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DYNAMICS OF A STOCHASTIC PREDATOR - PREY MODEL WITH THE BEDDINGTON - DEANGELIS FUNCTIONAL RESPONSE

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Abstract. We consider a stochastic predator - prey model with the Bedington - DeAngelis functional response. Firstly, we prove the existence, uniqueness and positivity of solutions. Then, the boundedness of moments of population are studied. Finally, we show the weak convergence of densities of prey to a singular measure in a special case, and give some upper growth and exponential death rates of population in some cases.

1. Introduction

In [24], Volterra considered the following model

$$\begin{cases} \dot{x}_t = (a_1 - c_1 y_t - b_1 x_t) x_t, \\ \dot{y}_t = (-a_2 + c_2 x_t - b_2 y_t) y_t, \end{cases} \tag{1.1}$$

where a_1, a_2, b_1, b_2, c_1 and c_2 are positive constants. In this model, x_t and y_t represent the population density of prey species and predator species at time t, respectively; a_1 is the intrinsic growth rate of prey in the absence of predator, a_2 is the death rate of predator in the absence of prey; b_1, b_2 measure the inhibiting effect of environment on two species; c_1, c_2 are coefficients of the effect of a species on the other. In this model, the effect of two species on the growth rates is linear, i.e., the predator consumes the prey with functional response of type $c_1 x_t y_t$ and contributes to its growth with rate $c_2 x_t y_t$. It is well - known that solutions of (1.1) are asymptotically stable.

After the appearance of this model, the dynamical relationship between predators and prey has been studied a lot by several authors. In those researches, they defined the average number of prey killed per individual predator per unit of time. This is called a functional response. One functional response can depend on only prey's density or both prey's and predator's densities; for example, a linear functional response in above type, a Holling's type II functional response $\frac{x_t y_t}{\alpha + \beta x_t}$ [22], a ratio - dependent one $\frac{x_t y_t}{\beta x_t + \gamma y_t}$ [2]. However, some biologists have argued that in many situations, especially when predators have to search for food, the functional response should depend on both prey's and predator's densities (see [1], [3],

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[10], [15], [16]). One of the most popular functional responses is the Beddington-DeAngelis functional response in type

$$\frac{x_t y_t}{\alpha + \beta x_t + \gamma y_t}.$$

This functional response was originally given by Beddington [5] and DeAngelis et al. [9], independently. After the appearance of these two investigations, there are many other ones for analogous systems with diffusion in a deterministic environment (see [6], [7], [8], [11], [13]). However, the deterministic environment is rarely the case in real life. Most natural environments are random environments. For the stochastic predator-prey models with the functional response of type $c_1 x_t y_t$, the following model has been studied by some authors

$$\begin{cases} dx_t = (a_1 - c_1 y_t - b_1 x_t) x_t dt + (\sigma_1 + \rho_1 x_t) x_t dw_t, \\ dy_t = (-a_2 + c_2 x_t - b_2 y_t) y_t dt + (\sigma_2 + \rho_2 y_t) y_t dw_t. \end{cases} \tag{1.2}$$

Here $\{w_t, t \geq 0\}$ is a Brownian motion defined on a complete probability space with filtration $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions (see [17], [18]). In [21], Rudnickii considers the model (1.2) with $\rho_1 = \rho_2 = 0$, i.e., the parameter a_i of (1.1) is stochastically perturbed: $a_i \rightsquigarrow a_i + \sigma_i \dot{w}_t$. The author shows that either the predator dies out and the prey grows stably or both prey and predator grow stably. In [19] and [20], Mao et al. consider (1.2) with $\sigma_1 = \sigma_2 = 0$, i.e., the parameter b_i of (1.1) is stochastically perturbed: $b_i \rightsquigarrow b_i + \rho_i \dot{w}_t$, and obtain some asymptotic moments and pathwise estimates.

