GL(1|1) Graph Connections

Andrea Bourque

Follow this and additional works at: https://repository.lsu.edu/honors_etd

Part of the Mathematics Commons

Recommended Citation
Bourque, Andrea, "GL(1|1) Graph Connections" (2022). Honors Theses. 240.
https://repository.lsu.edu/honors_etd/240

This Thesis is brought to you for free and open access by the Ogden Honors College at LSU Scholarly Repository. It has been accepted for inclusion in Honors Theses by an authorized administrator of LSU Scholarly Repository. For more information, please contact ir@lsu.edu.
GL(1|1) Graph Connections

by

Andrea Bourque

Undergraduate honors thesis under the direction of

Dr. Anton Zeitlin

Department of Mathematics

Submitted to the LSU Roger Hadfield Ogden Honors College in partial fulfillment of
the Upper Division Honors Program.

April, 2022

Louisiana State University
& Agricultural and Mechanical College
Baton Rouge, Louisiana
Summary

We establish a complete Gauss decomposition in $GL(1|1)$. This leads to a parametrization of the identity component of $GL(1|1)$. Using this, we construct coordinates on the moduli space of $GL(1|1)$ flat connections on Riemann surfaces using combinatorial structures related to ribbon graphs. We study the mapping class group action via flip transformations. Using the aforementioned coordinates for $GL(1|1)$, we present a convenient solution to the flipped graph connection. Finally, we discuss $gl(1|1)$ and write explicitly the invariant Poisson bivector on the space of $GL(1|1)$ graph connections.
## Contents

1 Introduction 2
   1.1 Acknowledgements 3

2 Supermathematics 4

3 Parametrization of $GL(1|1)$ 6

4 Fatgraphs and Flips 8

5 Graph Connections 11

6 Coordinates on the Moduli Space of $GL(1|1)$ Graph Connections 14

7 Convenient Coordinates for Flips 17

8 Fock-Rosly Poisson Structure 21

9 $gl(1|1)$ 23

10 Poisson Structure on the Space of $GL(1|1)$ Graph Connections 24
1 Introduction

The goal of this paper is to write down explicit formulas for certain aspects of $GL(1|1)$ graph connections. These formulas do not exist in current literature, and we hope that they will be employed by researchers in mathematical physics.

We first review the basics of supermathematics – enough to understand and work with $GL(1|1)$. Then, we discuss how certain geometric constructions can be encoded using combinatorics and algebra.

In particular, we can represent a punctured surface as a fatgraph, also known as a ribbon graph. This fatgraph gives a discrete, or combinatorial, view of the geometry on the surface. For instance, the mapping class group is a powerful invariant of a surface which acts on Teichmüller space. The mapping class group is generated by so called Whitehead flips, which act transitively on ideal triangulations of a surface. These ideal triangulations are dual to a trivalent fatgraph of the surface, so the Whitehead flips also act on fatgraphs. Furthermore, the flips act on graph connections, which are a certain assignment of group elements to the edges of a fatgraph. In order to be able to apply many consecutive flips on a fatgraph, we need convenient coordinates.

We can also use fatgraphs to get information on the moduli space of flat $G$ connections on a surface. It was shown by Fock and Rosly ([1]) that the moduli space of $G$ graph connections on a fatgraph $\tau$ is isomorphic to the moduli space of flat $G$ connections on the punctured surface $F$ corresponding to $\tau$. The equivalence is given by looking at the monodromy around punctures. In a graph connection, the monodromy is a product of group elements along a cycle of the fatgraph. Furthermore, they constructed a Poisson structure on the space of graph connections such that the vertex gauge group acts in a Poisson way, leading to a Poisson structure on the moduli space.

The first result is that there is the following parametrization of the identity component of $GL(1|1)$:

$$(a, b; \alpha, \beta) = a(1 + \frac{1}{2}b^2\alpha\beta) \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b^{-1} & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}.$$  

Using this parametrization, we introduce coordinates on the moduli space of $GL(1|1)$ graph connections. Then, we use the coordinates to write a flip transformation in which the parameters near the flipped edge do not "spread" around the graph. This allows for one to easily perform multiple flips successively and generate the mapping class group. We then give a basis for $\mathfrak{gl}(1|1)$ and relate it to our parametrization of $GL(1|1)$. Then, we write an $r$-matrix using this basis. Finally, we write an invariant Poisson bivector on the space of $GL(1|1)$ graph connections.
1.1 Acknowledgements

This work was done as an undergraduate research project, with a view towards both an Honors thesis and a paper to be published. While it was not supported by a grant, time spent working on the project was supported by a work study stipend. I give great thanks to my advisor, Dr. Anton Zeitlin, who has been guiding me for three years now.
2 Supermathematics

We begin by giving preliminary definitions in supermathematics. Briefly, structures are
given decompositions into "even" and "odd" elements, such that the odd elements have
some sort of opposite symmetry to the even elements. This idea comes from physics, in
which fermions behave in a much different way than bosons. For this reason, even elements
may be called bosonic, and odd elements may be called fermionic. The following definitions
and notations come from [2].

Definition 1. A superalgebra is an algebra $A$ together with a decomposition $A = A_0 \oplus A_1$
such that $A_i A_j \subset A_{i+j}$, with the convention $A_{1+1} = A_0$.

Remark. This is nothing but a $\mathbb{Z}/2\mathbb{Z}$ grading.

Definition 2. Given a superalgebra $A$, elements in $A_i$ are said to be homogeneous of
degree $i$. Degree 0 elements are called even or bosonic, and degree 1 elements are called odd or fermionic.

We use $|X| = i$ to denote that $X$ has degree $i$. A formula involving $|X|$ will implicitly
assume $X$ is homogeneous.

Definition 3. A superalgebra is supercommutative if $XY = (-1)^{|X||Y|}YX$ for all homo-
geneous elements $X, Y$.

In particular, odd elements anti-commute with each other, while even elements com-
mute with everything.

Remark. From now on, we will use the convention that odd elements are denoted by
Greek letters, and even elements are denoted by Latin letters.

Let $\mathbb{R}^{S[N]}$ the real Grassmann algebra with generators $1, \beta[i]$, for $i = 1, 2, ..., N$. The
generators have relations $1 \beta[i] = \beta[i] 1$ and $\beta[i] \beta[j] = - \beta[j] \beta[i]$. In particular, $(\beta[i])^2 = 0$.
We also use the notation $\beta[\lambda] = \beta[\lambda_1] \cdots \beta[\lambda_k]$ for an ordered multi-index $\lambda = \lambda_1, ..., \lambda_k$. Note
that if $\lambda$ has a repeated index, then $\beta[\lambda] = 0$. Then we can identify a multi-index with a
subset of $\{1, 2, ..., N\}$, with the empty set corresponding to the empty product, 1. Thus,
an element in $\mathbb{R}^{S[N]}$ can be written in the form $x = \sum_{\lambda} x_{\lambda} \beta[\lambda]$ for $x_{\lambda} \in \mathbb{R}$ as $\lambda$ runs over all
subsets of $\{1, 2, ..., N\}$.

