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# DOUBLE BARRIER BACKWARD DOUBLY STOCHASTIC DIFFERENTIAL EQUATIONS

TADASHI HAYASHI*

ABSTRACT. We prove the existence and uniqueness of a solution to double barrier backward doubly stochastic differential equations under some appropriate conditions, by means of the penalization method.

## 1. Introduction

Backward stochastic differential equations (BSDEs, for short) were introduced by Peng and Pardoux [23]. The interest in BSDEs comes from their connections with different mathematical fields, such as mathematical finance, stochastic control and partial differential equations. In these fields, numerous results have been obtained, for example, [8], [20], [27], [24], [26], [28] and [29]. We are interested in further extension of BSDEs, that is, double barrier backward doubly stochastic differential equations (DB-BDSDEs, for short).

DB-BDSDEs with the data $(f, g, \xi, L, U)$ are equations with two different directions of stochastic integrals, i.e., the equations involve both a standard (forward) stochsatic integral $dW_t$ and a "backward" stochastic integral $\overleftarrow{dB}_t$, for $t \in [0, T]$,

$$\begin{aligned} Y_t = \xi + & \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g(s, Y_s, Z_s) \overleftarrow{dB}_s - \int_t^T Z_s dW_s \\ & + (K_T^+ - K_t^+) - (K_T^- - K_t^-), \end{aligned} \tag{1.1}$$

$$L_t \le Y_t \le U_t \quad \text{and} \quad \int_0^T (Y_t - L_t) dK_t^+ = \int_0^T (U_t - Y_t) dK_t^- = 0,$$

where $\xi$ is a random variable, $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are two jointly measurable processes, and $W$ and $B$ are two mutually independent standard Brownian motions, with values in $\mathbb{R}$.

This class of equations is a joint version of backward doubly stochastic differential equations (BDSDEs, for short) and double barrier backward stochastic differential equations (DB-BSDEs, for short). The former has been introduced by Pardoux and Peng [25]. They have proved the extended Ito's formula, the connection with a class of systems of quasilinear SPDEs and the existence and uniqueness result of such PDEs. The latter has been done by Bahlali et al. [3], Cvitanić et

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al. [6] and Hamadène et al. [15]. In [3], the existence of a solution to DB-BSDEs is shown under the so-called Mokobodski condition:

*There exist two non-negative supermartingales* $h = (h_t)$ *and* $h' = (h'_t)$ *in* $\mathbb{S}^2([0,T];\mathbb{R})$ *(defined later), such that* $L_t \le h_t - h'_t \le U_t$, $0 \le t \le T$.

Also, in [6] and [15], the existence of a solution to DB-BSDEs is proved by the penalization (mentioned later) method by dealing with the regularity condition ($(B5)$ in Section 3).

A solution of DB-BDSDEs is a quadruple $(Y, Z, K^+, K^-)$ with values in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$ which satisfies (1.1). Here, Karouf [19] has proved the existence and uniqueness of a solution to DB-BDSDEs via the Mokobodski condition as we mention above, and the comparison principle. On the other hand, we have proved it under appropriate conditions including the regularity condition by using the penalization method. Roughly speaking, in our paper, the penalization means that a solution $Y_t$ to DB-BDSDE is approximated by a sequence of Single Barrier-BDSDEs (SB-BDSDEs, for short) $(Y^n_t)_{n\ge 1}$ which have unique solutions (In Section 2, we mention the result of SB-BDSDEs).

Moreover, the connection between SB-BDSDEs and single obstacle Stochastic Partial Differential Equations (SPDEs, for short) has been studied in [2] and [1]. It has been shown by using the idea of stochastic viscosity solution. Therefore, we have the belief that the existence of a stochastic viscosity solution to double obstacle SPDEs would be expressible as the limit of a sequence of stochastic viscosity solutions to single obstacle SPDEs corresponding to a sequence of SB-BDSDEs. Consequently, we think that our topic is interesting and meaningful.

The paper is organized as follows. In Section 2, we give some settings, or notations, definitions and assumptions. Moreover, we introduce single barrier backward doubly stochastic differential equations. In Section 3, we show our main result by proving several lemmas.

## 2. Settings and Result of SB-BDSDEs

**2.1. Settings.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $T > 0$ be fixed throughout this paper. Let $\{W_t, 0 \le t \le T\}$ and $\{B_t, 0 \le t \le T\}$ be two mutually independent standard Brownian motion processes, with values in $\mathbb{R}$, defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{N}$ denote the class of $\mathbb{P}$-null sets of $\mathcal{F}$. For each $t \in [0,T]$, we define $\mathcal{F}_t := \mathcal{F}^W_t \vee \mathcal{F}^B_{t,T}$. For any process $\{\eta_t; t \in [0,T]\}$ and any $0 \le s \le t \le T$, $\mathcal{F}^\eta_{s,t} = \sigma\{\eta_r - \eta_s; s \le r \le t\} \vee \mathcal{N}$ and $\mathcal{F}^\eta_t = \mathcal{F}^\eta_{0,t}$.

Note that $\{\mathcal{F}^W_{0,t}\}$ is increasing and $\{\mathcal{F}^B_{t,T}\}$ is decreasing, and the collection $\{\mathcal{F}_t; t \in [0,T]\}$ is neither increasing nor decreasing. So it does not constitute a filtration. Now, let us introduce some spaces:

We denote by $\mathbb{L}^2$ the space of $\mathbb{R}$-valued and $\mathcal{F}_T$-measurable random variable $\xi$ such that

$$||\xi||^2 = \mathbb{E}\left[|\xi|^2\right] < +\infty,$$

by $\mathbb{S}^2([0,T];\mathbb{R})$ the space of $\mathbb{R}$-valued and $\mathcal{F}_t$-measurable random processes $Y$ such that

$$||Y||^2 = \mathbb{E}\left[\sup_{0\le t\le T}|Y_t|^2\right] < +\infty,$$

and by $\mathbb{H}^2([0,T];\mathbb{R})$ the space of $\mathbb{R}$-valued and $\mathcal{F}_t$-measurable random processes $Z$ such that

$$||Z||^2 = \mathbb{E}\left[\int_0^T |Z_t|^2 dt\right] < +\infty.$$

**2.2. Result – SB-BDSDEs.** In this subsection, we present a result of the existence and uniqueness for single barrier backward doubly stochastic differential equations with the data $(f,g,\xi,L)$.

