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SYMMETRIC FUNCTIONS ALGEBRAS (SFA) III: STOCHASTIC AND CONSTANT ROW SUM MATRICES

PHILIP FEINSILVER*

ABSTRACT. We study the symmetric functions algebra based on the induced matrix map on the algebra of real $d \times d$ matrices. For fixed integer $N > 0$, the induced matrix map takes a matrix to the symmetric tensor power in degree N. It is determined by the action of the matrix on polynomials in d variables. The symmetric functions algebra has various bases which obey the identities of the standard algebra of symmetric functions. The case of symmetric tensor powers is taken up here when the initial element is a stochastic matrix, leading to the various bases consisting of nonnegative matrices with constant row sums . In this work, we determine row sums for the various bases: elementary, homogeneous, power sum, monomial, and Schur functions. The induced matrix map takes stochastic matrices to stochastic matrices so that in each degree N there is a corresponding Markov chain arising from a given stochastic matrix. With this interpretation, it is of interest to look at limit theorems for powers of the various elements that arise.

1. Introduction

Symmetric functions expressed in variables ("coordinate description") form five main families: elementary, homogeneous (complete), power sum, and monomial symmetric functions and Schur functions. The symmetric functions algebra $\text{SFA}(\phi, X)$ is determined by a multiplicative map ϕ defined on a matrix algebra, and an element X of that algebra. With these ingredients, we define in matrix terms families analogous to the families of symmetric functions but with no explicit variables appearing, i.e., a "coordinate-free description".

In this work, we look at SFA algebras for the specific case where ϕ is the induced matrix mapping (see below for details). Then we study the basic structures related to the SFA algebra for a stochastic matrix. In §3 matrices with constant row sums, CRS matrices, are considered. Nonnegativity and positivity properties are the subject of §4. Convergence results appear in §5. An analytic approach to the elementary functions is sketched out in §6, and then we take a look in §7 at some theorems involving matrices with zero row sums. In §8, some complements are indicated and we conclude with remarks on prospects for further developments. An Appendix indicates some details for the case where X is a reflection.

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2. Induced Matrices and Symmetric Functions Algebras

We start with a review of the basic objects of interest in this study.

Notation. We will be working with d variables, $\{x_1, x_2, \ldots, x_d\}$. Letters m and n always denote multi-indices, d-tuples. Other indices will be single indices unless otherwise indicated or clear from the context. Multi-index (compressed) notation will typically be employed, i.e., $x^m = x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d}$, $m! = m_1!\, m_2! \cdots m_d!$, $|m| = \sum_i m_i$, etc. And we will then be starting with real $d \times d$ matrices within the class of all real square matrices.

From $E(v) = \prod_{i=1}^d (1 + vx_i)$, the generating function for elementary symmetric functions, we have the generating function for the homogeneous symmetric functions

$$H(v) = E(-v)^{-1} = \prod_{i=1}^{d} \frac{1}{1 - vx_i} = \sum_{\ell \geq 0} v^\ell \mathrm{h}_\ell$$

which has a relation with the power sum functions p_i

$$P(v) = \sum_{i \geq 1} \frac{v^i}{i}\, \mathrm{p}_i = \log H(v) \text{ or } H(v) = e^{P(v)}$$

These relations may be interpreted as formal power series for the relations between the various sets of functions.

Perhaps the most natural basis for symmetric functions (polynomials in the variables $\{x_1, \ldots, x_d\}$), are the monomial functions indexed by partitions $\lambda = [\lambda_1 \lambda_2 \ldots \lambda_{L(\lambda)}]$

$$\mathrm{m}_\lambda = \text{minimal symmetric polynomial containing the monomial } x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_L^{\lambda_L}$$

where L is the length of λ, the number of positive parts.

Recall that a useful way to express a partition is by multiplicity notation

$$\rho = \rho(\lambda) = (1^{\rho_1} 2^{\rho_2} \ldots \ell^{\rho_\ell})$$

where ρ_i is the number of times that i appears in $\lambda \vdash \ell$. With this notation available, we can use multi-index notation to express the product

$$\mathrm{p}_\lambda = \mathrm{p}_{\lambda_1} \cdots \mathrm{p}_{\lambda_L} = \mathrm{p}_1^{\rho_1} \mathrm{p}_2^{\rho_2} \cdots \mathrm{p}_\ell^{\rho_\ell} = \mathrm{p}^\rho$$

with $\lambda \vdash \ell$, i.e., λ is a partition of $\ell = \sum j\, \rho_j$, noting that $\sum \rho_j = L$, length of λ.

And the Schur functions, or S-functions, indexed by partitions, may be defined by the expansion in products of power sums, cf., [11, p. 86, 6.2;14], [12, p. 114], see eq. (2.6) below. As well, recall the Jacobi-Trudi formulas

$$\{\lambda\} = \det(\mathrm{h}_{\lambda_i - i + j}) = \det(\mathrm{e}_{\lambda'_i - i + j}) \tag{2.1}$$

which apply in the matrix setting by first expanding and then substituting the matrix homogeneous/elementary functions in the resulting expressions.

Let us look at some basic features of elementary symmetric functions. Consider the product $\prod_{i=1}^N (1 + vX_i)$, the generating function for the elementary symmetric functions $\mathrm{e}_\ell(X_1, \ldots, X_N)$, $0 \leq \ell \leq N$, with $\mathrm{e}_0 = 1$. For a sequence $\{X_1, X_2, \ldots\}$, with $X_0 = 0$, writing $\Delta X_i = X_i - X_{i-1}$, we have

$$E_N = \prod_{i=1}^{N} (1 + v\,\Delta X_i) \;\Rightarrow \Delta E_N = v\,E_{N-1}\,\Delta X_N$$

a difference equation showing E_N as a discrete exponential.

Another property in the stochastic context is the following. Let $\{\Delta X_1, \Delta X_2, \ldots\}$ be independent, mean zero. Then $E_N = \prod_{i=1}^{N}(1 + v\,\Delta X_i)$ is a martingale, as we see that

$$\mathrm{E}(E_N|\mathcal{F}_{N-1}) = \mathrm{E}(1 + v\,\Delta X_N)\cdot E_{N-1} = E_{N-1}$$

accordingly. This formulation as a prototypical exponential martingale has been studied by Emery [6], [7].

Let $\{X_1, X_2, \ldots\}$ be independent, mean zero, with $\mathrm{var}(X_i) = \sigma_i^2$. Then the functionals $\mathrm{e}_\ell(X_1, \ldots, X_N)$ are an orthogonal family. We check, $\langle\ \rangle$ denoting expected value,

$$\begin{aligned}\langle E(v)E(w)\rangle &= \prod \langle (1 + (v+w)X_i + vwX_i^2)\rangle \\ &= \prod (1 + vw\,\sigma_i^2) = \sum_\ell (vw)^\ell\,\mathrm{e}_\ell(\sigma_1^2, \ldots, \sigma_N^2)\end{aligned}$$

in other words

$$\langle \mathrm{e}_i(X_1, \ldots, X_N)\mathrm{e}_j(X_1, \ldots, X_N)\rangle = \delta_{ij}\,\mathrm{e}_i(\sigma_1^2, \ldots, \sigma_N^2)\ .$$

We propose that more explorations involving various classes of symmetric functions in the stochastic setting is worthwhile. A well-known example is provided by the zonal polynomials, [12, Ch. 7], [10].

A general discussion of algebras $\mathrm{SFA}(\phi, X)$ is found in [8], with the case of induced matrices the focus of [9]. Now let us review the various bases of interest. The elements of the classes of symmetric functions in the SFA algebra are indicated by capital letters versus the lower case letters used above. The notation for generating functions will be modified correspondingly. Our starting point is the definition

$$\mathcal{E}(v) = \phi(I + vX)$$

where ϕ is a multiplicative map on a matrix algebra and X is a given element. In this work, we will use the induced matrix map for ϕ:

$$\phi(X) = \bar{X}$$

defined as follows.

Given a $d \times d$ matrix X, we have, for each $N > 0$, the *induced matrix*, $\bar{X}$, via the action on polynomials in d variables. Let

$$y_i = \sum_j X_{ij}x_j$$

and define the matrix elements/coefficients $\bar{X}$ by

$$y^m = y_1^{m_1}\cdots y_d^{m_d} = \prod_i (\sum_j X_{ij}x_j)^{m_i} = \sum_n \bar{X}_{mn}x^n \tag{2.2}$$

where $|m| = |n| = N$. Accompanying the induced matrix map is the Γ-map defined as the derivative at the identity in the direction X, i.e.,

$$\overline{e^{tX}} = e^{t\,\Gamma(X)}$$

So

$$\Gamma(X) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 \phi(e^{tX}) = \phi'(I)X$$

by the chain rule, where ϕ' is the jacobian of the map ϕ. And as well

$$\Gamma(X) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 \phi(I + tX)$$

Recall that Γ is a linear map on matrices.

We have

$$\begin{aligned}\mathcal{E}(v) = \overline{I + vX} &= I + \sum_{1\le\ell\le N} v^\ell \Gamma_\ell(X)\\ &= I + v\Gamma_1(X) + v^2\Gamma_2(X) + \cdots + v^N\Gamma_N(X)\\ &= I + vE_1 + v^2E_2 + \cdots + v^N E_N\end{aligned}$$

thus defining the elementary functions E_ℓ. In particular, $E_0 = I$, $E_1 = \Gamma_1(X) = \Gamma(X)$ and $E_N = \Gamma_N(X) = \bar{X}$.

There are five classes of interest:

$$\mathbb{G} = \{E_\ell\}_{0\le\ell\le N} \cup \{H_\ell\}_{\ell\ge1} \cup \{P_i\}_{i\ge1} \cup \{M_\lambda\}_{L(\lambda)\le N} \cup \{\,\{\lambda\}\,\}_{L(\lambda)\le N} \tag{2.3}$$

$$= \{M_\lambda\}_{L(\lambda)\le N} \cup \{\,\{\lambda\}\,\}_{L(\lambda)\le N} \tag{2.4}$$

Note that E's, H's, and P's are contained in the union of the monomial and S-function classes.

These elements are defined by identities given by the classical case of symmetric functions in underlying variables, where in this case the definitions are "coordinate-free" — determined by the relations among the generating functions.

So we have

$$\mathcal{H}(v) = \mathcal{E}(-v)^{-1} = \sum_{\ell\ge0} v^\ell H_\ell = e^{\mathcal{P}(v)}$$

where

$$\mathcal{P}(v) = \sum_{i\ge1} \frac{v^i}{i}\, P_i$$

while

$$(I - vX)^{-1} = \exp\Big(\sum_{i\ge1} \frac{v^i}{i}\, X^i\Big)$$

implies

$$\overline{(I - vX)^{-1}} = \exp\left(\sum_{i\ge1} \frac{v^i}{i}\, \Gamma(X^i)\right)$$

i.e.,

$$P_i = \Gamma(X^i)$$

For the monomial functions, we recall Theorem 3.12 of [8], which in this case reads

$$\overline{I + \sum_{1\le k\le u} c_k X^k} = \sum_{\mathcal{D}(u,N)} c^\rho M_\rho \tag{2.5}$$

where $\mathcal{D}(u,N)$ is the domain of partitions λ with $\lambda_1 \le u$ and $L(\lambda) \le N$.

It remains to define the Schur functions, S-functions. In this work the expansion most useful takes the form, for $\lambda \vdash \ell$,

$$\{\lambda\} = \sum_{\rho\vdash\ell} \chi^\lambda_\rho \, \frac{P^\rho}{z_\rho} \tag{2.6}$$

where $P^\rho = P_1^{\rho_1} \cdots P_\ell^{\rho_\ell}$, χ^λ_ρ denoting the character table of the symmetric group, $z_\rho = 1^{\rho_1} 2^{\rho_2} \cdots \ell^{\rho_\ell} \rho_1!\, \rho_2! \cdots \rho_\ell!$.

Here are some useful facts.