Definition 4. The degree of a term $x_{\lambda} \beta[\lambda] \in \mathbb{R}^{S[N]}$ is defined as the size of the multi-index $\lambda$.

Thus $\mathbb{R}^{S[N]}$ has a superalgebra structure given by the decomposition $\mathbb{R}^{S[N]} = \mathbb{R}_0^{S[N]} \oplus \mathbb{R}_1^{S[N]}$ into elements which are sums of terms of even (respectively, odd) degree. For example,
$1 + \beta[1,3] \in \mathbb{R}_0^{S[N]}$, $1.2 \beta[10] - 312 \beta[2,4,7] \in \mathbb{R}_1^{S[N]}$, and $(1 + \beta[1,3])(1.2 \beta[10] - 312 \beta[2,4,7]) \in \mathbb{R}_1^{S[N]}$. Since the $\beta$ generators anti-commute, $\mathbb{R}^{S[N]}$ is supercommutative.

Definition 5. The body map $\epsilon : \mathbb{R}^{S[N]} \rightarrow \mathbb{R}$ is the projection of an element onto its
coefficient of 1. The complementary soul map $s : \mathbb{R}^{S[N]} \rightarrow \mathbb{R}^{S[N]}$ is defined by $s(x) = x - \epsilon(x) \cdot 1$. 

– 4 –
Since there are \( N \) anti-commuting generators, we have that \( s(x)^{N+1} = 0 \). Thus \( \epsilon(x) \neq 0 \) if and only if \( x \) is invertible. Explicitly,

\[
x^{-1} = \frac{1}{\epsilon(x)} \left( 1 - \frac{s(x)}{\epsilon(x)} + \left( \frac{s(x)}{\epsilon(x)} \right)^2 + \ldots \left( \frac{s(x)}{\epsilon(x)} \right)^N \right).
\]

In a similar vain, we can use series expansions to define, say, \( \sqrt{x} \) for \( x \) with positive body, as the series will terminate.

**Remark.** Inequalities such as \( x > 0, x < 0, x \neq 0 \) for \( x \in \mathbb{R}^{S[N]} \) will be taken to be inequalities on the body \( \epsilon(x) \).

**Definition 6.** Given \( \mathbb{R}^{S[N]} = \mathbb{R}^{S[N]}_0 \oplus \mathbb{R}^{S[N]}_1 \), the **superspace** \( \mathbb{R}^{[1,1]} \) is defined as \( \mathbb{R}^{S[N]}_0 \times \mathbb{R}^{S[N]}_1 \).

In other words, we have a two-dimensional space with one even coordinate, and one odd coordinate. We can define more generally \( \mathbb{R}^{p|q} = (\mathbb{R}^{S[N]}_0)^p \times (\mathbb{R}^{S[N]}_1)^q \).

**Definition 7.** \( GL(1|1) \) is the group of invertible linear transformations of \( \mathbb{R}^{[1,1]} \). Elements of \( GL(1|1) \) can be identified as supermatrices of the form

\[
\begin{pmatrix}
a & \alpha \\
\beta & b
\end{pmatrix}
\]

for \( a, b \in \mathbb{R}^{S[N]}_0, \alpha, \beta \in \mathbb{R}^{S[N]}_1 \) with \( \epsilon(a), \epsilon(b) \neq 0 \).

**Remark.** In what follows, \( GL(1|1) \) refers to the identity component. In particular, the even entries \( a, b \) will have positive body.

We write the multiplication and inversion of elements in \( GL(1|1) \):

\[
\begin{pmatrix}
a & \alpha \\
\beta & b
\end{pmatrix}
\begin{pmatrix}
c & \gamma \\
\delta & d
\end{pmatrix} = \begin{pmatrix}
ac - \alpha \delta & a\gamma + d\alpha \\
c\beta + d\delta & bd - \beta\gamma
\end{pmatrix},
\]

\[
\begin{pmatrix}
a & \alpha \\
\beta & b
\end{pmatrix}^{-1} = \begin{pmatrix}
a^{-1}(1 - a^{-1}b^{-1}\alpha\beta) & -a^{-1}b^{-1}\alpha \\
-a^{-1}b^{-1}\beta & b^{-1}(1 + a^{-1}b^{-1}\alpha\beta)
\end{pmatrix}.
\]

The identity element of \( GL(1|1) \) is the usual \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

**Remark.** Note the minus signs in the multiplication formula. We use this convention as it matches up with properties of the Lie superalgebra. Other references may define supermatrix multiplication without these extra signs. This is fine, however; the two conventions can be identified with each other.
3 Parametrization of $GL(1|1)$

Our goal is to explicitly describe structures related to $GL(1|1)$ graph connections in convenient coordinates. For this, we introduce a parametrization which is based on a factorization which holds for all elements of $GL(1|1)$.

Any element in $GL(1|1)$ admits the following factorization:

\[
\begin{pmatrix}
  a & \alpha \\ \\
  \beta & b
\end{pmatrix} = \begin{pmatrix}
  1 & \frac{\alpha}{b} \\ \\
  0 & 1
\end{pmatrix} \begin{pmatrix}
  a + \frac{\alpha^2}{b} & 0 \\ \\
  0 & b
\end{pmatrix} \begin{pmatrix}
  1 & 0 \\ \\
  \frac{\beta}{b} & 1
\end{pmatrix}.
\]

Therefore, we can parametrize $GL(1|1)$ with coordinates $a, b, \alpha, \beta$ such that the corresponding element is

\[
\begin{pmatrix}
  1 & \alpha \\ \\
  0 & 1
\end{pmatrix} \begin{pmatrix}
  a & 0 \\ \\
  0 & b
\end{pmatrix} \begin{pmatrix}
  1 & 0 \\ \\
  \beta & 1
\end{pmatrix} = \begin{pmatrix}
  a - b\alpha \beta & b\alpha \\ \\
  b\beta & b
\end{pmatrix}.
\]

There are three types of matrices in this decomposition; we call them type $E^+, H, E^-$ from left to right, respectively. We write commutation relations among the different types:

\[
\begin{pmatrix}
  1 & \alpha \\ \\
  0 & 1
\end{pmatrix} \begin{pmatrix}
  a & 0 \\ \\
  0 & b
\end{pmatrix} \begin{pmatrix}
  1 & 0 \\ \\
  \beta & 1
\end{pmatrix} = (1 + \alpha \beta) \begin{pmatrix}
  a & 0 \\ \\
  0 & b
\end{pmatrix},
\]

\[
\begin{pmatrix}
  a & 0 \\ \\
  0 & b
\end{pmatrix} \begin{pmatrix}
  1 & \alpha \\ \\
  0 & 1
\end{pmatrix} = \begin{pmatrix}
  \frac{\alpha}{b} & 0 \\ \\
  0 & 1
\end{pmatrix} \begin{pmatrix}
  a & 0 \\ \\
  0 & b
\end{pmatrix},
\]

\[
\begin{pmatrix}
  1 & 0 \\ \\
  \beta & 1
\end{pmatrix} \begin{pmatrix}
  a & 0 \\ \\
  0 & b
\end{pmatrix} = \begin{pmatrix}
  a & 0 \\ \\
  0 & b
\end{pmatrix} \begin{pmatrix}
  1 & 0 \\ \\
  \frac{\beta}{b} & 1
\end{pmatrix}.
\]

