 \text{ (cf. Figure 9)}, \tag{3.4}$$

$$\int_{\mathrm{Q}_1\mathrm{Q}_2} g(x)v(x)\,dx_1 = 0 \qquad \text{for any } v \text{ which is a linear polynomial on } \mathrm{P}_1\mathrm{P}_2, \tag{3.5}$$

$$\frac{|g|^2_{H^{1/2}(\mathrm{P}_1\mathrm{P}_2)}}{\|g\|^2_{L_2(\mathrm{Q}_1\mathrm{Q}_2)}} = \frac{\alpha}{H}. \tag{3.6}$$

For $(h/H) \leq (1/8)$, we can now define $v_\dagger \in V_h(\Gamma)$ to be the discrete harmonic function which vanishes everywhere on Γ except the segment $\mathrm{P}_1\mathrm{P}_2$, where it is identical to the function g in Lemma 3.4. It follows from (1.1), Lemma 3.3, (3.4) and (3.6) that

$$a(v_\dagger, v_\dagger) \lesssim \frac{1}{H}(v_\dagger, v_\dagger)_{L_2(\mathrm{Q}_1\mathrm{Q}_2)}. \tag{3.7}$$

Given any decomposition

$$v_\dagger = v_H + \sum_{j=1}^{J} v_j \tag{3.8}$$

where $v_H \in V_H$ and $v_j \in V_j$ for $1 \leq j \leq J$, we have, since the overlap is minimal,

$$(v_\dagger - v_H)\big|_{\mathrm{Q}_1\mathrm{Q}_2} = v_{j_1}\big|_{\mathrm{Q}_1\mathrm{Q}_2} + v_{j_2}\big|_{\mathrm{Q}_1\mathrm{Q}_2}. \tag{3.9}$$

It follows from (3.5) and (3.9) that

$$(v_\dagger, v_\dagger)_{L_2(\mathrm{Q}_1\mathrm{Q}_2)} \leq (v_\dagger - v_H, v_\dagger - v_H)_{L_2(\mathrm{Q}_1\mathrm{Q}_2)} \lesssim \|v_{j_1}\|^2_{L_2(\mathrm{Q}_1\mathrm{Q}_2)} + \|v_{j_2}\|^2_{L_2(\mathrm{Q}_1\mathrm{Q}_2)}. \tag{3.10}$$

Let p_ℓ, $1 \leq \ell \leq L$, be the dyadic subdivision points on $\mathrm{Q}_1\mathrm{Q}_2$ induced by $\mathcal{T}_h$. Part (i) of Lemma 2.4 implies that

$$\|v_{j_1}\|^2_{L_2(\mathrm{Q}_1\mathrm{Q}_2)} + \|v_{j_2}\|^2_{L_2(\mathrm{Q}_1\mathrm{Q}_2)} \lesssim h\sum_{\ell=1}^{L}\left[v^2_{j_1}(p_\ell) + v^2_{j_2}(p_\ell)\right]. \tag{3.11}$$

Since the overlap is minimal, each p_ℓ belongs to a triangle $T_\ell \in \mathcal{T}_h$ where v_{j_2} vanishes at all the vertices except p_ℓ. These triangles appear as the shaded triangles in Figure 10.

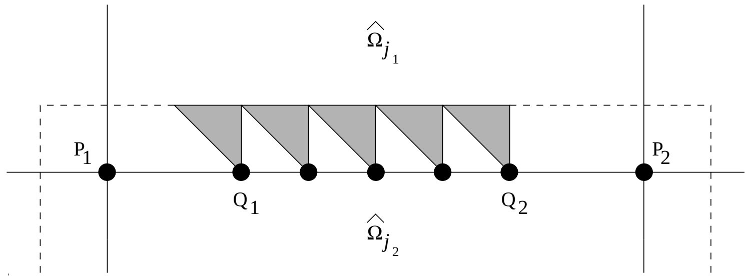

Figure 10.