Proposition 2.1. *In degree N, we have $\{\lambda\} = 0$ if $L(\lambda) > N$ and, as well, $M_\lambda = 0$ if $L(\lambda) > N$.*

Proof. Using the conjugate form of the Jacobi-Trudi identities, we see that if $L(\lambda) > N$, then $\lambda_1' > N$ and so $E_{\lambda_1'} = 0$ and the same for the rest of the top row. Hence $\{\lambda\}$ vanishes. For the monomial functions, we have the transition matrix $\hat{T}_{\lambda\mu}$ such that

$$M_\lambda = \sum_\mu \hat{T}_{\lambda\mu} E_\mu$$

where $\mu \trianglerighteq \lambda'$ (cf. [8, §2.3.4], [5, Problems 2.4, Ex. 2.9] for the inverse matrix). That is, $\mu_1 \ge \lambda_1' = L(\lambda) > N$ implies $E_{\mu_1} = 0$, so $E_\mu = 0$. □

And

Theorem 2.2. *Let the minimal polynomial of X have degree δ. Then the algebra* SFA$(^-, X)$ *is spanned by the $\binom{\delta+N-1}{N}$ elements $\{M_\lambda\}_{\lambda\in\mathcal{D}(\delta-1,N)}$ (identity is included).*

Proof. We have $X^\delta = \sum\limits_{\ell=0}^{\delta-1} a_{\delta\ell} X^\ell$ and, generally, $X^k = \sum\limits_{\ell=0}^{\delta-1} a_{k\ell} X^\ell$. Thus,

$$\begin{aligned} I + \sum_{k=1}^{u} c_k X^k &= I + \sum_{k=1}^{u} \sum_{\ell=0}^{\delta-1} c_k a_{k\ell} X^\ell \\ &= I + \sum_{\ell=0}^{\delta-1} (a_\ell \cdot c) X^\ell = (1 + (a_0 \cdot c))I + \sum_{\ell=1}^{\delta-1} (a_\ell \cdot c) X^\ell \end{aligned}$$

where the sequence of coefficients $a_\ell = (a_{1\ell}, \ldots, a_{u\ell})$, with the dot notation for clarity. Thus factoring out $1 + a_0 \cdot c$ and multiplying back in by $(1 + a_0 \cdot c)^N$ in degree N, we have

$$\overline{I+\sum_{1\le k\le u} c_kX^k} = \sum_{\mathcal{D}(\delta-1,N)} (1+a_0\cdot c)^{N-L(\lambda)}\Bigl(\prod_i (a_{\lambda_i}\cdot c)\Bigr)M_\lambda$$
$$= \sum_{\mathcal{D}(u,N)} c_\lambda M_\lambda$$

Comparing coefficients of monomials in the variables $\{c_1,\ldots,c_u\}$ shows that any M_λ, $\lambda\in\mathcal{D}(u,N)$, is a linear combination of elements in $\{M_\lambda\}_{\lambda\in\mathcal{D}(\delta-1,N)}$. To count how many elements are in $\mathcal{D}(\delta-1,N)$, write

$$\mathcal{D}(\delta-1,N)=\bigcup_{\ell=1}^N \mathcal{D}(\delta-1;\ell)$$

the union of partitions of exact length $\ell=1,2,\ldots$. The elements of $\mathcal{D}(u;\ell)$ are in 1-1 correspondence with monomials of homogeneous degree ℓ in u variables, $C=\{c_1,\ldots,c_u\}$, say, namely $\lambda\leftrightarrow c_{\lambda_1}c_{\lambda_2}\cdots$, the number of factors giving the homogeneous degree $L(\lambda)$ and the variables taken from the set C. Given such a monomial, reorder the subscripts in (weakly) decreasing order to find the corresponding partition λ. We know from the dimension of the matrices under the induced map that the number of such monomials is $\binom{u+\ell-1}{\ell}$. So we want, counting from $\ell=0$, corresponding to the identity matrix,

$$\sum_{\ell=0}^N \binom{\delta+\ell-2}{\ell} = \sum_{\ell=0}^N \frac{(\delta-1)_\ell}{\ell!}$$

which we note are the partial sums of the coefficients of the generating function $(1-x)^{-(\delta-1)}$ for which the generating function is thus

$$(1-x)^{-1}(1-x)^{-(\delta-1)}=(1-x)^{-\delta}$$

Hence the sum is equal to $\binom{\delta+N-1}{N}$ as stated. □

Corollary 2.3. *If $\delta=2$, then the elementary functions $\{E_\ell\}_{0\le\ell\le N}$, with $E_0=I$, are a basis for the algebra.*

This result suggests looking for a polynomial basis consisting of monomials in δ variables of homogeneous degree N. Starting with the minimal polynomial of X, applying the Gamma map to powers of X, we know that a power sum element P_j, $j\ge\delta$, is a linear combination of elements

$$\mathbb{P}_{0,\delta}=\{P_0,P_1,\ldots,P_{\delta-1}\}$$

with $P_0=\Gamma(I)=N\,\bar I$, effectively including the identity. This gives the required number of monomials: $\binom{\delta+N-1}{N}$. Alternatively, set

$$\mathbb{P}_\delta=\{P_1,P_2,\ldots,P_{\delta-1}\}$$

and define

$$\mathbb{P}_{\delta,N} = \text{monomials in the variables } \mathbb{P}_\delta \text{ of homogeneous degree at most } N$$

by taking out factors of the identity for positive degree, and including the identity as the degree 0 term.

We will show that we can express all elements $\{M_\lambda\}_{\lambda\in\mathcal{D}(\delta-1,N)}$ in terms of these monomials. First note that

$$M_\lambda = \zeta\, P_\lambda + \text{ terms of homogeneous degree at most } N$$

with $P_\lambda = P_{\lambda_1}\cdots P_{\lambda_L} \in \mathbb{P}_{\delta,N}$, where ζ is a known numerical factor. In fact, cf. [14], [19, Cor. 7.7.2], the terms after the P_λ term consist of terms P_μ where $\mu \trianglerighteq \lambda$, in particular $L(\mu) = \mu_1' \le \lambda_1' = L(\lambda)$, with P_λ of homogeneous degree $L(\lambda) \le N$. For terms with P_j, $j \ge \delta$, expand all such factors as linear combinations of variables $\mathbb{P}_{0,\delta}$. Upon expanding these factors, the homogeneous degree does not increase. We have

Proposition 2.4. *The SFA algebra has a polynomial basis consisting of monomials in the variables $\{P_1,\ldots,P_{\delta-1}\}$ of homogeneous degree at most N, including the identity.*

Corollary 2.5. *If $\delta = 2$, then the SFA algebra consists of polynomials in (the identity and) $\Gamma(X)$ of degree at most N.*

Example 2.6. Take $N = 3$, $\delta = 3$ and consider

$$M_{221} = \tfrac{1}{2}\, P_1P_2^2 - P_2P_3 - \tfrac{1}{2}\, P_1P_4 + P_5$$

and observe that upon expressing each of P_3, P_4, and P_5 as linear combinations of $\{I, P_1, P_2\}$ yields terms in $\mathbb{P}_{\delta,N}$ of degree at most $2 < N$.

Example 2.7. For $d = 2$ we have another approach. We look for structure constants for multiplication of the elements E_i

$$E_i\, E_j = \sum_\ell c_{ij}^\ell E_\ell$$

Assume X is not a multiple of the identity, with characteristic polynomial
$X^2 = \tau X - \Delta I$, $\tau = \operatorname{tr} X$, $\Delta = \det X$. We have

$$\overline{I+vX}\;\overline{I+wX} = \sum_{i,j} v^i w^j E_iE_j \tag{2.7}$$

which by the homomorphism property of the induced mapping equals

$$\begin{aligned}\overline{(I+vX)(I+wX)} &= \overline{I+(v+w)X+vwX^2}\\ &= \overline{I+(v+w)X+vw(\tau X-\Delta I)}\\ &= \overline{(1-vw\Delta)I+(v+w+vw\tau)X}\\ &= \sum_\ell (1-vw\Delta)^{N-\ell}(v+w+vw\tau)^\ell\, E_\ell \end{aligned}\tag{2.8}$$

by pulling out a factor of $1 - vw\Delta$ and multiplying back in as $(1 - vw\Delta)^N$ in degree N. Comparing coefficients of v^iw^j in (2.7) with those in (2.8) expanded out gives the structure constants for the algebra. We have, with summation over Greek indices understood,

$$\begin{aligned}(1-vw\Delta)^{N-\ell}&(v+w+vw\tau)^{\ell}\\ &=\binom{N-\ell}{\alpha}(-1)^{\alpha}v^{\alpha}w^{\alpha}\Delta^{\alpha}\binom{\ell}{\beta}(vw\tau)^{\ell-\beta}\binom{\beta}{\gamma}v^{\beta-\gamma}w^{\gamma}\end{aligned}$$

Identifying $i=\alpha+\ell-\gamma$, $j=\alpha+\ell-\beta+\gamma$, we have

$$c_{ij}^{\ell}=\sum_s\binom{N-\ell}{s}\frac{\ell!}{(i+j-\ell-2s)!\,(\ell+s-i)!\,(\ell+s-j)!}(-1)^s\tau^{i+j-\ell-2s}\Delta^s$$

As an example, let $X=\begin{pmatrix}1/2 & 1/2\\ 1 & 0\end{pmatrix}$. For $N=2$ we have the basis $E_0=I$,

$$E_1=\begin{pmatrix}1 & 1 & 0\\ 1 & 1/2 & 1/2\\ 0 & 2 & 0\end{pmatrix},\quad E_2=\bar{X}=\begin{pmatrix}1/4 & 1/2 & 1/4\\ 1/2 & 1/2 & 0\\ 1 & 0 & 0\end{pmatrix}$$

with

$$E_1^2=I+\tfrac{1}{2}E_1+2E_2,\quad E_1E_2=\tfrac{1}{2}E_1+E_2,\quad E_2^2=\tfrac{1}{4}(I+E_1+E_2)$$

as well, e.g., $H_5=\begin{pmatrix}\frac{47}{16} & \frac{41}{16} & \frac{1}{2}\\ \frac{41}{16} & \frac{85}{32} & \frac{25}{32}\\ 2 & \frac{25}{8} & \frac{7}{8}\end{pmatrix}=\frac{7}{8}I+\frac{25}{16}E_1+2E_2\,,$

$$M_{21}=\begin{pmatrix}\frac{3}{4} & 1 & \frac{1}{4}\\ 1 & \frac{3}{4} & \frac{1}{4}\\ 1 & 1 & 0\end{pmatrix}=\tfrac{1}{2}E_1+E_2,\qquad \{32\}=\begin{pmatrix}\frac{15}{16} & \frac{7}{8} & \frac{3}{16}\\ \frac{7}{8} & \frac{7}{8} & \frac{1}{4}\\ \frac{3}{4} & 1 & \frac{1}{4}\end{pmatrix}=\tfrac{1}{4}I+\tfrac{1}{2}E_1+\tfrac{3}{4}E_2$$

etc., noting that $M_{21}=E_1E_2$. In the $\mathbb{P}$-basis, we have, e.g., as polynomials in $P_1=\Gamma(X)=E_1$,

$$E_2=-\tfrac{1}{2}\,I-\tfrac{1}{4}\,P_1+\tfrac{1}{2}\,P_1^2\quad\text{and}\quad H_5=-\tfrac{1}{8}\,I+\tfrac{17}{16}\,P_1+P_1^2\ .$$

We can identify common eigenvectors of the SFA algebra as follows. Make the

Definition 2.8. Given a row vector ξ, a d-tuple, define the *multinomial extension of ξ in degree N* as the vector of coefficients of $(\xi_1x_1+\cdots+\xi_dx_d)^N$, using lexicographic order on the monomials in $\{x_1,\ldots,x_d\}$. Denote this extension by $\bar{\xi}^{(N)}$ or if N is understood, by $\bar{\xi}$.

In short, for a multi-index n, $\bar{\xi}_n=\dfrac{N!}{n!}\,\xi^n$.

Note that if we form the matrix Ξ with all rows equal to ξ, then $\bar{\Xi}$ in degree N will have identical rows all equal to $\bar{\xi}$.

Definition 2.9. Given a column vector θ, define the *monomial extension of* θ *in degree* N as the vector with components the monomials $\theta^n = \theta_1^{n_1}\theta_2^{n_2}\cdots\theta_d^{n_d}$, in lexicographic order. Denote this extension by $\bar{\theta}$ similarly to the multinomial extension, with an indication for the degree if necessary.

In other words, we have $\bar{\theta}_n = \theta^n$ for a multi-index n .

Note: The bar on a row vector refers to the multinomial extension, while on a column vector it indicates the monomial extension.

Proposition 2.10. Special eigenvectors

1. Let ξ be a left eigenvector of X with eigenvalue α. Then $\bar{\xi}$ is a left eigenvector of each E_ℓ with

$$\bar{\xi}E_\ell = \alpha^\ell \tbinom{N}{\ell}\bar{\xi}$$

hence $\bar{\xi}$ is a common (left) eigenvector of all elements of the SFA algebra.

2. Let θ be a right eigenvector of X with eigenvalue α. Then $\bar{\theta}$ is a right eigenvector of each E_ℓ with

$$E_\ell\bar{\theta} = \alpha^\ell \tbinom{N}{\ell}\bar{\theta}$$

hence $\bar{\theta}$ is a common (right) eigenvector of all elements of the SFA algebra.

Proof. Form the matrix Ξ as indicated above. Then we have $\Xi X = \alpha\Xi$ and $\Xi(I + vX) = (1 + v\alpha)\Xi$. Thus

$$\bar{\Xi}\;\overline{I + vX} = (1 + v\alpha)^N\bar{\Xi}$$

that is,

$$\bar{\Xi}\sum_\ell v^\ell E_\ell = \sum_\ell v^\ell \binom{N}{\ell}\alpha^\ell\bar{\Xi}$$

which shows the required result and for the right eigenvector we find directly, writing $Y = I + vX$,

$$\begin{aligned}\prod_i(\sum_j Y_{ij}\theta_j)^{m_i}) = \sum_n \bar{Y}_{mn}\bar{\theta}_n &= \sum_\ell v^\ell(E_\ell\bar{\theta})_m \\ &= \prod_i((1 + v\alpha)\theta_i)^{m_i} = (1 + v\alpha)^N\bar{\theta}_m \\ &= \sum_\ell v^\ell\alpha^\ell\binom{N}{\ell}\bar{\theta}_m\end{aligned}$$

and comparing coefficients of powers of v in the first and third lines above yields the result. ∎

2.1. Nilpotent matrices. Examples.

Example 2.11. Suppose $X^2 = 0$. Then we have $P_1 = \Gamma(X)$, $P_i = 0$, $i \geq 2$. So

$$e^{vX} = I + vX \quad \Rightarrow \quad \overline{e^{vX}} = e^{v\Gamma(X)} = \overline{I + vX}$$

Thus, denoting $\Gamma(X)$ by γ,

$$\mathcal{E}(v) = e^{v\gamma} \quad \text{and} \quad \mathcal{H}(v) = \mathcal{E}(-v)^{-1} = e^{v\gamma} = \mathcal{E}(v)$$

so that

$$E_\ell = H_\ell = \frac{\gamma^\ell}{\ell!}$$

And $E_\ell = 0$ for $\ell > N$ implies that γ is nilpotent: $\gamma^{N+1} = 0$. For S-functions, we have, for $\lambda \vdash \ell$,

$$\{\lambda\} = \sum_{\rho \vdash \ell} \chi^\lambda_\rho \frac{P_1^{\rho_1} P_2^{\rho_2} \cdots P_\ell^{\rho_\ell}}{z_\rho} = \chi^\lambda_{(1^\ell)} \frac{P_1^\ell}{\ell!} = \chi^\lambda_{(1^\ell)} \frac{\gamma^\ell}{\ell!}$$

the sum reducing to a single term. The coefficient $\chi^\lambda_{(1^\ell)}$ is the trace of the identity element for the representation $\{\lambda\}$ of the symmetric group S_ℓ, and thus is the degree of the representation. By the Frame-Robinson-Thrall formula, [19, Cor. 7.21.6], it is known that

$$\chi^\lambda_{(1^\ell)} = \frac{\ell!}{\prod(\text{hook lengths})}$$

thus

$$\{\lambda\} = \frac{\gamma^\ell}{\prod(\text{hook lengths})}$$

a combinatorial formula for the S-functions.