We also look at the multiplication amongst the types:

\[
\begin{pmatrix}
  1 & \alpha_1 \\ \\
  0 & 1
\end{pmatrix} \begin{pmatrix}
  1 & \alpha_2 \\ \\
  0 & 1
\end{pmatrix} = \begin{pmatrix}
  1 & \alpha_1 + \alpha_2 \\ \\
  0 & 1
\end{pmatrix},
\]

\[
\begin{pmatrix}
  1 & 0 \\ \\
  \beta_1 & 1
\end{pmatrix} \begin{pmatrix}
  1 & 0 \\ \\
  \beta_2 & 1
\end{pmatrix} = \begin{pmatrix}
  1 & 0 \\ \\
  \beta_1 + \beta_2 & 1
\end{pmatrix},
\]

\[
\begin{pmatrix}
  a_1 & 0 \\ \\
  0 & b_1
\end{pmatrix} \begin{pmatrix}
  a_2 & 0 \\ \\
  0 & b_2
\end{pmatrix} = \begin{pmatrix}
  a_1 a_2 & 0 \\ \\
  0 & b_1 b_2
\end{pmatrix}.
\]

Upon inverting an element in this parametrization, there is an extra factor gained:

\[
(1 + \frac{b}{a \alpha \beta}) \begin{pmatrix}
  1 & -\frac{b}{a \alpha} \\ \\
  0 & 1
\end{pmatrix} \begin{pmatrix}
  a^{-1} & 0 \\ \\
  0 & b^{-1}
\end{pmatrix} \begin{pmatrix}
  1 & 0 \\ \\
  -\frac{b}{a \beta} & 1
\end{pmatrix}.
\]

To deal with this extra factor, we instead take a slightly modified parametrization:

\[
a(1 + \frac{1}{2} b^2 \alpha \beta) \begin{pmatrix}
  1 & \alpha \\ \\
  0 & 1
\end{pmatrix} \begin{pmatrix}
  b^{-1} & 0 \\ \\
  0 & b
\end{pmatrix} \begin{pmatrix}
  1 & 0 \\ \\
  \beta & 1
\end{pmatrix}.
\]
Lemma 1. If \( \left( \begin{array}{cc} 1 & \alpha \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} a & 0 \\ 0 & b \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) = \left( \begin{array}{cc} 1 & \gamma \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} c & 0 \\ 0 & d \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ \delta & 1 \end{array} \right) \), then \( a = c, b = d, \alpha = \gamma, \beta = \delta \).

Proof. Multiplying the matrices on each side, we have four equations: \( a - b \alpha \beta = c - d \gamma \delta, ba = d \gamma, b \beta = d \delta, b = d \). Using \( b = d \) in the second and third gives \( \alpha = \gamma, \beta = \delta \). This information in the first equation gives \( a = c \). \( \square \)

Lemma 2. If \( a(1 + \frac{1}{2} b^2 \alpha \beta) \left( \begin{array}{cc} 1 & \alpha \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} b^{-1} & 0 \\ 0 & b \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ \beta & 1 \end{array} \right) = c(1 + \frac{1}{2} d^2 \gamma \delta) \left( \begin{array}{cc} 1 & \gamma \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} d^{-1} & 0 \\ 0 & d \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ \delta & 1 \end{array} \right) \), then \( a = c, b = d, \alpha = \gamma, \beta = \delta \).

Proof. Using Lemma 1, we have immediately \( \alpha = \gamma, \beta = \delta \). Then \( a(1 + \frac{1}{2} b^2 \alpha \beta)b^{\pm 1} = c(1 + \frac{1}{2} d^2 \alpha \beta)d^{\pm 1} \). Multiplying these two equations gives \( a^2(1 + b^2 \alpha \beta) = c^2(1 + d^2 \alpha \beta) \), while dividing the two equations gives \( b^2 = d^2 \). Using \( b^2 = d^2 \) in the first equation gives \( a^2 = c^2 \).

Now we have \( (a - c)(a + c) = 0 \). Since we have assumed (see Section 2) \( \epsilon(a) > 0, \epsilon(c) > 0 \), we have \( \epsilon(a + c) = \epsilon(a) + \epsilon(c) > 0 \) as well. Thus \( a + c \) is invertible, so \( a - c = 0 \). The same argument shows \( b - d = 0 \). \( \square \)

Lemma 3. Any element of \( GL(1|1) \) can be put in the form \( a(1 + \frac{1}{2} b^2 \alpha \beta) \left( \begin{array}{cc} 1 & \alpha \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} b^{-1} & 0 \\ 0 & b \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ \beta & 1 \end{array} \right) \).

Proof. Since we can always put an element in the form \( \left( \begin{array}{cc} 1 & \alpha \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} a & 0 \\ 0 & b \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ \beta & 1 \end{array} \right) \), it suffices to solve the equation
\[
\left( \begin{array}{cc} 1 & \alpha \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} a & 0 \\ 0 & b \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) = c(1 + \frac{1}{2} d^2 \gamma \delta) \left( \begin{array}{cc} 1 & \gamma \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} d^{-1} & 0 \\ 0 & d \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ \delta & 1 \end{array} \right)
\]
for \( c, d, \gamma, \delta \). By Lemma 1, we have \( \gamma = \alpha, \delta = \beta \), and thus \( a = c(1 + \frac{1}{2} d^2 \alpha \beta)d^{-1} \) and \( b = c(1 + \frac{1}{2} d^2 \alpha \beta)d \). Multiplying the equations gives \( ab = c^2(1 + d^2 \alpha \beta) \) and dividing gives \( d^2 = \frac{b}{a} \). Thus \( c^2 = ab(1 - \frac{b}{a} \alpha \beta) \). Since \( a, b \) have positive body, so too do \( c^2, d^2 \). Thus we can take square roots (see Section 2) and solve for \( c, d \) with positive body. \( \square \)

The above lemmas verify that the parametrization both obtains any element of \( GL(1|1) \) and is unique – assuming that we restrict to the connected component of the identity. We denote an element \( a(1 + \frac{1}{2} b^2 \alpha \beta) \left( \begin{array}{cc} 1 & \alpha \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} b^{-1} & 0 \\ 0 & b \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ \beta & 1 \end{array} \right) \) simply by \( (a, b; \alpha, \beta) \). We write the multiplication and inversion in these coordinates:
\[
(a, b; \alpha, \beta)(c, d; \gamma, \delta) = (ac(1 + \frac{1}{2} b^2 \gamma \beta)(1 + \frac{1}{2} b^2 d^2 \alpha \delta), bd; \alpha + b^{-2} \gamma, d^{-2} \beta + \delta),
\]
\[
(a, b; \alpha, \beta)^{-1} = (a^{-1}, b^{-1}; -b^2 \alpha, -b^2 \beta).
\]
4 Fatgraphs and Flips

We consider surfaces $F$ with genus $g \geq 0$ and $s \geq 1$ punctures, such that $2g - 2 + s > 0$. Put another way, the surface is required to have a negative Euler characteristic; $\chi = 2 - 2g - s$ for a punctured surface [3]. This requirement is necessary to have a trivalent fatgraph $\tau$, as we will explain later.