Hence, by Lemma 2.2, we have

$$\sum_{\ell=1}^{L} v_{j_2}^2(p_\ell) \lesssim \sum_{\ell=1}^{L} |v_{j_2}|_{H^1(T_\ell)}^2 \le |v_{j_2}|_{H^1(\Omega)}^2 = a(v_{j_2}, v_{j_2})\,. \tag{3.12}$$

Similarly we have

$$\sum_{\ell=1}^{L} v_{j_1}^2(p_\ell) \lesssim a(v_{j_1}, v_{j_1})\,. \tag{3.13}$$

Combining (3.7) and (3.10)–(3.13) we find

$$a(v_\dagger, v_\dagger) \lesssim \left(\frac{h}{H}\right) \left[a(v_{j_1}, v_{j_1}) + a(v_{j_2}, v_{j_2})\right]$$

for any decomposition of $v_\dagger$ given by (3.8), which implies (3.2) and hence the next lemma.

Lemma 3.5. For $(h/H) \le (1/8)$ we have

$$\lambda_{\min}(BA_h) \lesssim \left(\frac{h}{H}\right)\,. \tag{3.14}$$

Finally we can establish the estimate (1.13).

Theorem 3.6. There exists a positive constant c independent of h, H and J such that

$$\kappa(BA_h) \ge c\left(\frac{H}{h}\right)$$

holds for the second order model problem in the case of minimal overlap.

Proof. For $(h/H) \le (1/8)$ the estimate follows (3.1) and (3.14). On the other hand, the estimate follows from the trivial estimate $1 \le \kappa(BA_h)$ when $(h/H) \ge (1/8)$. □

Remark 3.7. It is easy to see that Theorem 3.6 can be applied to many other elements and that the estimate

$$\kappa(BA_h) \ge c\left(\frac{H}{\delta}\right) \tag{3.15}$$

is valid under the condition that (δ/h) is bounded. Note also that (3.15) can be extended to the second order model problem on the unit cube $(0,1)^3$.

Remark 3.8. The estimate (3.15) can also be extended to nonconforming finite elements.

Remark 3.9. When H is fixed and the overlap $\delta \approx h$, we have $\kappa(BA_h) \approx h^{-1}$, which is also the estimate for the condition number of the Schur complement in nonoverlapping domain decomposition algorithms for second order problems (cf. [1], [18], [5]).

4. The Fourth Order Case

We consider in this section the fourth order model problem where $V = H_0^2(\Omega)$, $a(\cdot,\cdot)$ is defined by (1.2) and the Hsieh-Clough-Tocher macro element is used. The overlap δ is again taken to be h.

The first lemma is established by the same argument in the proof of Lemma 3.1.

Lemma 4.1. The following estimate holds:

$$\lambda_{\max}(BA_h) \geq 1\,. \tag{4.1}$$

By (i) of Lemma 2.1, in order to show that $\lambda_{\min}(BA_h) \lesssim (h/H)^3$, it suffices to find one function $v_\dagger \in V_h$ such that

$$a(v_\dagger, v_\dagger) \lesssim \left(\frac{h}{H}\right)^3 \min_{\substack{v_\dagger = v_H + \sum_{j=1}^J v_j \\ v_H \in V_H,\, v_j \in V_j}} \Big[a(v_H, v_H) + \sum_{j=1}^J a(v_j, v_j)\Big]\,. \tag{4.2}$$

We will construct $v_\dagger$ as one of the *discrete biharmonic* functions associated with the nonoverlapping decomposition $\widehat{\Omega}_1, \ldots, \widehat{\Omega}_J$ (cf. Figure 3).

Let $\Gamma = \left(\bigcup_{j=1}^J \partial\widehat{\Omega}_j\right) \setminus \partial\Omega$ be the skeleton. The subspace $V_h(\Omega \setminus \Gamma)$ is defined by

$$V_h(\Omega \setminus \Gamma) = \{v \in V_h : v \text{ vanishes to the } \textit{first order} \text{ on } \Gamma\}\,.$$

The subspace $V_h(\Gamma)$ is then defined as in (3.3). The functions in $V_h(\Gamma)$ are known as discrete biharmonic functions and they are completely determined by their nodal values (i.e., derivatives up to order one at the vertices and normal derivatives at the midpoints (cf. Figure 2)) along Γ. The proof of the following property of discrete biharmonic functions can be found in [15], [16] and [7].