For example, with a typical hook indicated in red, with entries the hook length at each cell, for $\lambda = [4431]$, we have

7	5	4	2
6	4	3	1
4	2	1	
1			

with the product of the hook lengths equal to 161280.

For the monomial functions, we have

$$\overline{I + \sum_{k=1}^{u} c_k X^k} = \overline{I + c_1 X} = \sum_{\mathcal{D}(1,N)} c_1^\ell M_{(1^\ell)}$$

$$= I + \sum_{1 \leq \ell \leq N} c_1^\ell \Gamma_\ell(X) = I + \sum_{1 \leq \ell \leq N} c_1^\ell E_\ell$$

accordingly. Thus

$$M_\lambda = \begin{cases} E_\ell, & \text{if } \lambda = [1^\ell] \\ 0, & \text{otherwise} \end{cases}$$

Another nilpotent example:

Example 2.12. If $X^3=0$, then we have $P_i=0$ for $i\ge 3$ so that

$$\mathcal{E}(v)=\exp\bigl(vP_1-\frac{v^2}{2}P_2\bigr)=\sum_{\ell\ge 0}\frac{v^\ell}{\ell!}\,\mathrm{He}_\ell(P_1,P_2)$$

where $\mathrm{He}(x,t)$ are Hermite polynomials for variance t. And

$$\mathcal{H}(v)=\exp\bigl(vP_1+\frac{v^2}{2}P_2\bigr)=\sum_{\ell\ge 0}\frac{v^\ell}{\ell!}\,\mathrm{he}_\ell(P_1,P_2)$$

where $\mathrm{he}_\ell(x,t)$ are the Hermite moment polynomials

$$\mathrm{he}_\ell(x,t)=\int_{\mathbb{R}}(x+y)^\ell\,e^{-\frac{y^2}{2t}}\,\frac{dy}{\sqrt{2\pi t}}$$

For S-functions, we have, for $\lambda\vdash\ell$,

$$\{\lambda\}=\sum_{\rho\vdash\ell}\chi^\lambda_\rho\,\frac{P_1^{\rho_1}P_2^{\rho_2}}{2^{\rho_2}\,\rho_1!\,\rho_2!}$$

Noting that $\rho_1+2\rho_2=\ell$, substitute $\rho_2=s$, $\rho_1=\ell-2s$, yielding

$$\{\lambda\}=\sum_{0\le 2s\le\ell}\chi^\lambda_\rho\,\frac{P_1^{\ell-2s}P_2^s}{2^s\,(\ell-2s)!\,s!}=\frac{1}{\ell!}\sum_{0\le 2s\le\ell}\chi^\lambda_{(1^{\ell-2s}2^s)}\binom{\ell}{2s}\frac{(2s)!}{2^s\,s!}P_1^{\ell-2s}P_2^s$$

which checks with formulas for Hermite polynomials when specialized to the elementary and homogeneous functions.

For the monomial functions, we have, for $u\ge 2$,

$$\overline{I+\sum_{k=1}^{u}c_kX^k}=\overline{I+c_1X+c_2X^2}=\sum_{\mathcal{D}(2,N)}c_1^{\rho_1}c_2^{\rho_2}M_{(1^{\rho_1}2^{\rho_2})}\tag{2.9}$$

where the sum is over $\rho_1+\rho_2\le N$.

Another calculation of interest is to go back to the exponential definition of Γ. For $X^3=0$, we have

$$\overline{e^{tX}}=e^{t\Gamma(X)}=I+t\Gamma(X)+\frac{t^2}{2}\,\Gamma(X)^2+\cdots+\frac{t^\ell}{\ell!}\,\Gamma(X)^\ell+\cdots\tag{2.10}$$

on the one hand, and on the other, writing in terms of monomials, as in eq. (2.9),

$$\begin{aligned}\overline{e^{tX}}&=\overline{I+tX+t^2X^2/2}=I+\sum_{\mathcal{D}(2,N)}t^{\rho_1}\,\frac{t^{2\rho_2}}{2^{\rho_2}}\,M_\rho\\&=I+\sum_\ell\sum_{\substack{\rho_1+2\rho_2=\ell\\ \rho_1+\rho_2\le N}}t^\ell\frac{1}{2^{\rho_2}}\,M_\rho\end{aligned}$$

Comparing with eq. (2.10), we have, matching powers of t,

$$\frac{\Gamma(X)^\ell}{\ell!}=\sum_{\substack{\rho_1+2\rho_2=\ell\\ \rho_1+\rho_2\le N}}\frac{1}{2^{\rho_2}}\,M_\rho$$

Example 2.13. We can extend this last set of relations for general $X^{p+1} = 0$ and, hence, as well for general X. Start with eq. (2.10), continuing with

$$\begin{aligned}\overline{e^{tX}} &= \overline{I + tX + t^2X^2/2 + \cdots + \frac{t^\ell}{\ell!}X^\ell + \cdots} \\ &= I + \sum_\ell \sum_{\substack{|\lambda|=\ell \\ L(\lambda)\le N}} \frac{t^{|\lambda|}}{(1!)^{\rho_1}(2!)^{\rho_2}\cdots(\ell!)^{\rho_\ell}}\, M_\rho\end{aligned}$$

Note that the denominator is exactly $\lambda_1!\lambda_2!\cdots\lambda_L!$ so that in fact the coefficient of M_λ times $|\lambda|!$ is a multinomial coefficient. I.e., comparing powers of t,

$$\Gamma(X)^\ell = \sum_{\substack{|\lambda|=\ell \\ L(\lambda)\le N}} \frac{|\lambda|!}{\lambda_1!\lambda_2!\cdots\lambda_L!}\, M_\lambda \tag{2.11}$$

which is the multinomial theorem formulated in terms of symmetric functions. Recall the matrix mapping m_λ to $\mathrm{e}_\lambda = e_{\lambda_1}\cdots \mathrm{e}_{\lambda_L}$, [8, Prop. 2.4],[12, p. 102, 6.7 (i)],

$$\mathrm{e}_\lambda = \sum_\mu T_{\lambda\mu}\mathrm{m}_\mu$$

where $T_{\lambda\mu}$ is the number of $|\lambda|\times|\lambda|$, 0-1 matrices with row sums λ_i and columns sums μ_j. We have the expansion of e_1^ℓ in terms of monomial symmetric functions. That is, writing μ for λ above,

$$T_{(1^\ell)\mu} = \frac{\ell!}{\mu_1!\mu_2!\cdots\mu_L!}$$

the multinomial coefficient.

2.2. Symmetric functions algebra for special power sums. Let's look at the properties of an SFA for which the power sums have the form

$$P_i = s^i\, T \tag{2.12}$$

for a fixed T.

Definition 2.14. Define the class SPS(s,T) by the condition $X \in$ SPS(s,T) if and only if $P_i(X) = \Gamma(X^i) = s^i\, T$, for $i \ge 1$. This is the class of X with *special power sums with parameters s and T.*

We have, e.g., expressing elementary functions in terms of power sum functions,

$$\begin{aligned}E_2 &= \tfrac{1}{2}(P_1^2 - P_2) = \tfrac{1}{2}(T)(T-I)s^2 = s^2\binom{T}{2} \\ E_3 &= \tfrac{1}{6}\,P_1^3 - \tfrac{1}{2}\,P_1P_2 + \tfrac{1}{3}\,P_3 = \tfrac{1}{6}(T^3 - 3T^2 + 2T)\,s^3 = s^3\binom{T}{3}\end{aligned}$$

etc.

Generally, cf. [8, §4.6.1], [11, §7.2], [12, p. 45,4.], we have

Proposition 2.15. *If $X \in \mathrm{SPS}(s,T)$, then*

1. For $\lambda \vdash \ell$,

$$\{\lambda\} = s^\ell \sum_{\rho\vdash\ell} \chi^\lambda_\rho \frac{T^{L(\rho)}}{z_\rho} = s^\ell \frac{\prod\limits_{1\le i\le L} (T-i+1)_{\lambda_i}}{\prod(\text{hook lengths})}$$

which specializes to

2. $E_\ell = \{1^\ell\} = s^\ell \dfrac{T(T-1)\cdots(T-\ell+1)}{\ell!} = s^\ell \dbinom{T}{\ell}$

and

3. $H_\ell = \{\ell\} = s^\ell \dfrac{(T)_\ell}{\ell!}$

where we observe that the denominators for E_ℓ and H_ℓ are $\ell!$, the product of the corresponding hook lengths.

For the monomial functions, first note

Lemma 2.16. *If $X \in \mathrm{SPS}(s,T)$, then for any polynomial $f(X)$, we have*

$$\Gamma(f(X)) = f(s)\,T$$

Proof. Immediate. □

Now we can find the left-hand side in the basic relation (2.5)

$$\overline{I + \sum_{1\le k\le u} c_k X^k} = \sum_{\mathcal{D}(u,N)} c^\rho M_\rho$$

Write $g(X) = \sum\limits_{1\le k\le u} c_k X^k$, and consider

$$\begin{aligned}\overline{I+vg(X)} &= \exp\left(-\sum_{i\ge1}\frac{(-v)^i}{i}\,\Gamma(g(X)^i)\right)\\ &= \exp\big(-\sum_{i\ge1}\frac{(-v)^i}{i}\,g(s)^i\,T\big)\\ &= (1+v\sum_{1\le k\le u} c_k s^k)^T\end{aligned}$$

(cf. eq. (4.3) and Proposition 4.6 of [8]) which we expand as a binomial series and then expand powers of $g(s)$ yielding

$$\sum_j \binom{T}{j} v^j g(s)^j = \sum_j \binom{T}{j} v^j \sum_{\substack{\lambda\in\mathcal{D}(u;j)\\ \rho=\rho(\lambda)}} \frac{j!}{\rho!}\,c^\rho s^{|\lambda|}$$

noting that $j = L(\lambda)$, the length of λ, recalling the partition domain $\mathcal{D}(u;j) = \{\lambda\colon \lambda_1 \le u, L(\lambda) = j\}$. Note that once we have polynomial identities, we can let $v = 1$. And pulling out the coefficient of c^ρ we have

Theorem 2.17. *If X has special power sums, $P_i = s^i T$, then*

$$M_\lambda = \frac{L(\lambda)!}{\prod\limits_i \rho_i!} \binom{T}{L(\lambda)} s^{|\lambda|}$$

where $\rho = \rho(\lambda)$.

We check consistency,

$$M_{(1^\ell)} = E_\ell = \frac{\ell!}{\rho_1!}\binom{T}{\ell} s^\ell = s^\ell \binom{T}{\ell}$$

with $\rho_1 = \ell$, and

$$M_{(\ell)} = P_\ell = \frac{1!}{\rho_\ell!}\binom{T}{1} s^\ell = s^\ell\, T$$

with $\rho_\ell = 1$.

2.3. SPS mapping. The functions that appear for the SPS case are worth defining so as to be available for a variety of contexts. Referring to eq. (2.3),

Definition 2.18. Define the SPS-map $\psi(-;s,T)$ on $\mathbb{G}$ by

$$\psi(E_\ell;s,T) = s^\ell \binom{T}{\ell}, \qquad \psi(H_\ell;s,T) = s^\ell\, \frac{(T)_\ell}{\ell!}$$

$$\psi(P_i;s,T) = s^i\, T$$

$$\psi(M_\lambda;s,T) = \frac{L(\lambda)!}{\prod\limits_i \rho_i!} \binom{T}{L(\lambda)} s^{|\lambda|}$$

$$\psi(\{\lambda\};s,T) = s^{|\lambda|}\, \frac{\prod\limits_{1\le i\le L} (T-i+1)_{\lambda_i}}{\prod(\text{hook lengths})}$$

Notice that it suffices to define the map on the M_λ's and S-functions as the E's, H's, and P's are special cases of those families. From Proposition 2.15 and Theorem 2.17, we have, denoting elements of $\mathbb{G}$ by F, $F(X)$ denoting E_ℓ, H_ℓ, etc. in SFA$(^-, X)$,

Theorem 2.19. *If $X \in \mathrm{SPS}(s,T)$, then $F(X) = \psi(F;s,T)$ for all $F \in \mathbb{G}$.*

2.4. Symmetric functions algebra for an idempotent. If Ω is an idempotent, $\Omega^2 = \Omega$, then it will have special power sums, as

$$P_i = \Gamma(\Omega^i) = \Gamma(\Omega)$$

Thus, by the results of the previous section, with $s = 1$:

Proposition 2.20. *Let Ω be an idempotent. Then we have $P_i = \Gamma(\Omega)$, for $i \ge 1$, and $\Omega \in \mathrm{SPS}(1, \Gamma(\Omega))$, so that*

$$F(\Omega) = \psi(F;1,\Gamma(\Omega)), \text{ for all } F \in \mathbb{G}\ .$$

Another way to look at this case is to use the identity

$$e^{t\Omega} = I + (e^t - 1)\Omega$$

Write $e^t - 1 = v$, $e^t = 1 + v$ so that $(1+v)^\Omega = I + v\,\Omega$ and

$$\overline{(1+v)^\Omega} = (1+v)^{\Gamma(\Omega)} = \overline{I + v\,\Omega} = \sum_{\ell \geq 0} v^\ell \Gamma_\ell(\Omega) \tag{2.13}$$

yielding

$$\sum_{\ell \geq 0} \binom{\Gamma(\Omega)}{\ell} v^\ell = \sum_{\ell \geq 0} v^\ell \Gamma_\ell(\Omega)$$

That is,

$$\Gamma_\ell(\Omega) = E_\ell = \binom{\Gamma(\Omega)}{\ell} .$$

Proposition 2.21. *In degree N, we have the identity*

$$\Gamma(\Omega)(\Gamma(\Omega) - I)(\Gamma(\Omega) - 2I) \cdots (\Gamma(\Omega) - N\,I) = 0$$

in particular, for $\Omega \neq I$, $x(x-1)\cdots(x-N)$ is the minimal polynomial of $\Gamma(\Omega)$.