Now, we set the following assumptions:
(A1) The terminal condition $\xi \in \mathbb{L}^2$.
(A2) The functions $f:\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $g:\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are two jointly measurable processes, such that for any $(y,z)\in\mathbb{R}\times\mathbb{R}$,

$$f(\cdot,y,z),\ g(\cdot,y,z)\in\mathbb{H}^2.$$

Moreover, we assume that there exist constants $C>0$ and $0<\alpha<1$ such that for any $(\omega,t)\in\Omega\times[0,T],(y_1,z_1),(y_2,z_2)\in\mathbb{R}\times\mathbb{R}$,

$$|f(t,y_1,z_1)-f(t,y_2,z_2)|^2 \le C(|y_1-y_2|^2+|z_1-z_2|^2),$$
$$|g(t,y_1,z_1)-g(t,y_2,z_2)|^2 \le C|y_1-y_2|^2+\alpha|z_1-z_2|^2.$$

(A3) The reflecting barrier $L$ is $\mathcal{F}_t$-measurable and continuous real-valued processes which satisfy

$$\mathbb{E}\left[\sup_{0\le t\le T}|L_t^+|^2\right] < \infty,$$

where $L^+$ is positive part of $L$.
(A4) The barrier $L$ satisfies $L_T \le \xi$.

**Definition 2.1.** A *solution* for SB-BDSDE is a triplet $(Y,Z,K)$ in $\mathbb{S}^2([0,T];\mathbb{R})\times\mathbb{H}^2([0,T];\mathbb{R})\times\mathbb{S}^2([0,T];\mathbb{R}^+)$ and which satisfies

$$\begin{aligned} Y_t = \xi + &\int_t^T f(s,Y_s,Z_s)ds + \int_t^T g(s,Y_s,Z_s)\overleftarrow{dB}_s \\ -&\int_t^T Z_s dW_s + K_T - K_t, \end{aligned} \tag{2.1}$$
$$L_t \le Y_t \quad \text{and} \quad \int_0^T (Y_t - L_t)dK_t = 0.$$

The following result established by Aman et al. [2] and [1] for the existence and uniqueness of a solution to the SB-BDSDE (2.1) .

**Theorem 2.2.** *Under the assumptions* (A1) - (A4), *the SB-BDSDE* (2.1) *has a unique solution.*

## 3. Double Barrier Backward Doubly Stochastic Differential Equations

In this section, under the assumptions on $f$, $g$, $\xi$, $L$ and $U$ given below, we show the existence and uniqueness of the solution of the DB-BDSDE (1.1) via the penalization method (see Essaky et al. [10], Hamadène et al. [16] and Marzougue et al. [22]).

Here, we set the following assumptions:

(B1) The terminal condition $\xi \in \mathbb{L}^2$.

(B2) The functions $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are two jointly measurable processes, such that for any $(y,z) \in \mathbb{R} \times \mathbb{R}$,

$$f(\cdot, y, z),\ g(\cdot, y, z) \in \mathbb{H}^2.$$

Moreover, we assume that there exist constants $C > 0$ and $0 < \alpha < 1$ such that for any $(\omega, t) \in \Omega \times [0,T], (y_1, z_1), (y_2, z_2) \in \mathbb{R} \times \mathbb{R}$,

$$|f(t, y_1, z_1) - f(t, y_2, z_2)|^2 \le C(|y_1 - y_2|^2 + |z_1 - z_2|^2),$$

$$|g(t, y_1, z_1) - g(t, y_2, z_2)|^2 \le C|y_1 - y_2|^2 + \alpha|z_1 - z_2|^2,$$

and for any $(t, y, z) \in [0,T] \times \mathbb{R} \times \mathbb{R}$, $\mathbb{P}$-a.s.,

$$|f(t, y, z)| \le C(1 + |y| + |z|).$$

(B3) The two reflecting barriers $L$ and $U$ are $\mathcal{F}_t$-measurable and continuous real-valued processes which satisfy

$$\mathbb{E}\left[\sup_{0\le t\le T} |L_t^+|^2\right] + \mathbb{E}\left[\sup_{0\le t\le T} |U_t^-|^2\right] < \infty,$$

where $L^+$ and $U^-$ are positive and negative parts of $L$ and $U$, respectively.

(B4) The barriers $L$ and $U$ satisfy $L_t \le U_t$, $L_T \le \xi \le U_T$.

(B5) The barrier $U$ is *regular*, i.e., there exists a sequence of $\{U^n\}_{n\ge 1}$ such that

$$(i) \quad \forall t \le T, U^n \ge U^{n+1},\ \lim_{n\to\infty} U_t^n = U_t,$$

$$(ii) \quad \forall n \ge 1,\ \forall t \le T, U_t^n = U_0^n + \int_0^t a_s^n ds + \int_0^t b_s^n dW_s,$$

where $U_0^n \in \mathbb{L}^2$, $a^n$ and $b^n$ are $\mathcal{F}_t$-measurable and continuous processes such that

$$\sup_{n\ge 1} \sup_{0\le t\le T} |a_t^n| \le C^* \textit{ and } \mathbb{E}\left[\int_0^T |b_t^n|^2 ds\right]^{\frac{1}{2}} < +\infty.$$

Here, compared with $(A2)$ and $(B2)$, we see that the condition $|f(t, y, z)| \le C(1+ |y| + |z|)$ is not imposed in $(A2)$.