To such a surface $F$, we wish to associate a trivalent fatgraph $\tau$. Abstractly, we can define a fatgraph without any relation to a surface:

**Definition 8.** A fatgraph (also known as ribbon graph) is a graph with a cyclic ordering of the edges at each vertex. An orientation on a fatgraph is an assignment of direction to each edge of the graph.

From such a graph, one obtains a surface by replacing the edges by thin oriented strips and the vertices by disks (i.e. "fattening" the graph), and gluing the strips and disks according to the cyclic ordering. See Figure 1. We are also interested in the reverse process: obtaining a (trivalent) fatgraph from a punctured surface. We follow the exposition in [3].

**Definition 9.** An ideal arc of $F$ is (the isotopy class of) an embedded arc between punctures which is not homotopic to a point relative to the punctures (i.e. keeping the endpoints of the arc fixed throughout the homotopy).

**Definition 10.** An ideal triangulation of $F$ is a collection $\Delta$ of ideal arcs such that each region complementary to $\Delta$ is a triangle with vertices among the punctures. That is, the edges of such triangles are ideal arcs, and the vertices are (not necessarily distinct) punctures.

To any such ideal triangulation, we take the dual graph with each vertex representing a face, and an edge between vertices representing a shared arc between faces. Since the faces are triangles, each vertex will be trivalent. Furthermore, supposing $F$ is oriented, there will be a cyclic ordering inherited at each vertex of this graph. Hence, we have constructed a trivalent fatgraph. We furthermore note that this fatgraph is a deformation retract of the surface itself.

**Remark.** This construction is by no means unique. In fact, there is a method of changing one ideal triangulation into another, called a flip transformation. This will be explained later.

Now that we have seen both directions, we can safely identify a surface with a fatgraph. This is one of the key properties to be exploited in this work. Instead of geometry, we can deal with a discrete structure.

**Lemma 4.** The relationship $2g - 2 + s > 0$ is necessary to have a trivalent fatgraph.

**Proof.** Consider a graph with $e$ edges and $v$ vertices. If the graph is trivalent, then $e = 3v/2$. The Euler characteristic $\chi = 2 - 2g - s$ of $F$ is equal to $v - e = -v/2$. We then have $v = 2(2g - 2 + s)$. Since we have $v > 0$, we are done. \(\square\)
As an explicit example, consider a trivalent fatgraph with $v = 2$. Then $2g + s = 3$, so that we either have $g = 0, s = 3$ – a thrice-punctured sphere – or $g = 1, s = 1$ – a once-punctured torus. These two possible surfaces also correspond to the two possible cyclic orderings on a two-vertex trivalent graph. That is, either the two orderings are in the same direction (giving the sphere), or the opposite direction (giving the torus). See figure 2.

To illustrate the next definition, consider two triangles sharing an edge. They comprise a quadrilateral, with the shared edge being one of the diagonals. Replacing this diagonal by the other results again in two triangles sharing an edge.

**Definition 11.** Let $\Delta$ be an ideal triangulation, and let $e$ be an ideal arc in $\Delta$. Suppose that $e$ separates distinct triangles in $\Delta$, so that $e$ is the diagonal of a quadrilateral. Then a **flip** or **Whitehead move** at $e$ gives the ideal triangulation $\Delta_{e}$ with $e$ replaced by the other diagonal $f$ of the quadrilateral.

This flip transformation also acts on the dual graph of $\Delta$, so that we can talk about a flip of a fatgraph. This corresponds to contracting an edge and extending it out in a different direction. Figure 3 includes both a triangulation and its dual graph transforming under the flip.

Flips are important for a few reasons. First is "Whitehead’s classical fact" [3] that finite sequence of flips act transitively on the collection of ideal triangulations. This plays
into the discrete picture we would like to explore. The second importance of flips is that they give generators for the mapping class group of the surface, which in turn acts on the Teichmuller space of the surface.

Figure 3. Whitehead flip [3].
5 Graph Connections

Definition 12. Given a fatgraph $\tau$ with a chosen orientation and a group $G$, we define a $G$ graph connection on $\tau$ as follows. To each directed edge of $\tau$, we associate an element of $G$. Two graph connections are equivalent if there are group elements at each vertex such that the starting vertex of an edge corresponds to multiplication on the right, and the end vertex of an edge corresponds to multiplication by the inverse on the left. If the orientation of an edge is reversed, the corresponding group element is inverted.

There is another way to define a graph connection without choosing an orientation on $\tau$. Instead of oriented edges, we can consider ends of edges, also known as half-edges. Each half-edge is given a group element such that two half-edges which make a whole edge have inverse group elements [1]. However, the two are clearly equivalent, so we stick to the former.

As an explicit example of equivalent graph connections, we consider Figure 4. The group elements $a, b, c, d, e$ give an equivalent graph connection to $ah_v, bh_v, h_u^{-1}c, h_u^{-1}d, h_u^{-1}eh_u$,
Figure 6. Whitehead flip on a graph connection.

where \( h_u \) and \( h_v \) are the gauge elements at the vertices \( u \) and \( v \) respectively.

We are particularly interested in understanding the action of Whitehead flips on graph connections, as in Figure 7. We have the geometric requirement that the monodromy, given by multiplying elements around loops, is preserved by the flip. In the case of Figure 7, where we identify all the appropriate ends of edges on the outside of the figure, the monodromy equations are as follows:

\[
\begin{align*}
  aec &= a'f'c', \\
  bed &= b'f^{-1}d', \\
  aed &= a'd', \\
  d^{-1}c &= d'^{-1}f'c', \\
  bec &= b'c', \\
  ab^{-1} &= a'fb'^{-1}.
\end{align*}
\]

To "solve the flip" is to find a connection on the flipped fatgraph which can be expressed in terms of the group elements of the original graph connection. It turns out this can be done quite simply.

**Theorem 1.** Consider the flip transformation in Figure 7. Under the gauge transformations given by \( h_{u'} = e'c^{-1} \) and \( h_{v'} = (a')^{-1}a \), we have

\[
\begin{align*}
  a' &\mapsto a & b' &\mapsto be \\
  c' &\mapsto c & d' &\mapsto ed \\
  f &\mapsto e.
\end{align*}
\]

**Proof.** We use both the general rule of gauge transformations in our graph connections, and the monodromy equations. For instance, we have

\[
a' \mapsto a'h_{v'} = a'(a')^{-1}a = a.
\]
Figure 7. General solution of the flip transformation.