Proof. We know that $E_{N+1} = 0$ which reads

$$\frac{\Gamma(\Omega)(\Gamma(\Omega) - I) \cdots (\Gamma(\Omega) - (N-1)I)(\Gamma(\Omega) - N)}{(N+1)!} = 0$$

Since $\Gamma(\Omega)$ is not a multiple of the identity, this is the minimal polynomial for $\Gamma(\Omega)$. □

Remark 2.22. Note that $\Gamma(I) = N\bar{I}$ has minimal polynomial $x - N$.

As $E_N = \bar{\Omega}$, taking the first N factors in numerator and denominator in the above expression, we see that

$$\Gamma(\Omega)\,\bar{\Omega} = \bar{\Omega}\,\Gamma(\Omega) = N\bar{\Omega} . \tag{2.14}$$

Example 2.23. Note that if $X^2 = sX$, $s \neq 0$, then $X^i = s^{i-1}X$, $i \geq 1$. Now, we have $X = s\Omega$ for idempotent $\Omega = s^{-1}X$. Thus,

$$P_i = \Gamma(X^i) = s^i \Gamma(s^{-1}X) = s^i \Gamma(\Omega)$$

so that $X \in \mathrm{SPS}(s, \Gamma(\Omega))$.

2.5. Symmetric functions algebra for a reflection. The main families have an interesting structure for the case of a reflection, $X^2 = I$. See Appendix I for details.

3. Constant Row Sums

Notation. Define J to be the $d \times d$ all 1's matrix, with $\mathfrak{J}$, the corresponding $\binom{N+d-1}{N} \times \binom{N+d-1}{N}$ all ones matrix in degree N. The diagonal of $\bar{J}$ consists of the corresponding sequence of multinomial coefficients and the associated diagonal matrix is denoted $\mathcal{B}$. (See Lemma 3.5 below.)

Definition 3.1. Define CRS to be the class of matrices with constant row sums. For $X \in$ CRS we will typically denote the common row sum by s , the eigenvalue of X with eigenvector the corresponding all 1's vector. The subset of CRS consisting of entry-wise nonnegative matrices with positive constant row sums will be called *essentially stochastic* matrices. We denote this class by ES. The class of matrices with zero row sums is denoted ZRS.

Remark 3.2. See [1], [2] for details regarding CRS and ZRS families.

To say that $X \in$ CRS with common row sums s is equivalent to the relation

$$XJ = s\,J$$

And in that case, if in addition, X is nonnegative then X is essentially stochastic if and only if $s > 0$ and $s^{-1}\,X$ is stochastic.

We observe

Proposition 3.3. *1.* CRS *is a subalgebra of the algebra of $d \times d$ matrices.*

2. ES *is closed under addition, multiplication and scaling by positive numbers. In particular, it is a convex cone.*

We define the map $\mathfrak{s}\colon \mathrm{CRS} \to \mathbb{R}$ by

$$XJ = \mathfrak{s}(X)J$$

i.e., $\mathfrak{s}(X)$ is the common value of the row sums of X. We have

Proposition 3.4. *Properties of the map $\mathfrak{s}$.*

1. The map $\mathfrak{s}\colon \mathrm{CRS} \to \mathbb{R}$ is a multiplicative linear functional with kernel ZRS*.*

2. ZRS *is a maximal ideal of* CRS*.*

Proof. The first statement is clear. For #2, ZRS is clearly an ideal of CRS. Now take $X \notin$ ZRS. Then we have, with $\mathfrak{s}(X) \neq 0$,

$$Z = X - \mathfrak{s}(X)I \in \mathrm{ZRS} \quad \Rightarrow \quad I = \mathfrak{s}(X)^{-1}(X - Z) \in \text{ideal generated by } \{X\} \cup \mathrm{ZRS}$$

which thus is all of CRS. □

We have the basic

Lemma 3.5. *Let J denote the $d \times d$ all ones matrix. Let $\mathcal{B}$ denote the $\binom{N+d-1}{N} \times \binom{N+d-1}{N}$ diagonal matrix with entries $\mathcal{B}_{nn} = \frac{N!}{n!}$, multinomial coefficients. Then*

$$\bar{J} = \mathfrak{J}\mathcal{B}$$

where $\mathfrak{J}$ is the $\binom{N+d-1}{N} \times \binom{N+d-1}{N}$ all ones matrix.

Proof. We have

$$\prod_i \Big(\sum_j J_{ij} x_j\Big)^{m_i} = \prod_i \Big(\sum_j x_j\Big)^{m_i} = \sum_n \frac{N!}{n!} x^n = \sum_n \bar{J}_{mn} x^n$$

so that $\bar{J}$ has identical rows with $\bar{J}_{mn} = \frac{N!}{n!}$. Observe that the diagonal of $\bar{J}$ is $\mathcal{B}$. □

And:

Proposition 3.6. *Some properties of induced matrices.*

1. *If X is rank one, then so is $\bar{X}$ in every degree.*
2. *$X \in \mathrm{CRS}$ implies $\bar{X} \in \mathrm{CRS}$ in every degree. In particular, in degree N, $\mathfrak{s}(\bar{X}) = \mathfrak{s}(X)^N$.*
3. *$X \in \mathrm{ZRS}$ implies $\bar{X} \in \mathrm{ZRS}$ in every degree.*

Proof. For #1, write $X_{ij} = b_i c_j$. Then

$$\begin{aligned}\prod_i (\sum_j X_{ij} x_j)^{m_i} &= b^m \prod_i (\sum_j c_j x_j)^{m_i} \\ &= b^m (\sum_j c_j x_j)^N = b^m \sum_n \frac{N!}{n!} c^n x^n\end{aligned}$$

so that $\bar{X}_{mn} = b^m \, (\frac{N!}{n!} c^n)$. For #2 and #3, say $XJ = sJ$, then

$$\bar{X}\bar{J} = s^N \bar{J}$$

By the Lemma, we have

$$\bar{X}\mathfrak{J}\mathcal{B} = s^N \mathfrak{J}\mathcal{B}$$

and since $\mathcal{B}$ is invertible, we recover

$$\bar{X}\mathfrak{J} = s^N \mathfrak{J}$$

as required. Then #3 follows directly with $s = 0$. □

And to get started on row sums in the SFA, we have

Lemma 3.7. *Let $X \in \mathrm{CRS}$. Then*

1. *$\mathfrak{s}(\Gamma(X)) = N\mathfrak{s}(X)$.*
2. *$\mathfrak{s}(\Gamma_\ell(X)) = \binom{N}{\ell}\, \mathfrak{s}(X)^\ell$.*
3. *$\mathfrak{s}(P_i) = N\, \mathfrak{s}(X)^i$.*

Proof. We have $XJ = sJ$ so that $(I + vX)J = (1 + vs)J$ which implies

$$\begin{aligned}\overline{(I + vX)}\,\bar{J} &= \big(I + \sum_{\ell \geq 1} v^\ell \Gamma_\ell(X)\big)\bar{J} = (1 + vs)^N \bar{J} \\ &= (1 + \sum_{\ell \geq 1} \binom{N}{\ell} v^\ell s^\ell)\bar{J}\end{aligned}$$

noting that a common factor of $\mathcal{B}$ on the right cancels to convert $\bar{J}$ to $\mathfrak{J}$, the all ones matrix of the appropriate size. Comparing coefficients of powers of v yields #2 of which #1 is the special case $\ell = 1$. For #3, we have $P_i = \Gamma(X^i)$, with $\mathfrak{s}(X^i) = (\mathfrak{s}(X))^i$, and #1 applies to give the result. □

Note that #2 gives the result for the elementary functions

$$\mathfrak{s}(E_\ell) = \binom{N}{\ell} \mathfrak{s}(X)^\ell$$

with #3 the result for the power sum functions.

Theorem 3.8. *Let $X \in \mathrm{CRS}$. Then the elementary, homogeneous, power sum, monomial and Schur functions will all be in* CRS.

Proof. The fact that the elementary functions generate SFA as an algebra implies that any element of the algebra acts as a scalar multiple when multiplying $\mathfrak{J}$ on the left. Note that a similar argument holds using the fact that power sum functions generate SFA. □

Next, the results including the homogeneous, monomial, and S-functions.

3.1. Row sums for monomial and S-functions. We now find the row sums for the monomial and S-functions for $X \in$ CRS. The idea is that if we have an expression involving P_i's, then for finding the row sums when acting on $\mathfrak{J}$, we have, by Lemma 3.7, #3,

$$P_i\mathfrak{J} = Ns(X)^i\,\mathfrak{J}$$

so that effectively we are in the case of special power sums with $s = \mathfrak{s}(X)$, $T = N$. Now

$$P_\lambda\mathfrak{J} = N^{L(\lambda)}\,\mathfrak{s}(X)^{|\lambda|}\mathfrak{J}$$

as in the first equality in Proposition 2.15, #1. So, by the second equality there, we have the S-functions acting on $\mathfrak{J}$ according to SPS$(\mathfrak{s}(X), N)$ and hence, thinking of monomial functions expressed in terms of power sums, we have all of the basic families acting analogously.

Proposition 3.9. *If $X \in$ CRS, then, for $F \in \mathbb{G}$,*

$$\mathfrak{s}(F) = \psi(F; \mathfrak{s}(X), N)$$

invoking Theorem 2.17 for the monomial functions.

4. Nonnegativity

A major property of the various symmetric functions bases is that they are all nonnegative if X is nonnegative.

Definition 4.1. A matrix X is *nonnegative*, $X \geq 0$, if all of its entries are nonnegative, $X_{ij} \geq 0$, for all i, j.

A matrix X is *positive*, $X > 0$, if all of its entries are positive, $X_{ij} > 0$, for all i, j.

Further, we define relations between nonnegative matrices as follows

$$\begin{aligned} &A \supset B \quad \text{means } \exists\, \alpha > 0 \text{ and nonnegative } C \text{ such that } A = \alpha B + C \\ &A \geq B \quad \text{means } A_{ij} \geq B_{ij}, \forall i, j \\ &A > B \quad \text{means } A_{ij} > B_{ij}, \forall i, j \end{aligned}$$

In particular, if $B > 0$ and $A \supset B$, then $A > 0$.

4.1. Positivity properties of the induced map. First observe that

Proposition 4.2. *The induced map preserves positivity properties, that is,*

1. *If $X \geq 0$ then $\bar{X} \geq 0$.*
2. *If $X > 0$ then $\bar{X} > 0$.*

which follow immediately from the definition $(Xx)^m = \sum(\bar{X})_{mn}x^n$, eq. (2.2).

Note, however, that the Γ-maps, while preserving nonnegativity, do not preserve positivity, as Γ_ℓ shows transitions involving ℓ of the N possibilities. For example, for $N = 3$,

$$X = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \Rightarrow \quad \Gamma(X) = \Gamma_1(X) = \begin{pmatrix} 3 & 6 & 0 & 0 \\ 3 & 6 & 4 & 0 \\ 0 & 6 & 9 & 2 \\ 0 & 0 & 9 & 12 \end{pmatrix}$$

and generally, for a 2×2 matrix, $\Gamma(X)$ will be tridiagonal. On the other hand, we have

$$E_N = \Gamma_N(X) = \bar{X}$$

which *is* positive if X is.

4.2. Positivity properties of the monomial functions. The idea is to show nonnegativity of the M_λ's as that will imply nonnegativity for the remaining basic families and thus for the bases formed by products. Recall

$$\begin{aligned} E_\ell = M_{(1^\ell)} \quad &\text{and} \quad H_\ell = \sum_{|\lambda|=\ell} M_\lambda \\ P_i = M_{(i)} \quad &\text{and} \quad \{\lambda\} = \sum_\mu K_{\lambda\mu} M_\mu \end{aligned} \tag{4.1}$$

where the coefficients for the S-functions are the Kostka numbers $\{K_{\lambda\mu}\}$ counting types of tableaux [16, Definition 7.6].

And

Lemma 4.3. Nonnegativity of the monomial functions

If $X \geq 0$, then $M_\lambda \geq 0$ for every λ.

Proof. Use the basic relation (2.5)

$$\overline{I + \sum_{1 \leq k \leq u} c_k X^k} = \sum_{\mathcal{D}(u,N)} c^\rho M_\rho$$

here thinking of the c's as indeterminates. If $X \geq 0$, the left-hand side will be an expansion in the c's with nonnegative coefficents, which are precisely the monomial functions. □

And taking into account the above remarks, we have

Theorem 4.4. Preservation of nonnegativity

If $X \geq 0$, then all of the elements in the basic classes, $\mathbb{G}$, (2.3), are nonnegative. And hence the product families $\{E_\lambda\}$, $\{H_\lambda\}$, $\{P_\lambda\}$, that form linear bases for the algebra as well as the monomial and S-functions all consist of nonnegative matrices.

5. Convergence

First, recall some facts about the convergence of a Markov chain with transition matrix A.

Denote an invariant distribution for A by $\pi^\dagger$, i.e. $\pi^\dagger A = \pi^\dagger = (p_1, \ldots, p_d)$. The entries $p_i > 0$, for all i with $\sum_i p_i = 1$.