**Definition 3.1.** A *solution* to DB-BDSDE is a quadruple $(Y, Z, K^+, K^-)$ satisfying (1.1) such that $(Y, Z, K^+, K^-) \in \mathbb{S}^2([0,T];\mathbb{R}) \times \mathbb{H}^2([0,T];\mathbb{R}) \times \mathbb{S}^2([0,T];\mathbb{R}^+) \times \mathbb{S}^2([0,T];\mathbb{R}^+)$, and $K^\pm$ are two continuous and increasing processes with $K_0^\pm = 0$.

Next, our main result is the following.

**Theorem 3.2.** *Under the assumptions* $(B1)$ - $(B5)$*, the DB-BDSDE* (1.1) *has a unique solution* $(Y, Z, K^+, K^-)$.

The following lemmas are needed to prove Theorem 3.2.

**Lemma 3.3.** *Let* $(\bar{Y}^n, \bar{Z}^n, \bar{K}^n)$ *be the solution of the SB-BDSDE associated with* $(-C(1+|y|+|z|) - n(y-U_t)^+, g, \xi, L)$*, then there exists a positive constant* $C^*$ *such that*

$$\sup_{0\le t\le T} n(\bar{Y}^n_t - U_t)^+ \le C^*.$$

*Proof.* Here, we begin with considering $(\bar{Y}^{n,k}, \bar{Z}^{n,k})$, taking into account the approximation of SB-BDSDE by a sequence of BDSDE.

For each $n \ge 1$, and $k \ge 1$, let $(\bar{Y}^{n,k}, \bar{Z}^{n,k})$ be the solution of the following BDSDE

$$\begin{aligned}\bar{Y}^{n,k}_t = \xi - &\int_t^T C(1+|\bar{Y}^{n,k}_s| + |\bar{Z}^{n,k}_s|)ds - n\int_t^T (\bar{Y}^{n,k}_s - U_s)^+ ds \\ &+ k\int_t^T (\bar{Y}^{n,k}_s - L_s)^- ds + \int_t^T g(s, \bar{Y}^{n,k}_s, \bar{Z}^{n,k}_s)\overleftarrow{dB}_s - \int_t^T \bar{Z}^{n,k}_s dW_s,\end{aligned}$$

where $\lim_{k\to\infty} \bar{Y}^{n,k}_t = \bar{Y}^n_t$, for any $t \le T$, $\mathbb{P}$-a.s. (see Aman et al. [2], EL Karoui et al. [9] and Hamadène et al. [15]). We set $X^{n,k} := \bar{Y}^{n,k} - U^k$, then by $(B5)$, we have

$$\begin{aligned}X^{n,k}_t = \xi - U^k_T - &\int_t^T a^k_s ds - \int_t^T C(1+|\bar{Y}^{n,k}_s| + |\bar{Z}^{n,k}_s|)ds \\ &- n\int_t^T (X^{n,k}_s - (U_s - U^k_s))^+ ds + k\int_t^T (X^{n,k}_s - (L_s - U^k_s))^- ds \\ &+ \int_t^T g(s, \bar{Y}^{n,k}_s, \bar{Z}^{n,k}_s)\overleftarrow{dB}_s - \int_t^T (\bar{Z}^{n,k}_s - b^k_s) dW_s.\end{aligned}$$

For each $n \in \mathbb{N}$, let $\mathcal{D}_n$ denote the class of $\mathcal{F}_t$-measurable process taking values in $[0,n]$. For $\nu \in \mathcal{D}_n$ and $\mu \in \mathcal{D}_k$, we denote $R_t = e^{\int_0^t (\mu_s + \nu_s)ds}$. Applying the extended Ito's formula (see Pardoux and Peng [25]) to $R_t X^{n,k}_t$, we obtain

$$\begin{aligned}X^{n,k}_t = &\mathbb{E}[(\xi - U^k_T)\exp(-\int_t^T (\mu_r + \nu_r)dr) \\ &+ \int_t^T \exp(-\int_t^s (\mu_r + \nu_r)dr)(-a^k_s - C(1+|\bar{Y}^{n,k}_s| + |\bar{Z}^{n,k}_s|))ds \\ &+ \int_t^T \exp(-\int_t^s (\mu_r + \nu_r)dr)(\mu_s(L_s - U^k_s) + \nu_s(U_s - U^k_s))ds \\ &+ \int_t^T \exp(-\int_t^s (\mu_r + \nu_r)dr)(\mu_s(X^{n,k}_s - (L_s - U^k_s))) \\ &+ k(X^{n,k}_s - (L_s - U^k_s))^- + \nu_s(X^{n,k}_s - (U_s - U^k_s)) \\ &- n(X^{n,k}_s - (U_s - U^k_s))^+)ds|\mathcal{F}_t].\end{aligned}$$

Here, we use the facts that $W$ and $B$ are mutually independent and that the backward Ito integral term is a (backward) martingale (about backward martingale, see [11] and [21]). Therefore, we can show that (cf. [6] and [15])

$$
\begin{aligned}
X_t^{n,k} =& \operatorname*{ess\,sup}_{\mu\in\mathcal{D}_k} \operatorname*{ess\,inf}_{\nu\in\mathcal{D}_n} \mathbb{E}[(\xi - U_T^n)\exp(-\int_t^T(\mu_s+\nu_s)ds) \\
&+ \int_t^T \exp(-\int_t^s(\mu_r+\nu_r)dr)(-a_s^k - C(1+|\bar{Y}_s^{n,k}|+|\bar{Z}_s^{n,k}|) \\
&+ \mu_s(L_s - U_s^k) + \nu_s(U_s - U_s^k))ds|\mathcal{F}_t].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
X_t^{n,k} \le& \operatorname*{ess\,sup}_{\mu\in\mathcal{D}_k} \operatorname*{ess\,inf}_{\nu\in\mathcal{D}_n} \mathbb{E}\left[\int_t^T \exp(-\int_s^t(\mu_r+\nu_r)dr)|a_s^k|ds|\mathcal{F}_t\right] \\
\le& C^* \operatorname*{ess\,sup}_{\mu\in\mathcal{D}_k} \mathbb{E}\left[\int_t^T \exp(-\int_s^t(\mu_r+n)dr)ds|\mathcal{F}_t\right] \\
=& \frac{C^*}{n}e^{-n(T-t)}.
\end{aligned}
$$