In this paper, we consider a stochastic predator-prey model with the Beddington-DeAngelis functional response. The system is as follows

$$\begin{cases} dx_t = \left[x_t(a_1 - b_1 x_t) - \frac{c_1 x_t y_t}{\alpha + \beta x_t + \gamma y_t}\right] dt + \sigma_1 x_t dw_t, \\ dy_t = \left(-a_2 y_t + \frac{c_2 x_t y_t}{\alpha + \beta x_t + \gamma y_t} - b_2 y_t^2\right) dt + \sigma_2 y_t dw_t. \end{cases} \tag{1.3}$$

Throughout this paper, we suppose that all coefficients of this model are positive. The first aim of this paper is to study the existence, uniqueness and positivity of solutions to system (1.3) by using some Lyapunov functions and the technique of localization. The second problem that we shall investigate is the boundedness of moments and some upper-growth rates of positive solutions which play an important role in the population theory as well as in practice. In many cases, we need to know the extinction rate of each species in order to have a suitable policy in investment and to have timely measures to protect them from the extinct disaster. Therefore, we are also concerned with the asymptotic behavior of solutions to 0. We show that in some cases either both prey and predator die out or the predator dies out and the prey grows stably.

The paper is organized as follows. In section 2, we prove the existence, uniqueness and positivity of solution to (1.3). Section 3 produces asymptotic estimations of moments. In section 4, we get an outline of the singular case where either prey or predator is absent. We then show the weak convergence of the process $\ln x_t$ to a singular measure, and give the upper growth rate of population.

2. Existence and Uniqueness of Solution

In this section, we show that every solution of system (1.3) with positive initial value is positive and global. Denote $\mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i > 0, i = 1, \dots, n\}$ $(n = 1, 2)$. We have the following theorem.

Theorem 2.1. *For any given initial value $(x_0, y_0) \in \mathbb{R}^2_+$, there exists a unique solution (x_t, y_t) to (1.3) for $t \geq 0$. Further, with probability 1, $\mathbb{R}^2_+$ are positively invariant for (1.3), i.e., with any initial condition $(x_0, y_0) \in \mathbb{R}^2_+$, $(x_t, y_t) \in \mathbb{R}^2_+$ a.s. for all $t \geq 0$.*

Proof. Consider the following system

$$\begin{cases} d\xi_t = \left[(a_1 - \frac{\sigma_1^2}{2}) - b_1 \exp\{\xi_t\} - \frac{c_1 \exp\{\eta_t\}}{\alpha + \beta \exp\{\xi_t\} + \gamma \exp\{\eta_t\}}\right] dt + \sigma_1 dw_t, \\ d\eta_t = \left[-(a_2 + \frac{\sigma_2^2}{2}) + \frac{c_2 \exp\{\xi_t\}}{\alpha + \beta \exp\{\xi_t\} + \gamma \exp\{\eta_t\}} - b_2 \exp\{\eta_t\}\right] dt + \sigma_2 dw_t, \end{cases} \tag{2.1}$$

with an initial value $(\xi_0, \eta_0) = (\ln x_0, \ln y_0)$. Since the coefficients of (2.1) are locally Lipschitz continuous, there exists a unique local solution (ξ_t, η_t) to (2.1) for $t \in [0, \tau)$, where τ is the explosion time (see [4], [12]). Therefore, by Itô's formula, $(x_t, y_t) = (\exp\{\xi_t\}, \exp\{\eta_t\})$ is a unique positive local solution to (1.3) for $t \in [0, \tau)$ with the initial value (x_0, y_0). To show that this solution is global, we need to show that $\tau = \infty$ a.s. We use the technique of localization to do that (see [17], [18]). Let $k_0 > 0$ be sufficiently large for x_0 and y_0 lying within the interval $[\frac{1}{k_0}, k_0]$. Let us define a sequence of stopping times [17, Problem 2.7, p.7] for each integer $k \geq k_0$ by

$$\tau_k = \inf\left\{t \geq 0 : x_t \notin (\frac{1}{k}, k) \text{ or } y_t \notin (\frac{1}{k}, k)\right\}.$$

The convention here is that the infimum of the empty set is ∞. Since τ_k is nondecreasing as $k \to \infty$, there exists the limit $\tau_\infty = \lim_{k\to\infty} \tau_k$. It is clear $\tau_\infty \leq \tau$ a.s. Now, we will show that $\tau_\infty = \infty$ a.s. If this statement is false, then there exist $T > 0$ and $\varepsilon \in (0, 1)$ such that $\mathbb{P}\{\tau_\infty \leq T\} > \varepsilon$. Thus, by denoting $\Omega_k = \{\tau_k \leq T\}$, there exists $k_1 \geq k_0$ such that

$$\mathbb{P}(\Omega_k) \geq \varepsilon, \quad \text{for all } k \geq k_1. \tag{2.2}$$