Let u denote the all-ones vector. A is *ergodic*, if A^k converges as $k \to \infty$, $\lim_{k\to\infty} A^k = \Omega = u\pi^\dagger$, satisfying $\Omega^2 = \Omega = A\Omega = \Omega A$. Note that Ω is a stochastic rank one idempotent with constant rows.

The *state transition diagram*, S.T.D., of A is the directed graph with vertices $\{1, \ldots, d\}$ with an edge from i to j if $A_{ij} > 0$.

The notions of irreducibility and aperiodicity in terms of the S.T.D. may be taken as definitions for corresponding properties of the Markov chain. See [15, Ch 4, §3] for a detailed presentation from the graph perspective.

A is *irreducible* if for every pair (i, j) there is a path in the S.T.D. with initial vertex i and final vertex j. That is, the graph is strongly connected.

In the reducible case, a *communicating class*, C, is a set of states that forms a strongly connected component of the S.T.D. If there are no transient states, these classes comprise a partition of the set of states. This is the situation of interest here.

A *cycle* in the S.T.D. is a path with equal initial and final vertices.

For A irreducible, it is *aperiodic* if the greatest common divisor of all cycle lengths is 1. Otherwise it is *periodic*.

A nonnegative matrix A is *primitive* if for some power $k \in \mathbb{N}$ of A, $A^k > 0$, i.e., A^k has all positive entries. The basic results are these:

Theorem 5.1. [15, Ch. 3, Th. 2.1],[17, Th. 1.4]

A is irreducible and aperiodic if and only if it is primitive.

and

Theorem 5.2. [17, Th. 1.2], [3, Ch. 5, §2]

A is ergodic if and only if it is primitive.

For a general setting, see [3, Ch. 5, §2], where in the Appendix it is remarked that the implication “primitive implies ergodic” goes back to Markov [13].

Remark 5.3. There may be a transient class of states such that eventually the chain leaves it and enters a closed ergodic class of states. In that case, A will have a left eigenvector with eigenvalue 1, i.e., a left-invariant vector, with zero entries. We will not consider this case.

Definition 5.4. Given a nonnegative matrix $X \in \mathrm{ES}$, i.e., having positive constant row sums, denote by $\hat{X}$ the corresponding normalized matrix having constant row sums equal to 1.

In other words, set

$$\hat{X} = \frac{1}{\mathfrak{s}(X)}\, X$$

which is thus a stochastic matrix.

First, observe

Proposition 5.5. *If B is primitive and stochastic $A \supset B$, then A is ergodic.*

which follows immediately from the equivalence with primitivity.

Most important in the present context:

If A is stochastic, then $\bar{A}$ is stochastic.

as follows from nonnegativity and the CRS property $\mathfrak{s}(\bar{X}) = \mathfrak{s}(X)^N$. And from positivity and the homomorphism property $\overline{A^k} = (\bar{A})^k$ it follows that

$\bar{A}$ is ergodic, if A is.

In the following, A is a given stochastic matrix, cf. [20].

Proposition 5.6. *Let A be a primitive stochastic matrix with left eigenvector $\pi^\dagger$, so that $\lim\limits_{k\to\infty} A^k = \Omega = u\pi^\dagger$, where u is a column of 1's.* Then:

In degree N, we have the invariant distribution ${\pi^{(N)}}^\dagger$ given by the multinomial distribution for N independent trials with d choices per trial with probabilities given by $\pi^\dagger$. And $\lim\limits_{k\to\infty} \bar{A}^k = \bar{\Omega} = u_N {\pi^{(N)}}^\dagger$, where u_N is the corresponding all-ones vector. For any primitive stochastic B such that $\bar{B}$ has invariant distribution ${\pi^{(N)}}^\dagger$ it follows that $\lim\limits_{k\to\infty} \bar{B}^k = \bar{\Omega} = u_N {\pi^{(N)}}^\dagger$.

Proof. For a primitive stochastic matrix, the Perron-Frobenius Theorem, [17, Th. 1.1], implies that the invariant distribution, normalized left eigenvector for eigenvalue 1, is unique, with 1 a simple eigenvalue. Now if A is primitive and stochastic, so is $\bar{A}$. We have

$$A\Omega = \Omega A = \Omega = \Omega^2$$

so by the homomorphism property

$$\bar{A}\bar{\Omega} = \bar{\Omega}\bar{A} = \bar{\Omega} = \bar{\Omega}^2$$

with the rows of $\bar{\Omega}$ normalized left eigenvectors of $\bar{A}$ all equal. Again by the homomorphism property (and continuity) we have

$$\lim_{k\to\infty} A^k = \Omega \quad \Longrightarrow \quad \lim_{k\to\infty} \bar{A}^k = \bar{\Omega}$$

which follows as well by primitivity and uniqueness.

If B is primitive stochastic with $\bar{\Omega}\bar{B} = \bar{\Omega}$, then $\bar{B}$ is primitive, stochastic and by uniqueness, the limit of powers of $\bar{B}$ must be $\bar{\Omega}$.

As in the proof of Prop, 3.6, #1, we have with $\pi^\dagger = (p_1, \ldots, p_d)$,

$$\begin{aligned}\prod_i (\sum_j \Omega_{ij} x_j)^{m_i} &= \prod_i (\sum_j p_j x_j)^{m_i} \\ &= (\sum_j p_j x_j)^N = \sum_n \frac{N!}{n!} p^n x^n\end{aligned}$$

so that $\bar{\Omega}_{mn} = \frac{N!}{n!} p^n$, noting that $N!/n!$ is a multinomial coefficient in compressed notation, so the entries of each row of $\bar{\Omega}$ are precisely the multinomial probabilities as stated. $\square$

NOTE: If a matrix in ES is primitive, then the corresponding normalized matrix will be an ergodic stochastic matrix.

We will use the equivalence between ergodicity and primitivity for stochastic matrices, Theorem 5.2, as our basic tool.

Our main result in this section is

Theorem 5.7.
1. If A is ergodic, then in $\mathrm{SFA}(\bar{\ }, A)$, *every normalized basic element:* $\{\hat{E}_\ell\}$, $\{\hat{H}_\ell\}$, $\{\hat{P}_i\}$, $\{\hat{M}_\lambda\}$, $\{\{\hat{\lambda}\}\}$ *is ergodic. Hence each element of the corresponding product bases* $\{\hat{E}_\lambda\}$, $\{\hat{H}_\lambda\}$, $\{\hat{P}_\lambda\}$ *is ergodic. And every element of the semigroup generated by* $\mathbb{G}$ *is primitive.*

2. If $\lim_{k\to\infty} A^k = \Omega$, *then for every* $F \in \mathbb{G}$, *the elements of the basic classes,* $\lim_{k\to\infty} \hat{F}^k = \overline{\Omega}$, *the corresponding induced matrix.*

Proof. We will show primitivity below. We have $\bar{\Omega}$ as the limit of powers of A which commutes with F, hence $\bar{\Omega}$ commutes with each $F \in \mathbb{G}$. Since $F \in \mathrm{CRS}$ with positive row sums and each column of $\bar{\Omega}$ is a multiple of the all-ones vector, we have $\bar{\Omega}$ a stochastic idempotent with

$$F\bar{\Omega} = \mathfrak{s}(F)\bar{\Omega} = \bar{\Omega}F$$

That is, any row of $\bar{\Omega}$ is a normalized left eigenvector of eigenvalue 1 for the normalized $\hat{F}$. By uniqueness, Prop. 5.6, such a row is the invariant distribution for $\hat{F}$ and $\bar{\Omega}$ is the limit idempotent for powers of $\hat{F}$. □

We can check that $\bar{\Omega} \in \mathrm{SFA}(\bar{\ }, X)$:

Proposition 5.8. *For ergodic A, the limiting idempotent for $\bar{A}$, $\bar{\Omega}$, is an element of the SFA algebra.*

Proof. Write the minimal polynomial of A, $f(x)$, factoring out $x - 1$:

$$f(x) = (x - 1)q(x)$$

with $q(A) \neq 0$. Now any non-zero row of $q(A)$ is a left eigenvector of A for eigenvalue 1 and similarly each non-zero column of $q(A)$ is a right eigenvector. But $q(A)$ has constant row sums and is non-zero, hence every row of $q(A)$ is the same multiple of the invariant distribution for A. Hence $q(A)$ is a scalar multiple of Ω. We know that the induced matrix of any polynomial in A is a linear combination of monomials, M_λ, hence $\bar{\Omega}$ is an element of SFA. □

Definition 5.9. Call the least power of a primitive nonnegative matrix to have all positive entries the *index of primitivity*.

5.1. Ergodicity of the monomial functions. Now we use a strategy based on §4.2, eqs. (4.1). Once we show the $\hat{M}_\lambda$ are ergodic for all λ, then it will follow for all of the basic elements.

As noted above, if A is ergodic, then $\bar{A}$ will be ergodic: $A^r > 0$ implies $(\bar{A})^r > 0$.

Recall the normalizations, row sums, for the basic symmetric function matrices, Definition 2.18, with $s = 1$ and $T = N$. That is, they all have positive constant row sums for stochastic A.

Thus, if $B > 0$, then $FB > 0$ for any basic element $F \in \mathbb{G}$, since every such F has positive (constant) row sums. So we will show that for any M_λ, some power contains a term of the form $M_\mu \bar{A}$, for some μ. Hence $\hat{M}_\mu \bar{A}$ will be ergodic and hence the power of $\hat{M}_\lambda$ containing it will be ergodic.

Example 5.10. We start with showing that A ergodic implies $\hat{\Gamma}(X)$ ergodic. Setting $\ell = N$ in eq. (2.11), with stochastic A replacing X, shows that

$$\Gamma(A)^N \supset M_{(1^N)} = \bar{A}$$

and hence $\Gamma(A)$ is primitive. If A has index of primitivity p, then $\Gamma(A)^{pN} > 0$. Since A^i is ergodic for all $i > 0$, $P_i = \Gamma(A^i)$ is primitive and since the H's are positive combinations of products of P's, it follows they are primitive as well.

5.1.1. *Product of M_λ's.* First,

Remark 5.11. Recall, Prop. 2.1, that in degree N, M_λ and $\{\lambda\}$ vanish for all $L(\lambda) > N$.

We need some notations first.

Notation. Given two partitions λ and μ, form tuples of length $L(\lambda) + L(\mu)$ by appending $L(\mu)$ zeros to λ and $L(\lambda)$ zeros to μ. These tuples are denoted λ^+ and μ^+ respectively.

Given a tuple of nonnegative integers, put the values in weakly decreasing order, and strip trailing zeros to produce a partition. For a tuple ξ, write $\xi \sim \lambda$, where λ is the partition produced in this way. As well, we may write $\lambda \sim \xi$ if it is clear which is the tuple, say.

If ξ is an r-tuple, we denote the tuple resulting by permuting its entries by $\sigma \in S_r$, $\sigma\xi$.

Tuples are added component-wise. (Note. We are dropping commas from tuple and partition notation.)

Example 5.12. Given $\lambda = [33]$, $\mu = [111]$, we have $\lambda^+ = (33000)$ and $\mu^+ = (11100)$. If we form $\sigma\mu^+ = (01011)$, then we have $\lambda^+ + \mu^+ = (34011) \sim [4311]$.

Example 5.13. We illustrate with an example and then enunciate the general result.

$$\mathrm{m}_{211}\,\mathrm{m}_{22} = \mathrm{m}_{431} + \mathrm{m}_{4211} + \mathrm{m}_{332} + 2\,\mathrm{m}_{3221} + 3\,\mathrm{m}_{22211}$$

Using variables explicitly, with up to five variables we look at typical monomials in the product and by symmetry extend to the full monomial symmetric function. We have $\lambda^+ = (21100)$, $\mu^+ = (22000)$ corresponding to $x_1^2x_2x_3$ and $x_1^2x_2^2$ respectively. If we multiply these together we get $x_1^4x_2^3x_3$ giving the term m_{431}. Here are the remaining cases

21100	21100	21100	21100
+ 20020	+ 02200	+ 02020	+ 00022
$\sim$ 4211	$\sim$ 332	$\sim$ 3221	$\sim$ 22211
$x_1^2x_2x_3 \times x_1^2x_4^2$	$x_1^2x_2x_3 \times x_2^2x_3^2$	$x_1^2x_2x_3 \times x_2^2x_4^2$	$x_1^2x_2x_3 \times x_4^2x_5^2$
$x_1^4x_2x_3x_4^2$	$x_1^2x_2^3x_3^3$	$x_1^2x_2^3x_3x_4^2$	$x_1^2x_2x_3x_4^2x_5^2$
term in m_{4211}	term in m_{332}	term in m_{3221}	term in m_{22211}

Remark 5.14. Note that we are not deriving the coefficients, just which terms appear.

So to get the terms of the product of two monomial symmetric functions we have

Proposition 5.15. *For the product* $\mathrm{m}_\lambda \mathrm{m}_\mu = \sum_\nu c^\nu_{\lambda\mu}\, \mathrm{m}_\nu$*, the coefficient* $c^\nu_{\lambda\mu}$ *will be positive if there exists a permutation* σ *of* μ^+ *such that* $\lambda^+ + \sigma\mu^+ \sim \nu$*.*

Corollary 5.16. *In the product* $\mathrm{m}_\lambda \mathrm{m}_\mu$ *at least one term for every length* ℓ*,* $\max\{L(\lambda), L(\mu)\} \le \ell \le L(\lambda) + L(\mu)$*, occurs.*

In a sense, the corollary is clear first. Namely, by consecutively shifting non-zero values of μ^+ any length indicated in the corollary can be achieved. For example if $\lambda^+ = (543210000)$ and $\mu^+ = (432100000)$. First add $\lambda^+ + \mu^+$ for length 5. Then consecutively replace μ^+ by

$$(432001000),\ (430001200),\ (400001230),\ (000001234)$$

to get lengths 6 through 9.