The desired result follows. □

**Lemma 3.4.** *Let $(Y^n, Z^n, K^{n+})$ be the solutions of the SB-BDSDEs associated with $(f(t,y,z) - n(y-U_t)^+, g(t,y,z), \xi, L)$:*

$$
\begin{aligned}
Y_t^n =& \xi + \int_t^T f(s,Y_s^n,Z_s^n)ds - n\int_t^T (Y_s^n - U_s)^+ ds \\
&+ \int_t^T g(s,Y_s^n,Z_s^n)\overleftarrow{dB}_s - \int_t^T Z_s^n dW_s + K_T^{n+} - K_t^{n+}, \qquad (3.1)\\
L_t \le& Y_t^n \quad \text{and} \quad \int_0^T (Y_t^n - L_t)dK_t^{n+} = 0.
\end{aligned}
$$

*Then, there exists a positive constant $C$ such that*

$$
\mathbb{E}\left[\sup_{0\le t\le T}|Y_t^n|^2 + \int_0^T |Z_s^n|^2 ds + |K_T^{n+}|^2\right] \le C.
$$

*Moreover, there exist two $\mathcal{F}_t$-measurable processes $\{Y_t\}_{t\le T}$ and $\{K_t^+\}_{t\le T}$ such that $Y^n \searrow Y, K^{n+} \nearrow K^+$ and*

$$
\lim_{n\to\infty}\mathbb{E}\left[\int_0^T |Y_s^n - Y_s|^2 ds\right] = 0,
$$

$$
\lim_{n\to\infty}\mathbb{E}\left[\sup_{0\le t\le T}|K_t^{n+} - K_t^+|^2\right] = 0.
$$

*Proof.* Let $(\bar{Y}^n, \bar{Z}^n, \bar{K}^{n+})$ and $(\rho, \theta, \Pi^+)$ be the solutions of the SB-BDSDEs associated with $(-C(1+|y|+|z|) - n(y-U_t)^+, g(t,y,z), \xi, L)$ and $(-C(1+|y|+$

$|z|) - C^*, g(t,y,z), \xi, L)$:

$$\begin{aligned}\bar{Y}^n_t = \xi &- \int_t^T C(1+|\bar{Y}^n_s|+|\bar{Z}^n_s|)ds - n\int_t^T (\bar{Y}^n_s - U_s)^+ ds + \int_t^T g(s,\bar{Y}^n_s,\bar{Z}^n_s)\overleftarrow{dB}_s \\ &- \int_t^T \bar{Z}^n_s dW_s + \bar{K}^{n+}_T - \bar{K}^{n+}_t, \qquad (3.2)\\ L_t \leq \bar{Y}^n_t \quad &\text{and} \quad \int_0^T (\bar{Y}^n_t - L_t)d\bar{K}^{n+}_t = 0,\end{aligned}$$

$$\begin{aligned}\rho_t = \xi &- \int_t^T C(1+|\rho_s|+|\theta_s|)ds - \int_t^T C^* ds + \int_t^T g(s,\rho_s,\theta_s)\overleftarrow{dB}_s \\ &- \int_t^T \theta_s dW_s + \Pi^+_T - \Pi^+_t, \qquad (3.3)\\ L_t \leq \rho_t \quad &\text{and} \quad \int_0^T (\rho_t - L_t)d\Pi^+_t = 0.\end{aligned}$$

The comparison principle (cf. Karouf [19]) with (3.1) - (3.3) implies that $(Y^n)_{n\geq 1}$ (resp. $(K^{n+})_{n\geq 1}$) is the non-increasing (resp. non-decreasing) sequence of processes and $\forall n \geq 1$, $\mathbb{P}$-a.s, $Y^1 \geq Y^n \geq \bar{Y}^n \geq \rho$ and $K^{1+} \leq K^{n+} \leq \bar{K}^{n+} \leq \Pi^+$. Hence, there exist $Y$ and $K^+$ such that $\mathbb{P}$-a.s., for any $t \leq T$, $Y^n_t \searrow Y_t$ and $K^{n+}_t \nearrow K^+_t$ as $n \to \infty$. Now, according to EL Karoui et al. [9] and Hamadène et al. [15], we have

$$\mathbb{E}[(|K^{1+}_T|^2 + |\Pi^+_T|^2) + \int_0^T (|Z^1_t|^2 + |\theta_t|^2)dt + \sup_{0\leq t\leq T}(|Y^1_t|^2 + |\rho_t|^2)] \leq C. \qquad (3.4)$$

Moreover, taking account of the fact that $n \geq 1$, $\mathbb{P}$-a.s., $Y^1 \geq Y \geq \rho$, we obtain that $Y$ is also the limit

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^T |Y^n_s - Y_s|^2 ds\right] = 0,$$

by the dominated convergence theorem. Since the process $K^+$ is continuous and $K^+ \leq \Pi^+$, then by Dini's theorem, we have

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{0\leq t\leq T} |K^{n+}_t - K^+_t|^2\right] = 0.$$

Next, from (3.4) and the fact that $Y^1 \geq Y^n \geq \rho$ and $K^1_T \leq K^n_T \leq \Pi^+_T$ imply

$$\mathbb{E}[|K^{n+}_T|^2 + \sup_{0\leq t\leq T} |Y^n_t|^2] \leq C. \qquad (3.5)$$