Consider the following function $V(x, y) = x - \ln x + y - \ln y$. It is easy to see that $V \in C^2(\mathbb{R}^2_+, \mathbb{R}_+)$. If $(x_t, y_t) \in \mathbb{R}^2_+$, by using Itô's formula, we get

$$dV(x_t, y_t) = f(x_t, y_t)dt + g(x_t, y_t)dw_t, \tag{2.3}$$

where

$$\begin{aligned} g(x,y) =& \sigma_1(x-1) + \sigma_2(y-1), \\ f(x,y) =& \frac{\sigma_1^2 + \sigma_2^2}{2} - (x-1)(b_1 x - a_1) - b_2 y(y-1) - a_2(y-1) \\ &+ \frac{c_2 x(y-1) - c_1(x-1)y}{\alpha + \beta x + \gamma y}. \end{aligned} \tag{2.4}$$

We see easily from (2.4) that the function $f(x,y)$ is bounded above, say by M, in $\mathbb{R}^2_+$. Thus, since $(x_{t\wedge\tau_k}, y_{t\wedge\tau_k}) \in \mathbb{R}^2_+$ and (2.3), we have

$$\int_0^{T\wedge\tau_k} dV(x_t,y_t) \le \int_0^{T\wedge\tau_k} M dt + \int_0^{T\wedge\tau_k} g(x_t,y_t)dw_t.$$

Taking expectations yields

$$\mathbb{E}V(x_{T\wedge\tau_k}, y_{T\wedge\tau_k}) \le V(x_0,y_0) + M\mathbb{E}(T\wedge\tau_k) \le V(x_0,y_0) + MT. \tag{2.5}$$

On the other hand, for every $\omega \in \Omega_k$, either $x_{\tau_k}(\omega)$ or $y_{\tau_k}(\omega)$ belongs to the set $\{k, \frac{1}{k}\}$. Hence

$$V(x_{T\wedge\tau_k}(\omega), y_{T\wedge\tau_k}(\omega)) \ge \min\{k - \ln k, \frac{1}{k} - \ln\frac{1}{k}\} = \min\{k-\ln k, \frac{1}{k} + \ln k\}.$$

We therefore have from (2.2) that

$$\mathbb{E}V(x_{T\wedge\tau_k}, y_{T\wedge\tau_k}) \ge \mathbb{E}[1_{\Omega_k} V(x_{T\wedge\tau_k}, y_{T\wedge\tau_k})] \ge \varepsilon \min\{k-\ln k, \frac{1}{k} + \ln k\}.$$

It then follows from (2.5) that $V(x_0,y_0) + MT \ge \varepsilon\min\{k-\ln k, \frac{1}{k}+\ln k\}$. Letting $k\to\infty$ leads to a contradiction: $\infty > V(x_0,y_0)+MT = \infty$. Therefore, $\tau_\infty = \infty$ a.s. Thus $\tau = \infty$ and $(x_t,y_t) \in \mathbb{R}^2_+$ a.s. The proof is complete. □

3. Boundedness of Moments

Since system (1.3) does not have an explicit solution, the study of asymptotic moment behaviour is essential if we are to gain a deeper understanding of the underlying process. To do that we consider

$$\begin{aligned} LV(x,y) =& \frac{1}{2}\sigma_1^2 x^2 \frac{\partial^2 V}{\partial x^2} + \frac{1}{2}\sigma_2^2 y^2 \frac{\partial^2 V}{\partial y^2} + \sigma_1\sigma_2 xy \frac{\partial^2 V}{\partial x \partial y} \\ &+ f_1(x,y)\frac{\partial V}{\partial x} + f_2(x,y)\frac{\partial V}{\partial y}, \end{aligned}$$

the infinitesimal operator of (1.3), defined on the space $C^2(\mathbb{R}^2_+, \mathbb{R})$, where

$$\begin{aligned} f_1(x,y) &= x(a_1 - b_1 x) - \frac{c_1 xy}{\alpha+\beta x+\gamma y}, \\ f_2(x,y) &= -a_2 y + \frac{c_2 xy}{\alpha+\beta x+\gamma y} - b_2 y^2. \end{aligned}$$