Next, 

\[ b' \mapsto b' h_{u'} = b' c' c^{-1} = b e c c^{-1} = b e, \]

where we have used the monodromy equation \( b' c' = b e c. \) The rest of the computations are similar:

\[ c' \mapsto h_{u'}^{-1} c' = c(c')^{-1} c' = c, \]
\[ d' \mapsto h_{u'}^{-1} d' = a^{-1} a' d' = a^{-1} a e d = e d, \]
\[ f \mapsto h_{u'}^{-1} f h_{u'} = a^{-1} a' f c' c^{-1} = a^{-1} a e c c^{-1} = c. \]

The solution is clearly nice, and it works for any group. However, there is a lack of symmetry in the solution. Indeed, we have one direction of the graph staying the same, whereas the other direction gains extra terms. We would prefer a more symmetric solution. It certainly does not seem like this is a possible task when we only have five group elements. However, when the group elements themselves have coordinates, as in the case of a matrix group, we would like to have the flip not changing the coordinates that connect to the rest of the graph. This is the main problem we shall address in the case of \( GL(1|1). \)
6 Coordinates on the Moduli Space of $GL(1|1)$ Graph Connections

We fix a surface $F$ with genus $g \geq 0$ and $s \geq 1$ punctures such that $2g + s - 2 > 0$. We also fix a trivalent fatgraph $\tau \subset F$.

**Definition 13.** Given $F, \tau$ as above, we define a coordinate system $\tilde{C}(F, \tau)$ as follows:

- we assign to each edge in $\tau$ an orientation,
- we assign to each oriented edge $e$ in $\tau$ two positive even coordinates $a, b$ and two odd coordinates $\alpha, \beta$, written as an ordered tuple $(a, b; \alpha, \beta)$;
- reversing the orientation of an edge corresponds to the change of coordinates $(a, b; \alpha, \beta) \mapsto (a^{-1}, b^{-1}, -b^2\alpha, -b^2\beta)$;

**Definition 14.** Let $\vec{c} \in \tilde{C}(F, \tau)$ be a coordinate vector. Suppose $e_1, e_2, e_3$ are edges oriented towards a vertex $u$ such that, for each $i = 1, 2, 3$, $e_i$ has coordinates $(a_i, b_i; \alpha_i, \beta_i)$. Then a vertex rescaling at $u$ is one of three coordinate changes for the coordinates of each $e_i$, where $c$ is positive and even and $\gamma$ is odd:

- $(a_i, b_i; \alpha_i, \beta_i) \mapsto (a_i c^{-1}, b_i; \alpha_i, \beta_i)$;
- $(a_i, b_i; \alpha_i, \beta_i) \mapsto (a_i, b_i c^{-1}; c^2 \alpha_i, \beta_i)$;
- $(a_i, b_i; \alpha_i, \beta_i) \mapsto (a_i (1 + \frac{1}{2} b_i^2 \gamma \beta_i), b_i; \alpha_i - \gamma, \beta_i)$.

If $e_1, e_2, e_3$ are oriented away from a vertex $u$, then there is also a vertex rescaling at $u$ for odd $\gamma$:

- $(a_i, b_i; \alpha_i, \beta_i) \mapsto (a_i (1 - \frac{1}{2} b_i^2 \alpha_i \gamma), b_i; \alpha_i, \beta_i + \gamma)$.

**Definition 15.** Two coordinate vectors in $\tilde{C}(F, \tau)$ are **equivalent** if they are related by a finite number of vertex rescalings and/or reversal of edge orientations.
Theorem 2. Let $C(F, \tau) = \tilde{C}(F, \tau)/\sim$ be the quotient by the equivalence relation from Definition 3. Then $C(F, \tau)$ is in bijection to the moduli space of $GL(1|1)$ graph connections on $\tau$.

Proof. Definition 13 relates to the assignment of group elements to edges with orientation, and Definition 14 relates to the gauge transformations at each vertex. Definition 15 gives the equivalence relation analogous to the equivalence of graph connections.

Explicitly, the coordinates $(a, b; \alpha, \beta)$ correspond to

$$a(1 + \frac{1}{2} b^2 \alpha \beta) \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b^{-1} & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}$$

in $GL(1|1)$. The inverse of such an element is

$$a^{-1}(1 + \frac{1}{2} b^2 \alpha \beta) \begin{pmatrix} 1 & -b^2 \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b^2 \beta & 1 \end{pmatrix},$$

which corresponds to the coordinates $(a^{-1}, b^{-1}; -b^2 \alpha, -b^2 \beta)$. Therefore, the rule for reversing orientations in $C(F, \tau)$ matches the rule for reversing orientations of a graph connection on $\tau$.

Now consider a graph connection on $\tau$. For an edge $e$ directed towards a vertex $u$ in $\tau$, there is a gauge transformation $g_e \mapsto h_u^{-1} g_e$, where $g_e$ is a group element on $e$, and $h_u$ is a group element associated to the vertex $u$. Similarly, if $e$ is directed away from $u$, the gauge transformation is $g_e \mapsto g_e h_u$. This gauge transformation at $u$ acts on each edge adjacent to $u$. In the case of a trivalent $\tau$, there are three edges $e_1, e_2, e_3$ at a given vertex $u$. Then the vertex rescalings of Definition 14 correspond to gauge transformations at $u$, where $h_u$ is one of the following:

- $h_u = c,$
- $h_u = \begin{pmatrix} c^{-1} & 0 \\ 0 & c \end{pmatrix},$
- $h_u = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix},$
- $h_u = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}.$

Again, $c$ is positive even and $\gamma$ is odd. In the first three gauge transformations, we require that $e_1, e_2, e_3$ all point towards $u$, and for the fourth, we require that the edges all point away from $u$. The claim that the vertex rescalings correspond to gauge transformations follows from routine matrix multiplication in $GL(1|1)$. For instance, in the third scenario,
we have
\[
\begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}^{-1} a \left( 1 + \frac{1}{2} b^2 \alpha \beta \right) \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b^{-1} & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}
\]
\[
= a \left( 1 + \frac{1}{2} b^2 \gamma \beta \right) \left( 1 + \frac{1}{2} b^2 (\alpha - \gamma) \beta \right) \begin{pmatrix} 1 & (\alpha - \gamma) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b^{-1} & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}.
\]

This means that the new coordinates are \((a(1 + \frac{1}{2} b^2 \gamma \beta), b; \alpha - \gamma, \beta)\), which is exactly the transformation rule given in Definition 14.

Finally, the definition of the equivalence on \(\tilde{C}(F, \tau)\) corresponds to the equivalence relation on graph connections given by 1) reversing orientations and replacing the element by its inverse, and 2) gauge transformations at vertices. Even though we have only written down four types of gauge transformations, these gauge elements actually generate all possible transformations, since each of the four corresponding elements generate \(GL(1|1)\). 

\(\square\)
7 Convenient Coordinates for Flips

We now use the combinatorial moves to obtain a solution to the flip of a $GL(1|1)$ graph connection. In particular, we obtain a solution which leaves the fermionic ends fixed, as in Figure 9.

By fermionic ends, we refer to the fact that our decomposition into $E^+, H, E^-$ type matrices induce an orientation of the coordinates on an edge, as in Figure 8. Since the $a$ coordinate commutes with everything, it has no specific location on an edge.