Now applying Ito's formula with $|Y_t^n|^2$, we have

$$\begin{aligned}|Y_t^n|^2 + \int_t^T |Z_t^n|^2 ds = |\xi|^2 &+ 2\int_t^T Y_s^n f(s, Y_s^n, Z_s^n) ds - 2n\int_t^T Y_s^n (Y_s^n - U_s)^+ ds \\ &+ 2\int_t^T Y_s^n g(s, Y_s^n, Z_s^n) \overleftarrow{dB}_s - 2\int_t^T Y_s^n Z_s^n dW_s \\ &+ 2\int_t^T Y_s^n dK_s^{n+} + \int_t^T |g(s, Y_s^n, Z_s^n)|^2 ds.\end{aligned}$$

Next, taking expectation on both sides of the above equation and using the fact that $-nY_t^n(Y_t^n - U_t)^+ \leq nU_t^-(Y_t^n - U_t)^+$, for any $\epsilon > 0$, we get

$$\begin{aligned}&\mathbb{E}\left[|Y_t^n|^2\right] + \mathbb{E}\left[\int_t^T |Z_s^n|^2 ds\right] \\ \leq &\mathbb{E}\left[|\xi|^2\right] + 2\mathbb{E}\left[\int_t^T Y_s^n f(s, Y_s^n, Z_s^n) ds\right] - 2n\mathbb{E}\left[\int_t^T Y_s^n (Y_s^n - U_s)^+ ds\right] \\ &+ 2\mathbb{E}\left[\int_t^T Y_s^n dK_s^{n+}\right] + \mathbb{E}\left[\int_t^T |g(s, Y_s^n, Z_s^n)|^2 ds\right] \\ \leq &\mathbb{E}\left[|\xi|^2\right] + \frac{1}{\epsilon}\mathbb{E}\left[\int_t^T |Y_s^n|^2 ds\right] + \epsilon C\mathbb{E}\left[\int_t^T (|Y_s^n|^2 + |Z_s^n|^2) ds\right] \\ &+ \epsilon\mathbb{E}\left[\int_t^T |f(s,0,0)|^2 ds\right] + \mathbb{E}\left[\sup_{t \leq s \leq T} |Y_s^n|^2\right] + \mathbb{E}\left[|K_T^{n+}|^2\right] + \frac{1}{\epsilon}\mathbb{E}\left[\sup_{t \leq s \leq T} |U_s^-|^2\right] \\ &+ \epsilon\mathbb{E}\left[\int_t^T |n(Y_s^n - U_s)^+|^2 ds\right] + C\mathbb{E}\left[\int_t^T |Y_s^n|^2 ds\right] + \alpha\mathbb{E}\left[\int_t^T |Z_s^n|^2 ds\right] \\ &+ \mathbb{E}\left[\int_t^T |g(s,0,0)|^2 ds\right].\end{aligned}$$

Then, from (3.5)

$$\begin{aligned}\mathbb{E}\left[|Y_t^n|^2\right] + \mathbb{E}\left[\int_t^T |Z_s^n|^2 ds\right] \leq &\, Const. + (\epsilon C + \alpha)\mathbb{E}\left[\int_t^T |Z_s^n|^2 ds\right] \\ &+ \epsilon\mathbb{E}\left[\int_t^T |n(Y_s^n - U_s)^+|^2 ds\right].\end{aligned} \tag{3.6}$$

On the other hand,

$$\begin{aligned}n\int_0^T (Y_t^n - U_t)^+ dt = \xi + K_T^{n+} - Y_0^n &+ \int_t^T f(t, Y_t^n, Z_t^n) dt \\ &+ \int_0^T g(t, Y_t^n, Z_t^n) \overleftarrow{dB}_t - \int_0^T Z_t^n dW_t\end{aligned}$$

and then

$$\mathbb{E}\left[\int_0^T |n(Y_t^n - U_t)^+|^2 dt\right] \leq Const. + (\alpha+1)\mathbb{E}\left[\int_0^T |Z_s^n|^2 ds\right]. \tag{3.7}$$

Combining (3.6) and (3.7), and choosing $\epsilon$ as $\epsilon(C+\alpha+1)+\alpha<1$, we have the required result. □

**Lemma 3.5.** *We have*

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{0\leq t\leq T} |(Y_t^n - U_t)^+|^2\right] = 0.$$

*Proof.* Let $(\hat{Y}^n, \hat{Z}^n, \hat{K}^{n+})$ be the solution of SB-BDSDE associated with $(f(t,Y_t^n,Z_t^n) - n(y-U_t), g(t,Y_t^n,Z_t^n), \xi, L)$. For $t \leq T$, we have

$$\begin{aligned} \hat{Y}_t^n = \xi + & \int_t^T f(s,Y_s^n,Z_s^n)ds - n\int_t^T (\hat{Y}_s^n - U_s)ds \\ & + \int_t^T g(s,Y_s^n,Z_s^n)\overleftarrow{dB}_s - \int_t^T \hat{Z}_s dW_s + \hat{K}_T^{n+} - \hat{K}_t^{n+}, \end{aligned} \tag{3.8}$$

$$L_t \leq \hat{Y}_t^n \quad \text{and} \quad \int_0^T (\hat{Y}_t^n - L_t)d\hat{K}_t^{n+} = 0.$$

Then the comparison result with (3.1) and (3.8) implies that $Y^n \leq \hat{Y}^n$ and $d\hat{K}^{n+} \leq dK^{n+} \leq d\Pi^+$. Now let $\tau$ be an $\mathcal{G}_t$-stopping time ($\mathcal{G}_t = \mathcal{F}_t^W \vee \mathcal{F}_T^B$) such that $\tau \leq T$. Then, we can write (see [10], [15] and [22])

$$\begin{aligned} \hat{Y}_\tau^n = \mathbb{E}\Bigg[ & e^{-n(T-\tau)}\xi + \int_\tau^T e^{-n(s-\tau)}(f(s,Y_s^n,Z_s^n) + nU_s)ds \\ & + \int_\tau^T e^{-n(s-\tau)} d\hat{K}_s^{n+} | \mathcal{G}_\tau \Bigg]. \end{aligned}$$