Let θ_1, θ_2 be positive numbers, we put

$$\begin{aligned} d_2 &= \min\{\theta_1 b_1, \theta_2 b_2\}, \quad \theta = \frac{1}{\theta_1+\theta_2}, \\ d_1 &= \frac{1}{2}\sigma_1^2\theta_1(\theta_1-1) + \frac{1}{2}\sigma_2^2\theta_2(\theta_2-1) + \sigma_1\sigma_2\theta_1\theta_2 + \theta_1 a_1 + (\frac{c_2}{\beta} - a_2)\theta_2, \qquad (3.1) \\ \lambda_1 &= \left[\frac{d_1}{d_2} - (\theta_1+\theta_2)\{1-\ln(\theta_1+\theta_2)\}\right], \\ \lambda_2(x_0,y_0) &= \left[(\theta_1\ln x_0 + \theta_2 \ln y_0) + (\theta_1+\theta_2)\{1-\ln(\theta_1+\theta_2)\} - \frac{d_1}{d_2}\right]. \end{aligned}$$

Lemma 3.1. *For any positive numbers θ_1, θ_2, the solution (x_t, y_t) of (1.3) with initial value $(x_0,y_0) \in \mathbb{R}^2_+$ satisfies $\mathbb{E}(x_t^{\theta_1} y_t^{\theta_2}) < \infty$ for all $t \ge 0$.*

Proof. Define a function $V \in C^2(\mathbb{R}_+^2, \mathbb{R}_+)$ by $V(x,y) = x^{\theta_1} y^{\theta_2}$. For any $t \geq 0$, the Itô's formula shows that

$$dV(x_t, y_t) = LV(x_t, y_t)dt + (\theta_1\sigma_1 + \theta_2\sigma_2)V(x_t, y_t)dw_t. \tag{3.2}$$

It is easy to see that

$$\begin{aligned} LV(x,y) = \Big[&\frac{1}{2}\sigma_1^2\theta_1(\theta_1 - 1) + \frac{1}{2}\sigma_2^2\theta_2(\theta_2 - 1) + \sigma_1\sigma_2\theta_1\theta_2 + \theta_1(a_1 - b_1 x) \\ &- \theta_2(a_2 + b_2 y) + \frac{\theta_2 c_2 x - \theta_1 c_1 y}{\alpha + \beta x + \gamma y}\Big] V(x,y) \\ \leq {}& [d_1 - d_2(x+y)]V(x,y). \end{aligned} \tag{3.3}$$

For every integer $k \geq 1$, define a stopping time $\tau_k = \inf\{t \geq 0 : x_t + y_t \geq k\}$, then the sequence $\{\tau_k, k \geq 1\}$ is nondecreasing and by the positive invariance of (x_t, y_t) on $\mathbb{R}_+^2$, $\lim_{k\to\infty} \tau_k = \infty$ a.s. It then follows from (3.2) that

$$\begin{aligned} V(x_{t\wedge\tau_k}, y_{t\wedge\tau_k}) = V(x_0, y_0) &+ \int_0^{t\wedge\tau_k} LV(x_s, y_s)ds \\ &+ (\theta_1\sigma_1 + \theta_2\sigma_2)\int_0^{t\wedge\tau_k} V(x_s, y_s)dw_s. \end{aligned}$$

Taking expectations of both sides and using (3.3), we have

$$\begin{aligned} \mathbb{E}V(x_{t\wedge\tau_k}, y_{t\wedge\tau_k}) &\leq V(x_0, y_0) + d_1\mathbb{E}\int_0^{t\wedge\tau_k} V(x_s, y_s)ds \\ &\leq V(x_0, y_0) + d_1\int_0^t \mathbb{E}V(x_{s\wedge\tau_k}, y_{s\wedge\tau_k})ds. \end{aligned}$$