We give an overview of the following proof. We first employ the general solution of the flip transformation given in Section 5. Then, one by one, we apply vertex rescalings until we have an equivalent coordinate vector with the correct fermionic entries.
Theorem 3. Let \( \tau' \) represent the flip of \( \tau \) at an edge. Let \( \vec{c} \in C(F, \tau) \) and let \( \vec{c}' \) be the corresponding element in \( C(F, \tau') \). Choose coordinates for \( \vec{c} \) such that \( (a_i, b_i; \alpha_i, \beta_i) \) for \( i = 1, 2, 3, 4, 5 \) represent the edges where the flip occurs; edge \( i = 5 \) is the edge that is flipped. Then there are coordinates for \( \vec{c}' \) such that the corresponding edges \( (a'_i, b'_i; \alpha'_i, \beta'_i) \) for \( i = 1, 2, 3, 4, 5 \) have \( \alpha'_1 = \alpha_1, \alpha'_2 = \alpha_2, \beta'_3 = \beta_3, \beta'_4 = \beta_4 \).

**Proof.** Recall that the product of elements \((a, b; \alpha, \beta)\) and \((c, d; \gamma, \delta)\) is \((ac(1 + \frac{1}{2}\gamma\beta)(1 + \frac{1}{2}b^2d^2\alpha\delta), bd; \alpha + b^{-2}\gamma, d^{-2}\beta + \delta)\).

Applying this formula to the general solution of the flip means that we can first write \( \vec{c}' \) with coordinates

\[
\begin{align*}
(a_1, b_1; \alpha_1, \beta_1), \\
(a_2a_5(1 + \frac{1}{2}\alpha_5\beta_2)(1 + \frac{1}{2}b_2^2b_5^2\alpha_2\beta_5), b_2b_5; \alpha_2 + b_2^{-2}\alpha_5, b_5^{-2}\beta_2 + \beta_5), \\
(a_3, b_3; \alpha_3, \beta_3), \\
(a_4a_5(1 + \frac{1}{2}\alpha_4\beta_5)(1 + \frac{1}{2}b_4^2b_5^2\alpha_5\beta_4), b_4b_5; \alpha_5 + b_5^{-2}\alpha_4, b_4^{-2}\beta_5 + \beta_4), \\
(a_5, b_5; \alpha_5, \beta_5).
\end{align*}
\]

We see that \( \alpha_1 \) and \( \beta_3 \) are already in place, so we do not want to change these. We first change the orientations of the second, fourth, and fifth edges so that we may alter the \( \alpha \) term of the second and the \( \beta \) term of the fourth. This gives

\[
\begin{align*}
(a_2^{-1}a_5^{-1}(1 - \frac{1}{2}\alpha_5\beta_2)(1 - \frac{1}{2}b_2^2b_5^2\alpha_2\beta_5), b_2^{-1}b_5^{-1}; -b_2^2b_5^2\alpha_2 - b_2^2\alpha_5 - b_2^{-2}\beta_2 - b_2^{-2}b_5^2\beta_5), \\
(a_4^{-1}a_5^{-1}(1 - \frac{1}{2}\alpha_4\beta_5)(1 - \frac{1}{2}b_4^2b_5^2\alpha_5\beta_4), b_4^{-1}b_5^{-1}; -b_4^2\alpha_4 - b_4^2b_5^2\alpha_5 - b_4^{-2}\beta_4 - b_4^{-2}b_5^2\beta_5), \\
(a_5^{-1}, b_5^{-1}; -b_5^2\alpha_5, -b_5^2\beta_5).
\end{align*}
\]

Now that the second, third, and fifth edges are pointed towards a vertex, we apply a move which adds \( b_3^2\alpha_5 \) to the coordinates of these edges.

\[
\begin{align*}
(a_2^{-1}a_5^{-1}(1 - \frac{1}{2}\alpha_5\beta_2)(1 - \frac{1}{2}b_2^2b_5^2\alpha_2\beta_5)(1 + \frac{1}{2}\alpha_5(\beta_2 + b_2^2\beta_5)), b_2^{-1}b_5^{-1}; -b_2^2b_5^2\alpha_2 - b_2^2\beta_2 - b_2^{-2}b_3^2\beta_3), \\
(a_3(1 - \frac{1}{2}b_3^2b_5^2\alpha_3\beta_3), b_3; \alpha_3 + b_3^2\alpha_5, \beta_3), \\
(a_5^{-1}(1 + \frac{1}{2}b_3^2b_5^2\alpha_5\beta_5), b_5^{-1}; 0, -b_5^2\beta_5).
\end{align*}
\]

We note that \( (1 - \frac{1}{2}\alpha_5\beta_2)(1 + \frac{1}{2}\alpha_5(\beta_2 + b_2^2\beta_5)) = 1 + \frac{1}{2}b_2^2\alpha_5\beta_5 \) simplifies the coordinates of the second edge. Now, we can invert the second edge again to get \( (a_2a_5(1 - \frac{1}{2}b_2^2\alpha_5\beta_5)(1 + \frac{1}{2}b_2^2b_5^2\alpha_2\beta_5), b_2b_5; \alpha_2, \beta_2 - b_2^{-2}\beta_2). \)

We can now apply a move that adds \( b_3^2\beta_5 \) to the first, fourth, and fifth edge coordinates. This gives
Figure 10. The $b, \alpha, \beta$ coordinates of the fixed ends solution. Here $\beta'_1 = \beta_1 + b_2^2 \beta_5, \beta'_2 = \beta_5 - b_5^{-2} \beta_2, \alpha'_3 = \alpha_3 + b_2^2 \alpha_5, \alpha'_4 = b_5^2 \alpha_4 + \alpha_5$.

\[
(a_1(1 - \frac{1}{2}b_1^2 b_5^2 \alpha_1 \beta_5), b_1; \alpha_1, \beta_1 + b_2^2 \beta_5),
\]
\[
(a_4^{-1} a_5^{-1}(1 - \frac{1}{2} \alpha_4 \beta_5)(1 - \frac{1}{2} b_2^2 b_5^2 \alpha_5 \beta_4)(1 + \frac{1}{2} (\alpha_4 + b_5^2 \alpha_5) \beta_5), b_4^{-1} b_5^{-1} b_2^{-1} \alpha_4 - b_4^2 b_2^2 \alpha_5, -b_4^2 b_5^2 \beta_4),
\]
\[
(a_5^{-1}(1 + \frac{1}{2} b_5^2 \alpha_5 \beta_5), b_5^{-1}; 0, 0).
\]

Once again, there is a simplification $(1 - \frac{1}{2} \alpha_4 \beta_5)(1 + \frac{1}{2} (\alpha_4 + b_5^2 \alpha_5) \beta_5) = 1 + \frac{1}{2} b_5^2 \alpha_5 \beta_5$.

We invert the fourth edge to get $(a_4 a_5(1 + \frac{1}{2} b_2^2 b_5^2 \alpha_5 \beta_4)(1 - \frac{1}{2} b_2^2 \alpha_5 \beta_5), b_4 b_5; b_2^2 \alpha_4 + \alpha_5, \beta_4)$.