In particular, if $L(\lambda) = L$, then in ${\mathrm{m}_\lambda}^2$ terms of lengths from L to $2L$ occur, in ${\mathrm{m}_\lambda}^3$, terms of lengths from L to $3L$ occur, etc.

Notation. Introduce the all 1's index, $\varepsilon = (1, 1, \dots, 1)$ of length N, the degree of the induced matrices.

So

$$M_\varepsilon = \bar{A}\ .$$

We see that

Lemma 5.17. *Given* λ*,* $L(\lambda) \le N$*, the row sums of* M_λ *are invariant under a shift of* λ *by* ε*:*

$$\mathfrak{s}(M_\lambda) = \mathfrak{s}(M_{\lambda+\varepsilon})$$

Proof. Recall, Theorem 2.17, for stochastic A, $s(A) = 1$, with $L = L(\lambda)$,

$$\mathfrak{s}(M_\lambda) = \frac{L!}{\prod\limits_{i\ge 1} \rho_i!} \binom{N}{L} = \frac{N!}{(N-L)! \prod\limits_{i\ge 1} \rho_i!}$$

Think of λ as λ^+ here extended to length N by zeros as necessary. Then $N - L$ is effectively ρ_0, the multiplicity of 0. The denominator takes the form $\prod\limits_{i\ge 0} \rho_i!$. Now we see that adding ε to λ^+ keeps length N and keeps the set of multiplicities the same, while shifting indices of the multiplicities by one $\rho_i \to \rho_{i+1}$. Thus, the denominator keeps the same value and hence $\mathfrak{s}(M.)$ stays the same as well. □

We have the following

Theorem 5.18. *In degree* N*,*

1. *For any* λ*,* $L(\lambda) \le N$*, we have (padding* λ *with zeros as necessary),*

$$\bar{A}\, M_\lambda = M_{\lambda+\varepsilon}$$

with $L(\lambda + \varepsilon) = N$*.*

2. *If* $L(\lambda) = N$*, then* $M_\lambda = \bar{A}\, M_{\lambda-\varepsilon}$*.*

3. (Factorization) *Let* $\lambda = \dot{\lambda} + r\varepsilon$*,* $r \geq 0$*, where* $\dot{\lambda}$ *is minimal. Then* $M_\lambda = M_{\dot{\lambda}}\,\bar{A}^r$*.*

4. $\bar{A}^r = M_{r\varepsilon} = M_{(r^N)}$*.*

5. If A *is ergodic,* $A^k \underset{k\to\infty}{\to} \Omega$*, then* $\lim\limits_{k\to\infty} M_{\lambda+k\varepsilon} = \mathfrak{s}(M_\lambda)\,\bar{\Omega}$*.*

Proof. For #1, as in the discussion about Proposition 5.15 and the Corollary, we form λ^+ and ε^+, both of length $L(\lambda)+N$, by padding with zeros. For terms in the product, a permutation of ε^+ leaving the first $L(\lambda)$ ones in place gives the $\lambda+\varepsilon$ term. If a zero is transposed with one of the leading 1's under a non-zero λ_i, then $\lambda^+ + \varepsilon^+$ will yield a partition ν with $L(\nu) > N$, so the corresponding M_ν vanishes. By Lemma 5.17 above, the row sums do not change, so there is no additional factor as a coefficient other than one. Here is an example with $L(\lambda) = 2$, $N = 4$:

$$\begin{array}{r} 320000 \\ +\ \underline{111100} \\ \sim 4311 \end{array} \qquad \begin{array}{r} 320000 \\ +\ \underline{110011} \\ \sim 4311 \end{array} \qquad \begin{array}{r} 320000 \\ +\ \underline{101101} \\ \sim 42111 \end{array}$$

That is, for $N = 4$, $\bar{A}\,M_{(32)} = M_{(4311)}$. Now #2 follows by multiplying on the right-hand side according to #1. And #3 follows by iterating #2. For example, with $N = 4$, $M_{(3322)} = M_{(11)}\,\bar{A}^2 = E_2\,\bar{A}^2$. And #4 follows from $\bar{A} = M_\varepsilon$ and iterating. For #5, start with $M_{\lambda+\varepsilon} = M_\lambda\,\bar{A}$ then iteratively multiply by $\bar{A}$ yielding in the limit $M_\lambda\bar{\Omega}$. Since the columns of $\bar{\Omega}$ are multiples of the all 1's (column) vector, $M_\lambda\bar{\Omega}$ yields $\mathfrak{s}(M_\lambda)$ acting on every column. $\square$

Now we can show that M_λ is primitive if A is ergodic.

If $L(\lambda) = N$, factor as follows

$$M_\lambda = M_{\lambda-\varepsilon}\,\bar{A}$$

so that $\hat{M}_\lambda$ is ergodic and hence any element of $\mathbb{G}$ that contains M_λ is primitive.

If $L(\lambda) < N$, form ${M_\lambda}^p$ so that $p\,L(\lambda) \geq N$. It will contain at least one term of length N, say M_ν, with monomials of greater length vanishing in the SFA algebra. We have with $L(\nu) = N$,

$${M_\lambda}^p \supset M_{\nu-\varepsilon}\,\bar{A}$$

and primitivity/ergodicity follows.

Example 5.19. Here's an example. Along with the matrix is indicated the first power for which all the entries become positive. Take $N = 3$ for

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix} \text{ with } A^5 > 0$$

(For brevity, we will not write out all of the example matrices)

$$E_2 = M_{(11)} \quad \text{has exponent 5}$$

$$H_4 = \sum_{|\lambda|=4} M_\lambda \quad \text{has exponent 2}$$

$$P_2 = \Gamma(A^2) = M_{(2)} \quad \text{has exponent 6}$$

$$M_{(211)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 1/2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1/2 & 3/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1/4 & 0 & 1/2 & 0 & 1/2 & 1/4 & 1/2 & 1 & 0 \\ 0 & 0 & 1/4 & 1/2 & 1 & 0 & 1/2 & 3/4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3/2 & 0 & 0 & 3/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1 & 0 & 0 & 1/2 & 1/2 & 1/2 \\ 1/8 & & 0 & 3/8 & 1/2 & 1/2 & 1/8 & 1/2 & 1/2 & 0 \\ 0 & 3/8 & 3/8 & 3/4 & 3/4 & 0 & 3/8 & 3/8 & 0 & 0 \end{pmatrix}$$

$$\{22\} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 3 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1/2 & 1 & 0 & 1/2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 1/2 & 3/2 & 0 & 1 \\ 1/4 & 1/2 & 0 & 1/4 & 2 & 0 & 0 & 2 & 0 & 1 \\ 0 & 1/2 & 1/4 & 3/4 & 1/4 & 1 & 1/4 & 1 & 2 & 0 \\ 0 & 0 & 1/4 & 3/4 & 3/2 & 1/4 & 1/2 & 3/4 & 1 & 1 \\ 0 & 3/4 & 0 & 3/2 & 0 & 3/2 & 3/4 & 0 & 3/2 & 0 \\ 0 & 0 & 3/4 & 1/2 & 2 & 0 & 1/2 & 5/4 & 1/2 & 1/2 \\ 1/8 & 3/8 & 0 & 3/8 & 1 & 1 & 3/8 & 3/2 & 5/4 & 0 \\ 0 & 3/8 & 3/8 & 3/4 & 3/4 & 0 & 3/8 & 9/8 & 3/2 & 3/4 \end{pmatrix}$$

these last both having exponents 3. Note that $\mathfrak{s}(M_{(211)}) = 3$ and $\mathfrak{s}(\{22\}) = 6$. We have

$$\Omega = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \end{pmatrix}$$

and the first row of $\bar{\Omega}$, the multinomial extension of the first row of Ω,

$$\left(\frac{1}{125}, \frac{6}{125}, \frac{6}{125}, \frac{12}{125}, \frac{24}{125}, \frac{12}{125}, \frac{8}{125}, \frac{24}{125}, \frac{24}{125}, \frac{8}{125} \right)$$

with $\bar{\Omega}$ the common limiting idempotent for all of the above basic elements once normalized.

5.2. Some limit theorems. Further ergodic theorems.

5.2.1. *Assume that the powers of A converge.* Let $A^k \to \Omega$ as $k \to \infty$. Then

Proposition 5.20. *We have*

1. $\lim\limits_{k\to\infty} P_k = \Gamma(\Omega)$.

2. $\lim_{k\to\infty} \overline{I+vA^k} = (1+v)^{\Gamma(\Omega)}$*. In particular,* $\lim_{k\to\infty} \overline{I+A^k} = 2^{\Gamma(\Omega)}$*.*

Proof. For #1, we have

$$\lim_{k\to\infty} P_k = \lim_{k\to\infty} \Gamma(A^k) \to \Gamma(\Omega)$$

as stated. For #2,

$$\lim_{k\to\infty} \overline{I+vA^k} = \overline{I+v\Omega} = (1+v)^{\Gamma(\Omega)}$$

by eq. (2.13). Setting $v = 1$ yields the final statement. □

5.2.2. *General A.* For a general stochastic matrix, we have the Cesàro limit, [3, Th. 2.1],

$$\lim_{u\to\infty} \frac{1}{u} \sum_{k=1}^{u} A^k$$

and the Abel limit

$$\lim_{t\uparrow 1} (1-t)(I-tA)^{-1}$$

Looking entry-by-entry it follows from Abelian and Tauberian theorems, see [18, p. 282], that if either limit exists, then both exist and are equal.

Remark 5.21. Note that the Abel limit is particularly suited for symbolic programs such as Mathematica, Maple or Sympy.

In either case, we know that the limit, call it Ω, is a stochastic idempotent commuting with A. So the rows of Ω are left eigenvectors of A with eigenvalue 1.
We have

Theorem 5.22. Some ergodic theorems
Let Ω be the Cesàro limit of the powers of A.
1. We have the Cesàro limit

$$\lim_{u\to\infty} \frac{1}{u} \sum_{k=1}^{u} P_k = \Gamma(\Omega)\ .$$

2. In degree N, we have the extended Abel limit

$$\lim_{t\uparrow 1} (1-t)^N \sum_{\ell\geq 0} \frac{(N)_\ell}{\ell!}\, t^\ell \hat{H}_\ell = \overline{\Omega}\ .$$

3. We have for monomials

$$\lim_{u\to\infty} \left(\sum_{0\leq L\leq N} \frac{1}{u^L} \sum_{\substack{\sum \rho_j = L \\ \lambda_1 \leq u}} M_\rho \right) = 2^{\Gamma(\Omega)}$$

4. For the Γ_ℓ's,

$$\lim_{u\to\infty} \sum_{0\leq \ell\leq N} \frac{1}{u^\ell}\, \Gamma_\ell\big(\sum_{i=1}^{u} A^i\big) = 2^{\Gamma(\Omega)}$$

Proof. As in Prop. 5.20, #1, we have

$$\lim_{u\to\infty} \frac{1}{u}\sum_{k=1}^{u} P_k = \lim_{u\to\infty} \frac{1}{u}\sum_{k=1}^{u} \Gamma(A^k) = \Gamma(\Omega)$$

by linearity of Γ. For #2, consider the Abel limit under the induced mapping:

$$\lim_{t\uparrow 1} \overline{(1-t)(I-tA)^{-1}} = \lim_{t\uparrow 1}(1-t)^N \sum_{\ell\ge 0} t^\ell H_\ell = \lim_{t\uparrow 1}(1-t)^N \sum_{\ell\ge 0} \frac{(N)_\ell}{\ell!}\, t^\ell \hat{H}_\ell$$

recalling the normalization of the H's, (2.18). While, directly,

$$\lim_{t\uparrow 1} \overline{(1-t)(I-tA)^{-1}} = \bar{\Omega}$$

as required. For #3, we have

$$\lim_{u\to\infty} \overline{I + \frac{1}{u}\sum_{k=1}^{u} A^k} = \overline{I+\Omega} = 2^{\Gamma(\Omega)} \tag{5.1}$$

as in the second part of Prop. 5.20, #2. Apply the expansion eq. (2.5) to the left-hand side:

$$\overline{I + \frac{1}{u}\sum_{k=1}^{u} A^k} = \sum_{\mathcal{D}(u,N)} \frac{1}{u^{\sum \rho_j}} M_\rho$$

and the result follows. For #4, expand the left-hand side of eq. (5.1), using the generic form $\overline{I+vX} = \sum_\ell v^\ell \Gamma_\ell(X)$, with $X = \dfrac{1}{u}\sum\limits_{k=1}^{u} A^k$. Use the scaling property of Γ_ℓ to pull out a factor of $u^{-\ell}$. □

5.2.3. *Stochastic pencils I.* Recall the fact that an irreducible stochastic matrix with positive trace is primitive, [15, §3.1, Cor. 1.1]. Then we have

Proposition 5.23. *Let A be stochastic, irreducible with powers of A having Cesàro limit Ω. Let $p, q > 0$, $p+q=1$. Then*

1.
$$\lim_{k\to\infty} (qI+pA)^k = \Omega\ .$$

2.
$$\lim_{k\to\infty} \left(\overline{qI+pA}\right)^k = \bar{\Omega}\ .$$

3.
$$\lim_{k\to\infty} \left(\sum_{0\le\ell\le N} \binom{N}{\ell} p^\ell q^{N-\ell}\, \hat{\Gamma}_\ell(A) \right)^k = \bar{\Omega}\ .$$

Proof. First observe that $B = qI+pA$ is stochastic with $B\Omega = \Omega B = \Omega$. Since B is primitive, by uniqueness, Ω is the limit idempotent for powers of B. #2 follows by the homomorphism property. For #3, factor out q, apply the expansion

$$\overline{I+(p/q)A} = \sum_\ell (p/q)^\ell \Gamma_\ell(A)$$

with the factor of q scaling as q^N. Recalling the normalization of $\Gamma_\ell(A)$, (2.18), the result follows from #2. □

Of interest is that the limit is independent of $q > 0$.