Lemma 4.2. The following estimate holds:

$$|v|_{H^2(\widehat{\Omega}_j)} \approx \sum_{i=1,2} |v_{x_i}|_{H^{1/2}(\partial\widehat{\Omega}_j)} \qquad \text{for } 1 \leq j \leq J \quad \text{and} \quad \forall\, v \in V_h(\Gamma)\,.$$

Using Lemma 4.2 and referring to Figure 9, we obtain the following lemma by a simple calculation.

Lemma 4.3. Suppose that $v \in V_h(\Gamma)$ and $v\big|_\Gamma$ vanishes to the first order outside $\mathrm{Q}_1\mathrm{Q}_2$. Then we have

$$|v|_{H^2(\Omega)} \approx \sum_{i=1,2} |v_{x_i}|_{H^{1/2}(\mathrm{P}_1\mathrm{P}_2)} .$$

Note that the restriction of $v \in V_h$ to $\mathrm{P}_1\mathrm{P}_2$ is a C^1 function which is piecewise cubic.

In view of Lemma 4.3 we can again focus our construction to the reference interval $I = [0,1]$. Let $\mathcal{T}_\rho$ be a dyadic subdivision of I with mesh size ρ and $\mathcal{C}_\rho(I)$ be the space of C^1 functions which are piecewise cubic with respect to $\mathcal{T}_\rho$.

Since the dimension of the subspace $\{w \in \mathcal{C}_{1/8}(I) : w = 0 \text{ outside } (1/4, 3/4)\}$ of $\mathcal{C}_{1/8}(I)$ is six, there exists a nontrivial $g \in \mathcal{C}_{1/8}(I)$ with the following properties:

(i) $\hat{g} \in \mathcal{L}_{1/8}(I)$,

(ii) $\hat{g} = 0$ outside $(1/4, 3/4)$,

(iii) $\displaystyle\int_{1/4}^{3/4} x^k \hat{g}(x)\, dx = 0$ for $k = 0, 1, 2, 3$.

We denote by β the constant $(|\hat{g}'|^2_{H^{1/2}(I)} / \|\hat{g}\|^2_{L_2(I)})$, which is independent of h, H and J.

The following lemma is obtained by a scaling argument.

Lemma 4.4. There exists a C^1 function g defined on the line segment $\mathrm{P}_1\mathrm{P}_2$ (cf. Figure 9) which is piecewise cubic with respect to the dyadic subdivision induced by $\mathcal{T}_h$, for any $h \le (H/8)$, and which has the following properties:

$$g \quad \text{vanishes outside the line segment } \mathrm{Q}_1\mathrm{Q}_2 \text{ (cf. Figure 9)}, \tag{4.3}$$

$$\int_{\mathrm{Q}_1\mathrm{Q}_2} g(x) v(x)\, dx_1 = 0 \qquad \text{for any } v \text{ which is a cubic polynomial on } \mathrm{P}_1\mathrm{P}_2 , \tag{4.4}$$

$$\frac{|g_{x_1}|^2_{H^{1/2}(\mathrm{P}_1\mathrm{P}_2)}}{\|g\|^2_{L_2(\mathrm{Q}_1\mathrm{Q}_2)}} = \left(\frac{\beta}{H}\right)^3 . \tag{4.5}$$

For $(h/H) \le (1/8)$ we define $v_\dagger \in V_h(\Gamma)$ to be the discrete biharmonic function which vanishes to the first order everywhere on Γ except the segment $\mathrm{P}_1\mathrm{P}_2$. On $\mathrm{P}_1\mathrm{P}_2$ it satisfies the following conditions:

$$v_\dagger\big|_{\mathrm{P}_1\mathrm{P}_2} = g , \tag{4.6}$$

$$(v_\dagger)_{x_2}\big|_{\mathrm{P}_1\mathrm{P}_2} = 0 , \tag{4.7}$$

where g is the function in Lemma 4.4. It follows from (1.2), Lemma 4.3, (4.3) and (4.5)–(4.7) that

$$a(v_\dagger, v_\dagger) \lesssim \left(\frac{1}{H}\right)^3 (v_\dagger, v_\dagger)_{L_2(\mathrm{Q}_1\mathrm{Q}_2)} . \tag{4.8}$$