It is easily seen that

$$e^{-n(T-\tau)}\xi + \int_\tau^T e^{-n(s-\tau)} nU_s ds \to \xi \mathbf{1}_{\{\tau=T\}} + U_\tau \mathbf{1}_{\{\tau<T\}},$$

as $n\to\infty$, $\mathbb{P}$-a.s., in $\mathbb{L}^2$, since $\mathbb{E}\left[\sup_{0\leq t\leq T}|U_t|^2\right] < +\infty$.
Moreover, we have

$$\left|\int_\tau^T e^{-n(s-\tau)} f(s,Y_s^n,Z_s^n)ds\right| \leq \frac{1}{(2n)^{\frac{1}{2}}}\left\{\int_\tau^T |f(s,Y_s^n,Z_s^n)|^2 ds\right\}^{\frac{1}{2}}.$$

Then,

$$\int_\tau^T e^{-n(s-\tau)} f(s,Y_s^n,Z_s^n)ds \to 0,$$

as $n\to\infty$, $\mathbb{P}$-a.s., in $\mathbb{L}^2$.
In addition, we obtain

$$0 \leq \int_\tau^T e^{-n(s-\tau)} d\hat{K}_s^n \leq \int_\tau^T e^{-n(s-\tau)} d\Pi_s^+ \to 0,$$

as $n \to \infty$, $\mathbb{P}$-a.s., in $\mathbb{L}^1$. Consequently, it follows that

$$\hat{Y}^n_\tau \to \xi \mathbf{1}_{\{\tau = T\}} + U_\tau \mathbf{1}_{\{\tau < T\}},$$

as $n \to \infty$, $\mathbb{P}$-a.s., in $\mathbb{L}^1$. Therefore, $Y_\tau \le U_\tau$, $\mathbb{P}$-a.s.. From this fact and the section theorem, in Dellacherie and Meyer [7], P.220, it follows that $Y_t \le U_t$ for all $t \le T$, $\mathbb{P}$-a.s.. By Dini's theorem, we have $(Y^n_t - U_t)^+ \searrow 0$, $t \le T$, $\mathbb{P}$-a.s. The desired result follows by the dominated convergence theorem, since $(Y^n_t - U_t)^+ \le |Y^1_t| + |U_t|$. $\square$

**Lemma 3.6.** *There exist two $\mathcal{F}_t$-measurable processes $\{Z_t\}_{t \le T}$ and $\{K^-_t\}_{t \le T}$ such that*

$$\lim_{n \to \infty} \mathbb{E}\left[\int_0^T |Z^n_t - Z_t|^2 dt\right] = 0,$$

$$\lim_{n \to \infty} \mathbb{E}\left[\sup_{0 \le t \le T} |Y^n_t - Y_t|^2 + \sup_{0 \le t \le T} |K^{n-}_t - K^-_t|^2\right] = 0.$$

*Proof.* Using Ito's formula, we have

$$\begin{aligned}
&|Y^n_t - Y^m_t|^2 + \int_t^T |Z^n_t - Z^m_t|^2 ds \\
&= 2\int_t^T (Y^n_s - Y^m_s)(f(s, Y^n_s, Z^n_s) - f(s, Y^m_s, Z^m_s))ds \\
&\quad - 2\int_t^T (Y^n_s - Y^m_s)(n(Y^n_s - U_s)^+ - m(Y^m_s - U_s))^+ ds \\
&\quad + 2\int_t^T (Y^n_s - Y^m_s)(g(s, Y^n_s, Z^n_s) - g(s, Y^m_s, Z^m_s))\overleftarrow{dB}_s \\
&\quad - 2\int_t^T (Y^n_s - Y^m_s)(Z^n_s - Z^m_s)dW_s + 2\int_t^T (Y^n_s - Y^m_s)(dK^{n+}_s - dK^{m+}_s) \\
&\quad + \int_t^T |g(s, Y^n_s, Z^n_s) - g(s, Y^m_s, Z^m_s)|^2 ds.
\end{aligned}$$

Moreover, for any $\epsilon > 0$, we also have

$$\begin{aligned}
&\mathbb{E}\left[\int_t^T |Z^n_s - Z^m_s|^2 ds\right] \\
&\le \frac{1}{\epsilon}\mathbb{E}\left[\int_t^T |Y^n_s - Y^m_s|^2 ds\right] + \epsilon C \mathbb{E}\left[\int_t^T (|Y^n_s - Y^m_s|^2 + |Z^n_s - Z^m_s|^2)ds\right] \\
&\quad + 2\mathbb{E}\left[\int_t^T (Y^m_s - U_s)n(Y^n_s - U_s)^+ ds\right] + 2\mathbb{E}\left[\int_t^T (Y^n_s - U_s)m(Y^m_s - U_s))^+ ds\right] \\
&\quad + C\mathbb{E}\left[\int_t^T |Y^n_s - Y^m_s|^2 ds\right] + \alpha \mathbb{E}\left[\int_t^T |Z^n_s - Z^m_s|^2 ds\right],
\end{aligned}$$

since $\int_0^T (Y_s^n - Y_s^m)(dK_s^{n+} - dK_s^{m+}) \leq 0$. Hence, we get

$$\begin{aligned}
&(1-\epsilon C-\alpha)\mathbb{E}\left[\int_t^T |Z_s^n - Z_s^m|^2 ds\right] \\
&\qquad \leq (\frac{1}{\epsilon}+\epsilon C + C)\mathbb{E}\left[\int_t^T |Y_s^n - Y_s^m|^2 ds\right] \\
&\qquad + 2\left(\mathbb{E}\left[\sup_{0\leq t\leq T}((Y_t^m - U_t)^+)^2\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[(\int_t^T n(Y_s^n - U_s)^+ ds)^2\right]\right)^{\frac{1}{2}} \\
&\qquad + 2\left(\mathbb{E}\left[\sup_{0\leq t\leq T}((Y_t^n - U_t)^+)^2\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[(\int_t^T m(Y_s^m - U_s)^+ ds)^2\right]\right)^{\frac{1}{2}}.
\end{aligned}$$