5.3. Induced boundary. In the case where A is not ergodic, e.g., if A is periodic, we still have the Cesàro/Abel limit(s). With, as usual, $\hat{E}_\ell$ denoting $\Gamma_\ell(A)$ normalized, consider the union over ℓ of the sets of limit points of $(\hat{E}_\ell)^k$ as the *induced boundary*. As ℓ varies, the Cesàro limits will vary accordingly as well as the limit points of the set of powers of $\hat{E}_\ell$.

We illustrate with an example to show how periodicity comes into play.

Notation. Denote the Cesàro/Abel limit of powers of a stochastic matrix X by $\mathrm{cs}(X)$.

Example 5.24. Take the circulant

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

with eigenvalues 3rd roots of unity. For $N = 3$, we have

$$\Gamma_3(A) = \bar{A} = \begin{pmatrix} 0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&1&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&1&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0 \end{pmatrix}$$

The Cesàro limit $\mathrm{cs}(A)$ consists of three rows all equal to $(1/3, 1/3, 1/3)$ and the induced matrix of the Cesàro limit consists of 10 rows all equal to

$$\begin{pmatrix} 1/27 & 1/9 & 1/9 & 1/9 & 2/9 & 1/9 & 1/27 & 1/9 & 1/9 & 1/27 \end{pmatrix}$$

The interesting case is the Cesàro limit of $\bar{A}$:

$$\mathrm{cs}(\bar{A}) = \begin{pmatrix} 1/3&0&0&0&0&0&1/3&0&0&1/3 \\ 0&1/3&0&0&0&1/3&0&1/3&0&0 \\ 0&0&1/3&1/3&0&0&0&0&1/3&0 \\ 0&0&1/3&1/3&0&0&0&0&1/3&0 \\ 0&0&0&0&1&0&0&0&0&0 \\ 0&1/3&0&0&0&1/3&0&1/3&0&0 \\ 1/3&0&0&0&0&0&1/3&0&0&1/3 \\ 0&1/3&0&0&0&1/3&0&1/3&0&0 \\ 0&0&1/3&1/3&0&0&0&0&1/3&0 \\ 1/3&0&0&0&0&0&1/3&0&0&1/3 \end{pmatrix}$$

showing how reducibility appears, for example, in the periodic case. Note this matrix has rank 4, indicating multiplicity 4 for eigenvalue 1 for $\bar{A}$.

Remark 5.25. If A is periodic with period δ, say, by [17, §1.4, Th. 1.7], the δ^{th} roots of unity will be eigenvalues. By Proposition 2.10, for $\ell = \delta$, these lead to eigenvectors with eigenvalue 1 for normalized E_δ. This provides multiplicity at least δ for eigenvalue 1 for $\hat{E}_\delta$. As seen in the example above, the multiplicity may exceed δ, corresponding to combinations of eigenvalues evaluating to 1.

Proposition 5.26. *Let* $\Omega = \text{cs}(A)$. *Then*

$$\text{cs}(\hat{E}_\ell)\,\bar{\Omega} = \bar{\Omega}\,\text{cs}(\hat{E}_\ell) = \bar{\Omega}\ .$$

Proof. Start with

$$(I + vA)\Omega = \Omega(I + vA) = (1 + v)\Omega$$

and map to induced matrices, which yields, taking coefficients of powers of v,

$$E_\ell\bar{\Omega} = \bar{\Omega}E_\ell = \binom{N}{\ell}\bar{\Omega}\ .$$

Normalize to $\hat{E}_\ell$. Taking powers and the Cesàro limit yields the result. □

So that $\text{cs}(\hat{E}_\ell)$ is a stochastic idempotent commuting with $\bar{\Omega}$.

6. Analytic approach to Gamma maps

In this section W will denote any $d \times d$ matrix. In the definition of induced mapping, note that if we set $f(x) = x^m = x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d}$, we see that the entries $\bar{W}_{mn}$ are the Taylor coefficients of the expansion of $f(Wx)$. For the Γ-maps we have

$$f((I + vW)x) = f(\ldots, x_i + v\sum_j W_{ij}x_j, \ldots)$$

We want the Taylor expansion in powers of v for general f, then specialize to $f(x) = x^m$. We have the expansion

$$((I + vW)x)^m = \sum_\ell v^\ell \sum_n (E_\ell)_{mn} x^n$$

In general, start with

$$\frac{\mathrm{d}}{\mathrm{d}v} f((I + vW)x) = \sum_i \frac{\partial f}{\partial x_i}(Wx)_i = (\sum_{i,j} W_{ij}x_j D_i)\, f \tag{6.1}$$

using D_i to denote the corresponding partial derivative. The second derivative is found similarly, now with derivatives $D_i f$ playing the role of f:

$$\frac{\mathrm{d}^2}{\mathrm{d}v^2} f((I + vW)x) = \sum_{i_1,j_1} W_{i_1j_1}x_{j_1} \sum_{i_2,j_2} W_{i_2j_2}x_{j_2}\, D_{i_1}D_{i_2}\, f$$

noting that $\frac{\mathrm{d}}{\mathrm{d}v}$ commutes with all x-variables and derivatives.

We treat the x-variables as a row vector and the D-variables as a column vector. So we may write $\sum_{i,j} W_{ij}x_j D_i$ in the form

$$xW^\top D$$

with $W^\top$ denoting the transpose of W. Now group indices to form tensor indices, indicating Kronecker products, denoted by Greek letters:

$$\alpha = (i_1, i_2, \dots)\,, \quad \beta = (j_1, j_2, \dots)$$

So, for the second-order term we have

$$\sum_{i_1,j_1} W_{i_1j_1} x_{j_1} \sum_{i_2,j_2} W_{i_2j_2} x_{j_2}\, D_{i_1} D_{i_2}\, f = (x\otimes x)_\beta\, (W^\top \otimes W^\top)_{\beta\alpha} (D\otimes D)_\alpha$$

which leads to the general form, for $f(x) = x^m$:

Proposition 6.1. *For the matrix elements of* Γ_ℓ*, we have*

$$\sum_n \Gamma_\ell(W)_{mn}\, x^n = \frac{1}{\ell!}\; x^{\otimes \ell}\, (W^\top)^{\otimes \ell}\, D^{\otimes \ell}\, x^m$$

superscripts indicating iterated tensor (Kronecker) products.

In particular, for $\Gamma(W)$ the first derivative gives the result:

Proposition 6.2. *For the* Γ*-map, we have*

$$\sum_{i,j} W_{ij} x_j D_i\, x^m = \sum_n \Gamma(W)_{mn}\, x^n$$

where on the left is a first-order operator, vector field. Hence we can exponentiate acting on a general f as follows. (Note that here x behaves as a column vector.)

Corollary 6.3. *Let* $u(t) = f(e^{tW} x)$*. Then we have*

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \sum_{i,j} W_{ij} x_j \frac{\partial u}{\partial x_i}$$

with $u(0) = f(x)$*.*

This corresponds to the exponential definition of Γ: $\overline{e^{tW}} = e^{t\Gamma(W)}$.

Overall this approach illustrates how the sequence $E_1, E_2, \dots$ is built.

7. Zero Row Sums

The importance of zero row sums appears when considering one-parameter semigroups of stochastic matrices. Let $Q = (q_{ij})$ be given such that $A(t) = e^{tQ}$ is stochastic for $t > 0$. We assume that $A(t)$ is irreducible for all $t > 0$, with Q irreducible in the sense that the off-diagonal elements have a strongly connected S.T.D. In particular, Q has no zero rows. Then considering

$$Q = \lim_{t\to 0} \frac{A(t) - I}{t}$$

shows that $Q \in$ ZRS and considering diagonal and off-diagonal elements separately that

(1) $q_{ii} < 0,\ \forall i, 1 \le i \le d$.
(2) $q_{ij} \ge 0,\ \forall i \ne j, 1 \le i, j \le d$.

They are part of a special class of matrices called ML matrices in [17, §2.3]. Let us say

$Q \in \mathrm{ZRS}'$, if Q is an irreducible zero row sum matrix generating a stochastic semigroup.

By [17, §2.3, Th. 2.7], $Q \in \mathrm{ZRS}'$, in particular, irreducible, if and only if e^{tQ} has positive entries for all $t > 0$. The invariant distribution of $A(1)$ is the same for all $A(t)$, $t > 0$, namely a left null vector of Q. We will look at pencils below that imply uniqueness up to scaling, i.e., uniqueness of an invariant distribution for $A(t)$.

Proposition 7.1. *If* $Q \in \mathrm{ZRS}'$*, then* $\Gamma(Q) \in \mathrm{ZRS}'$.

Proof. The exponential definition of $\Gamma(Q)$, $\overline{e^{tQ}} = e^{t\Gamma(Q)}$, shows that $e^{t\Gamma(Q)}$ is a stochastic semigroup of positive matrices. As noted above, it follows that $\Gamma(Q)$ is irreducible as well. □

Remark 7.2. Note that generally for any $Z \in \mathrm{ZRS}$, recalling eq. (2.18), all of the basis elements, (2.3), of $\mathbb{G} \subset \mathrm{SFA}(^{-}, Z)$ are in ZRS.

7.1. Stochastic pencils II.

7.1.1. *Pencils of the form* $I + vQ$. Here we consider $Q \in \mathrm{ZRS}'$. Then $I + vQ$ is stochastic with a nonzero trace for all sufficiently small $v > 0$. As noted previously, nonzero trace implies primitivity. Let Ω be the limiting stochastic idempotent of powers of $I + v_0Q$, for some $v_0 > 0$. We have

$$\Omega(I + v_0Q) = \Omega = (I + v_0Q)\Omega \tag{7.1}$$

i.e.,

$$\Omega Q = Q\Omega = 0 \tag{7.2}$$

so that rows of Ω are equal to a positive null eigenvector of Q, unique up to scale and the columns multiples of the all-ones vector. Equations (7.1) and (7.2) show that

$$\Omega(I + vQ) = \Omega = (I + vQ)\Omega \tag{7.3}$$

for all v, especially for all sufficiently small v so that $I+vQ$ is stochastic. Applying the induced matrix map to eq. (7.3) yields

$$E_\ell \bar{\Omega} = 0 = \bar{\Omega} E_\ell$$

for $1 \le \ell \le N$. In particular,

$$\Gamma(Q)\bar{\Omega} = 0 = \bar{\Omega}\Gamma(Q)$$

so that $\bar{\Omega}$ is the invariant idempotent for the stochastic semigroup generated by $\Gamma(Q)$.

We have the limit

$$\lim_{k\to\infty} (I + vQ)^k = \Omega \tag{7.4}$$

is independent of $0 < v < v'$ for some $v' > 0$. Ω commutes with the stochastic semigroup e^{tQ} satisfying $\Omega e^{tQ} = \Omega$. For the induced matrices in degree N, for sufficiently small $v > 0$, eq. (7.4) yields

$$\Big(\sum_{0\le\ell\le N} v^\ell\, \Gamma_\ell(Q)\Big)^k \underset{k\to\infty}{\longrightarrow} \bar{\Omega}\ .$$

with $\Gamma_\ell(Q) \in \mathrm{ZRS}$, for $\ell > 0$.

7.1.2. *Pencils of the form $\mathcal{J} + vZ$.* Here $Z \in$ ZRS, not necessarily generating a stochastic semigroup. With J the $d \times d$ all-ones matrix, we have the stochastic idempotent

$$\mathcal{J} = \frac{1}{d}\, J$$

For all sufficiently small $|v|$, $P = \mathcal{J} + vZ$ is primitive stochastic. Let us calculate the powers P^k and the limit as $k \to \infty$. First, note the properties

$$\mathcal{J}^2 = \mathcal{J} \qquad \text{and} \qquad Z\mathcal{J} = 0$$

so we have

$$\begin{aligned}
P^2 &= (\mathcal{J} + vZ)(\mathcal{J} + vZ) = \mathcal{J} + v\mathcal{J}Z + v^2 Z^2 \\
P^3 &= (\mathcal{J} + vZ)(\mathcal{J} + v\mathcal{J}Z + v^2Z^2) = \mathcal{J} + v\mathcal{J}Z + v^2\mathcal{J}Z^2 + v^3 Z^3 \\
&\vdots \\
P^k &= \mathcal{J} \sum_{0 \le \ell < k} v^\ell Z^\ell + v^k Z^k \\
&\vdots
\end{aligned}$$

Now assume v small with $v^k Z^k \to 0$ as $k \to \infty$ so that the geometric series $\sum\limits_{k \ge 0} v^k Z^k$ converges accordingly. Then we have

$$\lim_{k\to\infty} (\mathcal{J} + vZ)^k = \mathcal{J}\,(I - vZ)^{-1}$$

Recall the diagonal matrix $\mathcal{B}$ of multinomial coefficients, and the corresponding all-ones matrix $\mathfrak{J}$, whence

$$\bar{\mathcal{J}} = \frac{1}{d^N}\, \mathfrak{J}\mathcal{B}$$

So

$$\lim_{k\to\infty} \overline{(\mathcal{J} + vZ)}^{\,k} = \frac{1}{d^N}\, \mathfrak{J}\mathcal{B}\, \overline{(I - vZ)^{-1}} = \frac{1}{d^N}\, \mathfrak{J}\mathcal{B} \sum_{\ell \ge 0} v^\ell H_\ell(Z)$$

with $H_\ell(Z)$ the homogeneous functions in the SFA algebra generated by Z. Observe that $d^{-N}\mathfrak{J}\mathcal{B}X$, say, averages the columns of X according to the uniform multinomial distribution with probabilities along each row of $d^{-N}\mathfrak{J}B$. We can indicate this averaging by

$$\lim_{k\to\infty} \overline{(\mathcal{J} + vZ)}^{\,k} = \sum_{\ell \ge 0} v^\ell \langle H_\ell(Z) \rangle_{\text{uniform multinomial}}$$

with each averaged matrix having identical rows. Note that the limit

$$\Omega = \mathcal{J}\,(I - vZ)^{-1}$$

is a stochastic idempotent as well as $\bar{\Omega}$.