Given any decomposition of $v_\dagger$ defined by (3.8), we have, by (4.4),

$$(v_\dagger, v_\dagger)_{L_2(Q_1Q_2)} \leq (v_\dagger - v_H, v_\dagger - v_H)_{L_2(Q_1Q_2)} \lesssim \|v_{j_1}\|^2_{L_2(Q_1Q_2)} + \|v_{j_2}\|^2_{L_2(Q_1Q_2)} . \tag{4.9}$$

Let p_ℓ, $1 \leq \ell \leq L$ be the dyadic subdivision points on Q_1Q_2 induced by $\mathcal{T}_h$. It follows from (ii) of Lemma 2.4 that

$$\begin{aligned} \|v_{j_1}\|^2_{L_2(Q_1Q_2)} + \|v_{j_2}\|^2_{L_2(Q_1Q_2)} &\lesssim h \sum_{\ell=1}^{L} \left[v^2_{j_1}(p_\ell) + v^2_{j_2}(p_\ell)\right] \\ &\quad + h^3 \sum_{\ell=1}^{L} \left[(v_{j_1})^2_{x_1}(p_\ell) + (v_{j_2})^2_{x_1}(p_\ell)\right] . \end{aligned} \tag{4.10}$$

Since the overlap is minimal, each p_ℓ belongs to a triangle $T_\ell \in \mathcal{T}_h$ where all the nodal values of v_{j_2} vanish on the side opposite to p_ℓ (cf. Figure 10). Hence, by Lemma 2.3, we have

$$\sum_{\ell=1}^{L} v^2_{j_2}(p_\ell) \lesssim h^2 \sum_{\ell=1}^{L} |v_{j_2}|^2_{H^2(T_\ell)} \leq h^2 |v_{j_2}|^2_{H^2(\Omega)} = h^2 a(v_{j_2}, v_{j_2}) , \tag{4.11}$$

$$\sum_{\ell=1}^{L} (v_{j_2})^2_{x_1}(p_\ell) \lesssim \sum_{\ell=1}^{L} |v_{j_2}|^2_{H^2(T_\ell)} \leq |v_{j_2}|^2_{H^2(\Omega)} = a(v_{j_2}, v_{j_2}) , \tag{4.12}$$

and similarly,

$$\sum_{\ell=1}^{L} v^2_{j_1}(p_\ell) \lesssim h^2 a(v_{j_1}, v_{j_1}) , \tag{4.13}$$

$$\sum_{\ell=1}^{L} (v_{j_1})^2_{x_1}(p_\ell) \lesssim a(v_{j_1}, v_{j_1}) . \tag{4.14}$$

Combining (4.8)–(4.14) we find

$$a(v_\dagger, v_\dagger) \lesssim \left(\frac{h}{H}\right)^3 \left[a(v_{j_1}, v_{j_1}) + a(v_{j_2}, v_{j_2})\right] ,$$

which implies (4.2) and hence the following lemma.

Lemma 4.5. For $(h/H) \leq (1/8)$ we have

$$\lambda_{\min}(BA_h) \lesssim \left(\frac{h}{H}\right)^3 . \tag{4.15}$$

Using (4.1) and (4.15) and the argument in the proof of Theorem 3.6 we obtain the following theorem.

Theorem 4.6. There exists a positive constant c independent of h, H and J such that

$$\kappa(BA_h) \geq c\left(\frac{H}{h}\right)^3$$

holds for the fourth order model problem in the case of minimal overlap.

Remark 4.7. It is easy to see that Theorem 4.6 can be applied to many other elements and that the estimate

$$\kappa(BA_h) \geq c\left(\frac{H}{\delta}\right)^3 \tag{4.16}$$

is valid under the condition that (δ/h) is bounded. The estimate (4.16) can also be extended to nonconforming finite elements. When H is fixed and the overlap $\delta \approx h$, we have $\kappa(BA_h) \approx h^{-3}$, which is also the estimate for the condition number of the Schur complement in nonoverlapping domain decomposition algorithms for fourth order problems (cf. [9], [5]).

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