We can invert the fifth edge to match the original choice of orientation, giving the final coordinates for $c'$:

\[
(a_1(1 - \frac{1}{2} b_1^2 b_5^2 \alpha_1 \beta_5), b_1; \alpha_1, \beta_1 + b_2^2 \beta_5),
\]
\[
(a_2 a_5(1 + \frac{1}{2} b_2^2 b_5^2 \alpha_2 \beta_3)(1 - \frac{1}{2} b_2^2 \alpha_5 \beta_5), b_2 b_5; \alpha_2, \beta_5 - b_5^{-2} \beta_2),
\]
\[
(a_3(1 - \frac{1}{2} b_3^2 b_5^2 \alpha_3 \beta_5), b_3; \alpha_3 + b_2^2 \alpha_5, \beta_3),
\]
\[
(a_4 a_5(1 + \frac{1}{2} b_4^2 b_5^2 \alpha_4 \beta_4)(1 - \frac{1}{2} b_2^2 \alpha_5 \beta_5), b_4 b_5; b_2^2 \alpha_4 + \alpha_5, \beta_4),
\]
\[
(a_5(1 - \frac{1}{2} b_5^2 \alpha_5 \beta_5), b_5; 0, 0).
\]

These coordinates have the desired fixed fermionic ends.

There are a few attractive features of the obtained solution. For instance, the middle edge is purely bosonic. Another is that the expression $a_5(1 - \frac{1}{2} b_5^2 \alpha_5 \beta_5)$ shows up in three
of the edges, which could be used to simplify the solution even further. For convenience, the solution is presented in Figure 10.
8 Fock-Rosly Poisson Structure

We now divert from the flips to discuss in detail the Poisson structure that can be endowed on the space of graph connections. It can be given in a way such that the gauge transformations at the vertices act in a Poisson way. The quotient space under this action, the moduli space of graph connections, then has a Poisson structure. We first give an overview of the construction from [1].

We fix a surface $F$ with genus $g \geq 0$ and $s \geq 1$ punctures such that $2g + s - 2 > 0$. We also fix a trivalent fatgraph $\tau \subset F$. Let $A^l$ be the space of graph connections on $\tau$. Let $Gl$ be the gauge group; that is, a direct product of the group $G$ with itself, one copy for each vertex in $\tau$.

We have seen that $Gl$ acts on $A^l$, which gives the equivalence of graph connections. In order for the action to be a Poisson map, $Gl$ needs a Poisson-Lie structure. Then $G$ itself needs a Poisson structure. This can be attained by choosing an $r$-matrix, which involves the Lie algebra $g$ of $G$ [5].

Definition 16. An $r$-matrix is a linear map $R : g \to g$ such that the bracket $[X, Y]_R = [X, RY] + [RX, Y]$ for $X, Y \in g$ satisfies the Jacobi identity.

Given a basis of $g$, we can identify $g$ and its dual $g^*$. Furthermore, there is a natural identification of $g \otimes g$ with $\text{Hom}(g^*, g)$. Then we can identify $R \in \text{Hom}(g, g)$ with an $r \in g \otimes g$.

The $r$-matrix gives a Poisson structure on $g$, which can be integrated to a Poisson-Lie structure on the identity component of $G$ [5]. The Poisson-Lie structures on each copy of $G$ need to be compatible in order for $Gl$ to have an appropriate Poisson-Lie structure. It is sufficient to require that the $r$-matrices have the same symmetric part.

Fix a nondegenerate invariant bilinear pairing on $g$. Let $\{e_i\}$ be a basis for $g$ which is orthonormal with respect to the pairing. Each $e_i$ gives rise to a vector field on $G$, $X_i(g) = L_i(g) - R_i(g^{-1})$, where $L_i, R_i$ are the left and right invariant vector fields on $G$ defined by $e_i$. The motivation between this vector field is that it is tangent to the image of $G$ in the map $G \to G \times G, g \mapsto (g, g^{-1})$.

Now, consider the fatgraph $\tau$. For each vertex, we fix an $r$-matrix. Using the basis $\{e_i\}$, we consider it as a matrix with coefficients $r_{ij}$. We require that the symmetric part $\frac{1}{2}(r_{ij} + r_{ji})$ is the same at each vertex.

The final thing needed to write the Poisson structure is to fix a linear order at each vertex of $\tau$, not just a cyclic one. This is called a ciliation in [1]. Since $\tau$ is trivalent, this means each vertex has edges 1, 2, 3 on it. For a vertex of $\tau$, we write the labels of the edges as elements of the vertex. Then the Poisson bivector is given in [1] as

$$B = \sum_{n \in V(\tau)} \left( \sum_{\alpha, \beta \in n, \alpha < \beta} r_{ij}(n) X_\alpha^i \wedge X_\beta^j + \frac{1}{2} \sum_{\alpha \in n} r_{ij}(n) X_\alpha^i \wedge X_\alpha^j \right).$$

Here, $V(\tau)$ are the vertices of $\tau$, and $X_\alpha^i$ is the vector field $X_i$ on the edge $\alpha$ which is incident with $n \in V(\tau)$. It should also be noted that $X \wedge Y = X \otimes Y - Y \otimes X$, without
a $\frac{1}{2}$ factor. Furthermore, there is a relation between vector fields on different vertices. If $\alpha \in n \in V(\tau)$ and $\beta \in m \in V(\tau)$ are both an edge between $m$ and $n$, then $X_{i}^{\alpha} = -X_{i}^{\beta}$. 
Definition 17. A Lie superalgebra is a super vector space together with a bilinear bracket $[\cdot, \cdot]$ such that $|[X, Y]| = |X| + |Y|$, $[X, Y] = -(-1)^{|X||Y|}[Y, X]$, and $(-1)^{|X||Z|}[X, [Y, Z]] + (-1)^{|Z||Y|}[Z, [X, Y]] + (-1)^{|Y||X|}[Y, [Z, X]] = 0$.

$\mathfrak{gl}(1|1)$ is the Lie superalgebra of $GL(1|1)$. Many Lie theoretic results carry over to the super-scenario. Of particular interest to us is that we can integrate $\mathfrak{gl}(1|1)$ to get the connected component of the identity in $GL(1|1)$. In particular, a Poisson structure on $\mathfrak{gl}(1|1)$ can be integrated to a Poisson-Lie group structure on $GL(1|1)$.

We choose a basis for $\mathfrak{gl}(1|1)$ given by $E, N, \Psi^\pm$ such that $E$ is central, $[N, \Psi^\pm] = \pm \Psi^\pm$, and $\{\Psi^+, \Psi^-\} = E$ [6]. Note that we use the anti-commutator $\{\cdot, \cdot\}$ when we are explicitly taking the bracket of two odd elements in the Lie superalgebra. Furthermore, we can identify these elements as matrices

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad \Psi^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \Psi^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. $$

We treat these as usual matrices, not supermatrices. For example, we have

$$\{\Psi^+, \Psi^-\} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = E.$$ 

Under this identification, we can relate the basis of $\mathfrak{gl}(1|1)$ to our parametrization of $GL(1|1)$. The identity of $GL(1|1)$ in our parametrization is $(1, 1; 0, 0)$. Then $(1+da, 1; 0, 0) - (1, 1; 0, 0) = da \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = daE$, so $E = \frac{\partial}{\partial a}$. Using sufficiently small $db$, we can write

$$(1, 1 + db; 0, 0) - (1, 1; 0, 0) = db \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -2dbN, \quad \text{so} \quad -2N = \frac{\partial}{\partial b}. $$

Similarly, we get

$$\frac{\partial}{\partial \alpha} = \Psi^+, \quad \frac{\partial}{\partial \beta} = \Psi^-.$$
10 Poisson Structure on the Space of GL(1|1) Graph Connections

We now apply the previous formalism to the case of GL(1|1) graph connections. The basis \{e_i\} we use is, in order, \(E, N, \Psi^+, \Psi^−\). First, we need an \(r\)-matrix.