Now we proceed to some further remarks and prospects.

8. Complements and Prospects

8.1. Cauchy identity for S-functions. In this section we will work with a general SFA(ϕ, X) algebra.

There are several variations, [12, 4.2', 4.3'], on expanding the product

$$\prod_{i,j}(1+x_iy_j)$$

among them the expansion in terms of S-functions [12, 4.3'],

$$\prod_{i,j}(1+x_iy_j)=\sum_\lambda \mathrm{s}_\lambda(x)\,\mathrm{s}_{\lambda'}(y) \tag{8.1}$$

where λ' indicates the conjugate partition to λ. We want to find identities for symmetric functions not explicitly involving variables, then interpret elementary symmetric functions in the y-variables as arbitrary coefficients $\{c_k\}$. We have the parallel definitions

$$E(v)=\prod_j(1+vx_j)=1+\sum_{\ell\ge 1}v^\ell \mathrm{e}_l(x)$$

and

$$\mathcal{E}(v)=\phi(I+vX)=\sum_{\ell\ge 0}v^\ell E_\ell$$

for the SFA algebra. First

$$\prod_{i,j}(1+x_iy_j)=\prod_j E(y_j)=\sum_\lambda \mathrm{e}_\lambda(y)\,\mathrm{m}_\lambda(x)$$

which translates to

$$\prod_j \mathcal{E}(y_j)=\prod_j \phi(I+y_jX)=\phi(I+\sum_{k\ge 1}\mathrm{e}_k(y)X^k)$$

yielding the identity

$$\phi(I+\sum_{k\ge 1}c_kX^k)=\sum_\lambda c_\lambda M_\lambda$$

cf. eq. (2.5), considering the partition λ whether given by parts or multiplicities.

Now

Definition 8.1. Given a partition λ and a sequence of variables $x=\{x_i\}$, define the Jacobi-Trudi determinant

$$\mathrm{JT}(\lambda,x)=\det(x_{\lambda_i-i+j})$$

where $x_0=1$ and $x_k=0$ for $k<0$.

Thus we may write eqs. (2.1) in the form

$$\{\lambda\}=\mathrm{JT}(\lambda,\mathrm{h})\quad\text{and}\quad\{\lambda'\}=\mathrm{JT}(\lambda,\mathrm{e})\ .$$

Recall the Kostka matrix, [12, Ch. 6], the coefficients in the expansion $\mathrm{s}_\lambda=\sum\limits_\mu K_{\lambda\mu}\mathrm{m}_\mu$.

Proposition 8.2. *We have*

$$\phi(I + \sum_{k\geq 1} c_k X^k) = \sum_\lambda \{\lambda\}\, \mathrm{JT}(\lambda, c)$$

and

$$\phi(I + \sum_{k\geq 1} c_k X^k) = \sum_\lambda \{\lambda\} \sum_\mu c_\mu (K^{-1})_{\mu\lambda}\ .$$

Proof. For the first equation, we have

$$\prod_{i,j}(1 + x_i y_j) = \sum_\lambda \mathrm{s}_\lambda(x)\, s_{\lambda'}(y) = \sum_\lambda \mathrm{s}_\lambda(x)\, \mathrm{JT}(\lambda, \mathrm{e}(y))$$

which translates to the SFA as required.

For the second equation, proceed as follows, changing notation accordingly,

$$\begin{aligned}\phi(I + \sum_{k\geq 1} c_k X^k) &= \sum_\mu c_\mu M_\mu \\ &= \sum_\mu c_\mu \sum_\lambda (K^{-1})_{\mu\lambda}\{\lambda\} \\ &= \sum_\lambda \{\lambda\} \sum_\mu c_\mu (K^{-1})_{\mu\lambda}\end{aligned}$$

as stated. □

This verifies $\mathrm{JT}(\lambda, c) = \sum_\mu c_\mu (K^{-1})_{\mu\lambda}$, cf. [12, Ch. 6, Table 1] (transition matrix expressing S-functions in terms of homogeneous functions).

Remark 8.3. Note that $(K^{-1})_{\mu\lambda} \neq 0$ implies $\mu \trianglerighteq \lambda$, i.e., μ is greater than or equal to λ in dominance order.

8.2. Prospects.

8.2.1. *Periodic case.* As remarked in §5.3, for periodic A, it is not clear what the behavior of powers of the elementary functions will be. And of course, a description of the behavior of powers of elements in all of the basic families is of interest.

8.2.2. *Non-homogeneous products.* It would be interesting to discover limiting behavior of products E_λ, H_λ, etc., as $|\lambda|$ and/or $L(\lambda)$ goes to infinity. What is wanted is a description of a sequence of partitions λ so that the corresponding products E_λ, say, converge. Behavior of products of elements from $\mathbb{G}$ could be explored as well.

8.2.3. *Non-commutative case for Lie algebras.* We know that the Γ-map preserves Lie products, but it is not clear how the various SFA algebras generated by elements of a given Lie algebra are related.

8.2.4. *Questions for various* SFA(ϕ, X) *systems.* One could study further properties for systems with general ϕ or assuming some special properties. As well, one could prescribe conditions on X, as we did in this work, considering stochastic matrices for the case of ϕ being the induced matrix map.

Study of SFA's for certain specifications of ϕ's is definitely of interest.

Any information on S-functions in any of the situations mentioned would be most welcome.

9. Appendix I

Here we look in detail when $X^2 = I$. First we have

Proposition 9.1. *For $X^2 = I$, we have the power sums*

$$P_{\text{odd}} = \Gamma(X) \quad \text{and} \quad P_{\text{even}} = N\,\bar{I}$$

for even indices, recalling that $\Gamma(I) = N\bar{I}$.

(Note: subsequently we drop the explicit bar on the identity.)

Definition 9.2. For a partition ρ, denote the number of odd parts by $\mathcal{O}(\rho)$, and denote the number of even parts by $\varepsilon(\rho)$.

We have the power sum expansions

$$E_\ell = \sum_{\rho\vdash\ell}(-1)^{\ell-L(\rho)}\,\frac{N^{\varepsilon(\rho)}\,\Gamma(X)^{\mathcal{O}(\rho)}}{z_\rho}$$

$$H_\ell = \sum_{\rho\vdash\ell}\frac{N^{\varepsilon(\rho)}\,\Gamma(X)^{\mathcal{O}(\rho)}}{z_\rho}$$

where $z_\rho = 1^{\rho_1}2^{\rho_2}\cdots\rho_1!\rho_2!\cdots$ as usual, with similar expansions holding for all S-functions $\{\lambda\}$.

9.1. Structure constants for the elementary functions. Since the minimal polynomial is of degree 2, we look for structure constants for multiplication of the E's,

$$E_iE_j = \sum_\ell c_{ij}^\ell E_\ell$$

Form the product

$$\begin{aligned}\overline{(I+vX)(I+wX)} &= \overline{I+(v+w)X+vwX^2}\\ &= \overline{(1+vw)I+(v+w)X)}\\ &= \sum_\ell (1+vw)^{N-\ell}(v+w)^\ell\,E_\ell\end{aligned}$$

Using Greek letters to indicate summation, the coefficient of E_ℓ expands to

$$\binom{N-\ell}{\alpha}v^\alpha w^\alpha\binom{\ell}{\beta}v^{\ell-\beta}w^\beta$$

which gives the equations

$$i = \alpha + \ell - \beta \quad \text{and} \quad j = \alpha + \beta$$

yielding

$$\alpha = \frac{i+j-\ell}{2} \quad \text{and} \quad \beta = \frac{\ell - i + j}{2}$$

Proposition 9.3. *For multiplication of the E's we have the relations*

$$E_i E_j = \sum_\ell \binom{N-\ell}{\frac{i+j-\ell}{2}} \binom{\ell}{\frac{\ell-i+j}{2}} E_\ell$$

where the sum is over ℓ such that $\ell \equiv i + j \pmod 2$.

9.2. Identification of E and H families and their relationship. Introduce the idempotents

$$\Omega_+ = (I + X)/2 \quad \text{and} \quad \Omega_- = (I - X)/2$$

They satisfy

$$\begin{aligned} \Omega_+ + \Omega_- &= I \\ \Omega_+ \Omega_- &= 0 \\ \Omega_+ - \Omega_- &= X \end{aligned}$$
$$\Gamma(\Omega_+) = (NI + \Gamma(X))/2 \quad \text{and} \quad \Gamma(\Omega_-) = (NI - \Gamma(X))/2$$

We have

$$\begin{aligned} I + vX &= (I + v\,\Omega_+)(I - v\,\Omega_-) \\ &= (I+v)^{\Omega_+} (I-v)^{\Omega_-} \end{aligned}$$

recalling eq. (2.13) for idempotents. Hence

$$\begin{aligned} \overline{I + vX} &= (I+v)^{\Gamma(\Omega_+)} (I-v)^{\Gamma(\Omega_-)} \\ &= (I+v)^{(NI+\Gamma(X))/2} (I-v)^{(NI-\Gamma(X))/2} \qquad (9.1) \\ &= (1-v^2)^{N/2} \left(\frac{1+v}{1-v}\right)^{\Gamma(X)/2} \qquad (9.2) \end{aligned}$$

From eq. (9.1) we have by binomial expansion:

Proposition 9.4. *The E's are given by the polynomial expressions*

$$E_\ell = \sum_j \binom{(NI + \Gamma(X))/2}{\ell - j} \binom{(NI - \Gamma(X))/2}{j} (-1)^j$$

in terms of the power sums NI and $\Gamma(X)$. It follows that all elements of the SFA *algebra are polynomials in $\Gamma(X)$.*

And from eq. (9.2), we have

$$\mathcal{H}(v) = \mathcal{E}(-v)^{-1} = (1-v^2)^{-N/2} \left(\frac{1+v}{1-v}\right)^{\Gamma(X)/2} = (1-v^2)^{-N}\, \mathcal{E}(v)$$

Expanding the initial factor

$$(1-v^2)^{-N} = \sum_k \frac{(N)_k}{k!}\, v^{2k}$$

we have

Proposition 9.5. *The H's are given in terms of the elementary functions as*

$$H_\ell = \sum_{0\le 2k\le l} \frac{(N)_k}{k!}\, E_{\ell-2k}$$

9.3. Identification of the monomial functions. Start with eq. (2.5)

$$\overline{I + \sum_{1\le k\le u} c_k X^k} = \sum_{\mathcal{D}(u,N)} c^\rho M_\rho$$

and write

$$I + \sum_{k\ge 1} c_k X^k = I + c_{\mathcal{O}} X + c_{\mathcal{E}} I$$

where $c_{\mathcal{O}} = c_1 + c_3 + \cdots$, the sum of the odd-indexed c's and $c_{\mathcal{E}}$ the analogous sum of even-indexed c's starting with c_2. We have

$$\overline{I + \sum_{1\le k\le u} c_k X^k} = \overline{(1+c_{\mathcal{E}})I + c_{\mathcal{O}} X} = \sum_\ell (1+c_{\mathcal{E}})^{N-\ell} c_{\mathcal{O}}^\ell\, E_\ell = \sum_{\mathcal{D}(u,N)} c^\rho M_\rho$$

Expanding:

$$(1+c_{\mathcal{E}})^{N-\ell} = (1+c_2+c_4+\cdots)^{N-\ell} = \sum_\rho \frac{(N-\ell)!}{\rho_0!\rho_2!\cdots}\, c_2^{\rho_2} c_4^{\rho_4}\cdots$$

and

$$(c_{\mathcal{O}})^\ell = (c_1+c_3+\cdots)^\ell = \sum_\rho \frac{\ell!}{\rho_1!\rho_3!\cdots}\, c_1^{\rho_1} c_3^{\rho_3}\cdots$$

where ρ_0 corresponds to choosing the "1" in expanding the power of the sum of even coefficients, i.e., $\rho_0 = N-\ell-\rho_2-\cdots = N-L(\rho)$, noting that ℓ is the number of odd parts.

Matching coefficients of c^ρ, we have

Proposition 9.6. *The monomial functions are given by*

$$M_\rho = \frac{(N-o(\rho))!}{\rho_0!\rho_2!\cdots}\,\frac{o(\rho)!}{\rho_1!\rho_3!\cdots}\, E_{o(\rho)}$$

where $\rho_0 = N - L(\rho)$ *and* $o(\rho)$ *counts the number of odd parts of* ρ.

Example 9.7. We see that any M_ρ where ρ has only even parts is a multiple of E_0, the identity. E.g., with $N=4$,

$$M_{222} = 4\,E_0 = 4\,I\ ,\quad M_{2222} = E_0 = I$$
$$M_{3211} = 3\,E_3\ ,\quad M_{3221} = 2\,E_2$$

Note that if X is stochastic, we can check that the coefficients agree with the row sums of the corresponding elements.

9.4. Probabilistic interpretation. The expression, cf. (9.1),

$$(1+v)^{(N+x)/2}(1-v)^{(N-x)/2} \tag{9.3}$$

is indeed the generating function for elementary symmetric functions in the steps $X_i = \pm 1$ for a random walk on the line:

$$\prod_i (1 + v\, X_i)$$

where N is the number of steps and x is the position relative to the starting point. If we denote by p the number of positive steps out of N and n the number of negative steps, we have

$$N = \mathrm{p} + \mathrm{n}$$
$$x = \mathrm{p} - \mathrm{n}$$

which converts to eq. (9.3). Modulo a change of variables, these are *Krawtchouk polynomials* in the variables (x, N). Note that to recover our case from [4, 18.23.3], set there $p = 1/2$ and x there is our n, the number of negative jumps.

We see that in the SFA context, $\Gamma(X)$ plays the role of a quantum random walk.

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