Using Gronwall's inequality gives $\mathbb{E}V(x_{t\wedge\tau_k}, y_{t\wedge\tau_k}) \leq V(x_0, y_0)\exp\{d_1 t\}$. Since $V(x_{t\wedge\tau_k}, y_{t\wedge\tau_k}) > 0$, letting $k \to \infty$ and using Fatou's lemma, we have $\mathbb{E}V(x_t, y_t) \leq V(x_0, y_0)\exp\{d_1 t\}$ for all $t \geq 0$. We complete the proof. □

Theorem 3.2. *Let θ_1, θ_2 be positive numbers. For any initial value $(x_0, y_0) \in \mathbb{R}_+^2$, the solution (x_t, y_t) of (1.3) satisfies*

$$\mathbb{E}(x_t^{\theta_1} y_t^{\theta_2}) \leq \exp\{\lambda_1 + \lambda_2\exp\{-d_2 t\}\}, \qquad \text{for all } t \geq 0.$$

Consequently, $\limsup_{t\to\infty} \mathbb{E}(x_t^{\theta_1} y_t^{\theta_2}) \leq \exp\{\lambda_1\}$.

Proof. The inequality $V(x,y) = x^{\theta_1} y^{\theta_2} \leq (x+y)^{\theta_1+\theta_2}$ implies $x + y \geq V^\theta(x,y)$. Then, from (3.3) we get

$$LV(x,y) \leq [d_1 - d_2 V^\theta(x,y)]V(x,y). \tag{3.4}$$

By Lemma 3.1,

$$\mathbb{E}\left[V^{1+\theta}(x_t, y_t)\right] = \mathbb{E}\left[x_t^{\theta_1(1+\theta)} y_t^{\theta_2(1+\theta)}\right] < \infty, \qquad \text{for all } t \geq 0.$$

Thus, using the Hölder's inequality yields

$$[\mathbb{E}V(x_t, y_t)]^{1+\theta} \leq \mathbb{E}\left[V^{1+\theta}(x_t, y_t)\right].$$

It then follows from (3.2) and (3.4) that for any $t \geq 0$ and $h > 0$,

$$\begin{aligned}\mathbb{E}V(x_{t+h}, y_{t+h}) - \mathbb{E}V(x_t, y_t) &\leq \int_t^{t+h} \left[d_1 \mathbb{E}V(x_s, y_s) - d_2 \mathbb{E}V^{1+\theta}(x_s, y_s)\right] ds \\ &\leq \int_t^{t+h} \left[d_1 \mathbb{E}V(x_s, y_s) - d_2 \left[\mathbb{E}V(x_s, y_s)\right]^{1+\theta}\right] ds. \quad (3.5)\end{aligned}$$

Put $v(t) = \mathbb{E}V(x_t, y_t)$, then $0 < v(t) < \infty$ for all $t \geq 0$ and the continuity of $v(t)$ in t follows from the continuity of the solution (x_t, y_t) and the dominated convergence theorem. We define the right upper derivative of $v(t)$ by

$$D^+v(t) = \limsup_{h\to 0} \frac{v(t+h) - v(t)}{h}\cdot$$

From (3.5), we have

$$\frac{v(t+h) - v(t)}{h} \leq \frac{1}{h} \int_t^{t+h} \left[d_1 v(s) - d_2 v^{1+\theta}(s)\right] ds.$$

Let $h \to 0$, we obtain $D^+v(t) \leq v(t)[d_1 - d_2 v^\theta(t)]$ for all $t \geq 0$. Therefore,

$$\begin{aligned}D^+[\exp\{d_2 t\} \ln v(t)] &= d_2 \exp\{d_2 t\} \ln v(t) + \exp\{d_2 t\} \frac{D^+v(t)}{v(t)} \\ &\leq d_2 \exp\{d_2 t\} \ln v(t) + \exp\{d_2 t\}[d_1 - d_2 v^\theta(t)] \\ &= d_1 \exp\{d_2 t\} + d_2 \exp\{d_2 t\}[\ln v(t) - v^\theta(t)].\end{aligned}$$

It is easy to see that $\ln x - x^\theta \leq -\frac{1}{\theta}(1 + \ln\theta)$ for all $x > 0$, then

$$D^+[\exp\{d_2 t\} \ln v(t)] \leq \left[d_1 - \frac{1}{\theta}(1 + \ln\theta) d_2\right] \exp\{d_2 t\}.$$