Using Lemma 3.4 and 3.5, we obtain

$$\lim_{n\to\infty}\mathbb{E}\left[\int_0^T |Z_t^n - Z_t|^2 dt\right] = 0.$$

Now, going back to Ito's formula, it follows that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq t\leq T}|Y_s^n - Y_s^m|^2\right] \\
&\leq (1+2C)\mathbb{E}\left[\int_t^T |Y_s^n - Y_s^m|^2 ds\right] \\
&\quad + 2\left(\mathbb{E}\left[\sup_{0\leq t\leq T}((Y_t^m - U_t)^+)^2\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[(\int_t^T n(Y_s^n - U_s)^+ ds)^2\right]\right)^{\frac{1}{2}} \\
&\quad + 2\left(\mathbb{E}\left[\sup_{0\leq t\leq T}((Y_t^n - U_t)^+)^2\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[(\int_t^T m(Y_s^m - U_s)^+ ds)^2\right]\right)^{\frac{1}{2}} \\
&\quad + (C+\alpha)\mathbb{E}\left[\int_t^T |Z_s^n - Z_s^m|^2 ds\right],
\end{aligned}$$

which implies

$$\lim_{n\to\infty}\mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t^n - Y_t|^2\right] = 0.$$

Here, we set

$$K_t^{n-} = \int_0^t n(Y_s^n - U_s)^+ ds,$$

and

$$\begin{aligned}
K_t^- = Y_t - Y_0 + &\int_0^t f(s,Y_s,Z_s)ds + \int_0^t g(s,Y_s,Z_s)\overleftarrow{dB}_s \\
&- \int_0^t Z_s dW_s + K_t^+ - K_0^+.
\end{aligned}$$

For $t \leq T$ and $n \geq 1$, we obtain

$$\begin{aligned}\int_0^t n(Y_s^n - U_s)^+ ds - K_t^- = (Y_t^n - Y_t) - (Y_0^n - Y_0) \\ + \int_0^t (f(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s))ds \\ + \int_0^t (g(s, Y_s^n, Z_s^n) - g(s, Y_s, Z_s))\overleftarrow{dB}_s \\ - \int_0^t (Z_s^n - Z_s)dW_s + (K_t^{n+} - K_t^+) - (K_0^{n+} - K_0^+).\end{aligned}$$

By Lemma 3.4 and the same discussion as above, we deduce that

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{0\leq t\leq T} |\int_0^t n(Y_s^n - U_s)^+ ds - K_t^-|^2\right] = 0.$$

□

Next, we prove Theorem 3.2.

**(Proof of Theorem 3.2)**
Let us prove that the process $(Y, Z, K^+, K^-)$ is a solution to the DB-BDSDE (1.1). Obviously the process satisfies

$$\begin{aligned}Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)\overleftarrow{dB}_s - \int_t^T Z_s dW_s \\ +(K_T^+ - K_t^+) - (K_T^- - K_t^-).\end{aligned}$$

Now, $Y_t^n \geq L_t$ and $\mathbb{E}\left[\sup_{0\leq t\leq T}(Y_s^n - U_s)^+\right] \to 0$ as $n \to \infty$, then $L_t \leq Y_t \leq U_t$, for any $t \leq T$.

We show that $\int_0^T (Y_s - L_s)dK_s^+ = \int_0^T (U_s - Y_s)dK_s^- = 0$, $\mathbb{P}$-a.s. We see

$$\begin{aligned}\int_0^T (Y_s - L_s)dK_s^+ &= \int_0^T (Y_s - Y_s^n)dK_s^+ + \int_0^T (Y_s^n - L_s)dK_s^+ \\ &= \int_0^T (Y_s - Y_s^n)dK_s^+ + \int_0^T (Y_s^n - L_s)(dK_s^+ - dK_s^{n+})\end{aligned}$$

Let $\omega$ be fixed. It follows from Lemma 3.6 that for any $\epsilon > 0$, there exists $n_0(\omega)$ such that for any $n \geq n_0(\omega)$, $t \leq T$, $Y_t(\omega) \leq Y_t^n(\omega) + \epsilon$. Hence we have

$$\int_0^T (Y_s - Y_s^n)dK_s^+ \leq \epsilon K_T^+(\omega). \tag{3.9}$$

On the other hand, since the function $(Y_t^n(\omega) - L_t(\omega))_{t\leq T}$ is continuous, then there exists a sequence of step functions $(f^m(\omega))_{m\geq 0}$, which converges uniformly on $[0, T]$ to $Y_t^n(\omega) - L_t(\omega)$. That is, there exists $m_0(\omega) \geq 0$ such that for $m \geq m_0(\omega)$, we have $t \le T$, $|Y_t^n(\omega) - L_t(\omega) - f_t^m(\omega)| < \epsilon$. Therefore it follows that

$$\begin{aligned}\int_0^T (Y_s^n - L_s)d(K_s^+ - K_s^{n+}) &= \int_0^T (Y_s^n - L_s - f_s^m)d(K_s^+ - K_s^{n+}) \\ &\quad + \int_0^T f_s^m d(K_s^+ - K_s^{n+}) \\ &\le \int_0^T (\epsilon + f_s^m)d(K_s^+ - K_s^{n+}) \\ &\le \int_0^T f_s^m d(K_s^+ - K_s^{n+}) \\ &\quad + \epsilon(K_T^+(\omega) + K_T^{n+}(\omega))\end{aligned} \tag{3.10}$$

However, the right-hand side converges to $2\epsilon K_T^+(\omega)$, as $n \to \infty$, since $f^m(\omega)$ is a step function and then $\int_0^T f_s^m d(K_s^+ - K_s^{n+}) \to 0$. Therefore we have

$$\int_0^T (Y_s - L_s)dK_s^+ \le 3\epsilon K_T^+,$$

from (3.9) and (3.10). As $\epsilon$ is arbitrary and $Y \ge L$ then

$$\int_0^T (Y_s - L_s)dK_s^+ = 0.$$

Moreover, thanks to Lemma 3.6 again, we get

$$\int_0^T (U_s - Y_s^n)n(Y_s^n - U_s)^+ ds \to \int_0^T (U_s - Y_s)dK_s^-, \ as\ n \to \infty,$$