**Lemma 5.** \(r \in \mathfrak{gl}(1|1) \otimes \mathfrak{gl}(1|1)\) given by \(r = E \otimes E + N \otimes N - \Psi^+ \otimes \Psi^+ - \Psi^− \otimes \Psi^−\) is an \(r\)-matrix.

**Proof.** \(r\) induces a self-linear map \(R\) on \(\mathfrak{gl}(1|1)\) that maps an element \(X = X_0 + X_1\) to \(X_0 - X_1\), where \(X_i\) denotes the even/odd parts of \(X\) for \(i = 0, 1\) respectively. Let \([X, Y]_R = [X, RY] + [RX, Y]\). Then

\[
[X, Y]_R = [X_0 + X_1, Y_0 - Y_1] + [X_0 - X_1, Y_0 + Y_1] = 2[X_0, Y_0] - 2[X_1, Y_1].
\]

Recall that the bracket of two even elements or two odd elements is even, so \([X, Y]_R\) is always even. It follows that the nested bracket \([X, [Y, Z]]_R = 4[X_0, [Y_0, Z_0] - [Y_1, Z_1]]\).

Furthermore, the bracket of two odd elements in the case of \(\mathfrak{gl}(1|1)\) is either 0 or \(\pm E\), so in any case \([X_0, [Y_1, Z_1]] = 0\). Thus \([X, [Y, Z]]_R = 4[X_0, [Y_0, Z_0]],\) so the Jacobi identity for \([\cdot, \cdot]_R\) follows from the Jacobi identity for \([\cdot, \cdot]\). \(\square\)

Note that this \(r\)-matrix has simple coefficients in our basis: \(r^{11} = r^{22} = -r^{33} = -r^{44} = 1\), with all other terms 0. We also note that the Poisson bivector needs to be slightly modified in the case of dealing with a Lie superalgebra. In particular, if \(X_i\) and \(X_j\) are vector fields corresponding to odd superalgebra elements \(e_i, e_j\), then we replace the wedge product by the symmetric product \(X \circ Y = X \otimes Y + Y \otimes X\).

Since the Poisson structure is given by a bivector which is the sum of bivectors at each vertex, it suffices to write explicitly the bivector at a single vertex. Ordering the edges as 1, 2, 3, we have

\[
P = E^1 \wedge E^2 + E^1 \wedge E^3 + E^2 \wedge E^3 + N^1 \wedge N^2 + N^1 \wedge N^3 + N^2 \wedge N^3
\]
\[
- (\Psi^+ \circ \Psi^2 + \Psi^+ \circ \Psi^3 + \Psi^+ \circ \Psi^+ + \Psi^+ \circ \Psi^+ + \Psi^+ \circ \Psi^- + \Psi^- \circ \Psi^+ + \Psi^- \circ \Psi^-)
\]
\[
- (\Psi^+ \circ \Psi^1 + \Psi^+ \circ \Psi^2 + \Psi^+ \circ \Psi^3 + \Psi^- \circ \Psi^- + \Psi^- \circ \Psi^- + \Psi^- \circ \Psi^-).
\]

Thus, to find the bivector for the whole graph one writes the bivectors at each vertex, makes the necessary relations between vector fields, and then sums. As an explicit example, we compute the Poisson bivectors for the thrice punctured sphere and once punctured torus. In Figure 11, the linear order at each vertex gives the relations \(X_b^1 = -X_a^3, X_b^2 = -X_a^2, X_b^3 = -X_a^1\), where \(a, b\) refer to the two vertices. Since each vector field gets negated, and the terms come in pairs, we suppress the minus signs. Furthermore, all the \(X_b^i\) are in terms of \(X_a^i\), so we suppress the subscript. Thus we can write \(P_b\) by just swapping 1 and 3:

\[
P_b = E^3 \wedge E^2 + E^3 \wedge E^3 + E^2 \wedge E^1 + N^3 \wedge N^2 + N^3 \wedge N^1 + N^2 \wedge N^1
\]
\[
- (\Psi^3 \circ \Psi^2 + \Psi^3 \circ \Psi^3 + \Psi^3 \circ \Psi^1 + \Psi^2 \circ \Psi^+ + \Psi^+ \circ \Psi^+ + \Psi^+ \circ \Psi^- + \Psi^- \circ \Psi^+ + \Psi^- \circ \Psi^-)
\]
\[
- (\Psi^3 \circ \Psi^2 + \Psi^3 \circ \Psi^2 + \Psi^2 \circ \Psi^3 + \Psi^2 \circ \Psi^3 + \Psi^2 \circ \Psi^- + \Psi^- \circ \Psi^- + \Psi^- \circ \Psi^-).
\]
Figure 11. Ciliated fatgraph for thrice punctured sphere.

Figure 12. Ciliated fatgraph for once punctured torus.

Note that all the $E$ and $N$ terms in $P_b$ are negations of the $E$ and $N$ terms in $P_a$, since the wedge is antisymmetric. However, the $\Psi^\pm$ terms stay the same. Thus, the Poisson bivector for the thrice punctured sphere is

\[
P = -2(\Psi^{+1} \circ \Psi^{+2} + \Psi^{+1} \circ \Psi^{+3} + \Psi^{+2} \circ \Psi^{+3} + \Psi^{-1} \circ \Psi^{-2} + \Psi^{-1} \circ \Psi^{-3} + \Psi^{-2} \circ \Psi^{-3})
- 2(\Psi^{+1} \otimes \Psi^{+1} + \Psi^{+2} \otimes \Psi^{+2} + \Psi^{+3} \otimes \Psi^{+3} + \Psi^{-1} \otimes \Psi^{-1} + \Psi^{-2} \otimes \Psi^{-2} + \Psi^{-3} \otimes \Psi^{-3}).
\]

One can do the same procedure for the one punctured torus. From the linear order, the bivectors at each vertex are the same. The total bivector is then

\[
P = 2(E^1 \wedge E^2 + E^1 \wedge E^3 + E^2 \wedge E^3 + N^1 \wedge N^2 + N^1 \wedge N^3 + N^2 \wedge N^3)
- 2(\Psi^{+1} \circ \Psi^{+2} + \Psi^{+1} \circ \Psi^{+3} + \Psi^{+2} \circ \Psi^{+3} + \Psi^{-1} \circ \Psi^{-2} + \Psi^{-1} \circ \Psi^{-3} + \Psi^{-2} \circ \Psi^{-3})
- 2(\Psi^{+1} \otimes \Psi^{+1} + \Psi^{+2} \otimes \Psi^{+2} + \Psi^{+3} \otimes \Psi^{+3} + \Psi^{-1} \otimes \Psi^{-1} + \Psi^{-2} \otimes \Psi^{-2} + \Psi^{-3} \otimes \Psi^{-3}).
\]
References