$\mathbb{P}$-a.s.. Because of

$$\begin{aligned}&\int_0^T (U_s - Y_s^n)dK_s^{n-} - \int_0^T (U_s - Y_s)dK_s^- \\ &= \int_0^T (U_s - Y_s^n)dK_s^{n-} - \int_0^T (U_s - Y_s^n)dK_s^- \\ &\quad + \int_0^T (U_s - Y_s^n)dK_s^- - \int_0^T (U_s - Y_s)dK_s^- \\ &= \int_0^T (U_s - Y_s^n)(dK_s^{n-} - dK_s^-) + \int_0^T (Y_s - Y_s^n)dK_s^-,\end{aligned}$$

and taking into account the same argument as above, it follows that

$$\int_0^T (U_s - Y_s)dK_s^- = 0.$$

Next, we prove the uniqueness of the solution. If $(Y', Z', K'^{+}, K'^{-})$ is another solution, then using Ito's formula, we obtain

$$
\begin{aligned}
|Y_t - Y_t'|^2 + &\int_t^T |Z_s - Z_s'|^2 ds \\
=2 &\int_t^T (Y_s - Y_s')(f(s, Y_s, Z_s) - f(s, Y_s', Z_s'))ds \\
&+ 2\int_t^T (Y_s - Y_s')(g(s, Y_s, Z_s) - g(s, Y_s', Z_s'))\overleftarrow{dB}_s \\
&- 2\int_t^T (Y_s - Y_s')(Z_s - Z_s')dW_s \\
&+ 2\int_t^T (Y_s - Y_s')(dK_s^+ - dK'^+_s) - 2\int_t^T (Y_s - Y_s')(dK_s^- - dK'^-_s) \\
&+ \int_t^T |g(s, Y_s, Z_s) - g(s, Y_s', Z_s')|^2 ds.
\end{aligned}
$$

Here, $\int_0^T (Y_s - Y_s')(dK_s^+ - dK'^+_s) \leq 0$ and $\int_0^T (Y_s - Y_s')(dK_s^- - dK'^-_s) \geq 0$, then

$$
\begin{aligned}
&|Y_t - Y_t'|^2 + \int_t^T |Z_s - Z_s'|^2 ds \\
&\leq 2\int_t^T (Y_s - Y_s')(f(s, Y_s, Z_s) - f(s, Y_s', Z_s'))ds \\
&\quad + 2\int_t^T (Y_s - Y_s')(g(s, Y_s, Z_s) - g(s, Y_s', Z_s'))\overleftarrow{dB}_s \\
&\quad - 2\int_t^T (Y_s - Y_s')(Z_s - Z_s')dW_s \\
&\quad + \int_t^T |g(s, Y_s, Z_s) - g(s, Y_s', Z_s')|^2 ds.
\end{aligned}
$$

Then we get

$$
\begin{aligned}
\mathbb{E}\left[|Y_t - Y_t'|^2\right] + \mathbb{E}\left[\int_t^T |Z_s - Z_s'|^2 ds\right] &\leq (\frac{1}{\epsilon} + \epsilon C + C)\mathbb{E}\left[\int_t^T |Y_s - Y_s'|^2 ds\right] \\
&\quad + (\epsilon C + \alpha)\mathbb{E}\left[\int_t^T |Z_s - Z_s'|^2 ds\right],
\end{aligned}
$$

where $0 < \alpha < 1$. Consequently,

$$
\begin{aligned}
\mathbb{E}\left[|Y_t - Y_t'|^2\right] + (1 - \epsilon C - \alpha)\mathbb{E}&\left[\int_t^T |Z_s - Z_s'|^2 ds\right] \\
&\leq (\frac{1}{\epsilon} + \epsilon C + C)\mathbb{E}\left[\int_t^T |Y_s - Y_s'|^2 ds\right].
\end{aligned}
$$

From Gronwall's lemma, $\mathbb{E}\left[|Y_t - Y'_t|^2\right] = 0$, for $0 \leq t \leq T$. And hence

$$\mathbb{E}\left[\int_0^T |Z_s - Z'_s|^2 ds\right] = 0.$$

In that, it follows that $Y = Y'$ and $Z = Z'$.
Finally, we prove that $K^+ = {K'}^+$ and $K^- = {K'}^-$. For any $t \leq T$,

$$\int_0^T (Y_s - L_s) dK_s^+ = \int_0^T (Y'_s - L_s) d{K'}_s^+ = \int_0^T (Y_s - L_s) d{K'}_s^+,$$

Therefore, we have $\int_0^T (Y_s - L_s)(dK_s^+ - d{K'}_s^+) = 0$. Since $Y \geq L$, we deduce

$$K^+ = {K'}^+.$$

Next, for any $t \leq T$,

$$\int_0^T (U_s - Y_s) dK_s^- = \int_0^T (U_s - Y'_s) d{K'}_s^- = \int_0^T (U_s - Y_s) d{K'}_s^-,$$

we also have $\int_0^T (U_s - Y_s)(dK_s^- - d{K'}_s^-) = 0$.
Since $Y \leq U$, we deduce

$$K^- = {K'}^-.$$

Theorem 3.2 is proved.

**Notes - Examples**

(1)A convertible bond gives the holder of the contract the right to convert a bond into a stock and the issuer of the contract the right to recall the bond. It is a so-called Dynkin game. The contract value is defined as a solution to the optimal stopping problem. The value function can be characterized as a solution to double barrier BSDE (cf. [4], [5], [12], [13] and [17]).

(2)Also, double barrier BSDE can be used to solve starting and stopping problems (two modes switching) , see [14] and [18].

Therefore, as DB-BDSDE is a extension of DB-BSDE, various applications can be expected.

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