Taking integrations of both sides yields

$$\begin{aligned}\exp\{d_2 t\} \ln v(t) &\leq \ln v(0) + \left[\frac{d_1}{d_2} - \frac{1}{\theta}(1 + \ln\theta)\right] [\exp\{d_2 t\} - 1] \\ &= \lambda_2 + \lambda_1 \exp\{d_2 t\}.\end{aligned}$$

Consequently, $\ln v(t) \leq \lambda_1 + \lambda_2 \exp\{-d_2 t\}$ which proves the first statement of the theorem. Letting $t \to \infty$ in the latter inequality, we get the second one. The proof is complete. □

Theorem 3.3. *For any positive numbers* $\theta_1, \theta_2, \varsigma_1, \varsigma_2$, *there exist positive constants* $K_i = K_i(\theta_1, \theta_2, \varsigma_1, \varsigma_2)$ $(i = 1, 2)$ *such that for any initial value* $x_0 \in \mathbb{R}^2_+$, *the solution of* (1.3) *has the following properties*

(i) $\limsup_{t\to\infty} \frac{1}{t}\mathbb{E}\left[\varsigma_1 x_t^{\theta_1} + \varsigma_2 y_t^{\theta_2}\right] \leq K_1$.

(ii) *If* $\theta_i \in (0, 1) (i = 1, 2)$ *then* $\limsup_{t\to\infty} \frac{1}{t}\mathbb{E}\left[\int_0^t (\varsigma_1 x_s^{\theta_1} + \varsigma_2 y_s^{\theta_2}) ds\right] \leq K_2$.

Proof. Consider a function $V : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ defined by $V(x, y) = \varsigma_1 x^{\theta_1} + \varsigma_2 y^{\theta_2}$. For any $t \geq 0$, the Itô's formula shows that

$$dV(x_t, y_t) = LV(x_t, y_t)dt + (\theta_1 \sigma_1 \varsigma_1 x_t^{\theta_1} + \theta_2 \sigma_2 \varsigma_2 y_t^{\theta_2}) dw_t, \quad (3.6)$$

where

$$LV(x,y) = \Big[\frac{1}{2}\theta_1(\theta_1-1)\sigma_1^2\varsigma_1 x^{\theta_1} + \frac{1}{2}\theta_2(\theta_2-1)\sigma_2^2\varsigma_2 y^{\theta_2} + \theta_1\varsigma_1 x^{\theta_1}(a_1 - b_1 x - \frac{c_1 y}{\alpha+\beta x+\gamma y}) + \theta_2\varsigma_2 y^{\theta_2}(-a_2 + \frac{c_2 x}{\alpha+\beta x+\gamma y} - b_2 y)\Big]. \tag{3.7}$$

It is easy to get from (3.7) that there exists $K_1 = K_1(\theta_1,\theta_2,\varsigma_1,\varsigma_2)$ such that $LV(x,y) \le K_1$ for all $(x,y) \in \mathbb{R}_+^2$. Then, from (3.6) we have

$$V(x_t,y_t) \le V(x_0,y_0) + K_1 t + \int_0^t (\theta_1\sigma_1\varsigma_1 x_s^{\theta_1} + \theta_2\sigma_2\varsigma_2 y_s^{\theta_2}) dw_s.$$

Taking expectations of both sides yields $\mathbb{E}V(x_t,y_t) \le V(x_0,y_0) + K_1 t$ for all $t \ge 0$, which completes the proof of Part (i). To prove Part (ii), we see from (3.7) that for any $\theta_1,\theta_2 \in (0,1)$,

$$\lim_{||(x,y)||\to\infty} [LV(x,y) + V(x,y)] = -\infty.$$

Then, there exists $K_2 = K_2(\theta_1,\theta_2,\varsigma_1,\varsigma_2)$ such that $LV(x,y) + V(x,y) \le K_2$ for all $(x,y) \in \mathbb{R}_+^2$. We therefore have from (3.6) that

$$V(x_t,y_t) \le V(x_0,y_0) + \int_0^t \{K_2 - V(x_s,y_s)\} ds + \int_0^t (\theta_1\sigma_1\varsigma_1 x_s^{\theta_1} + \theta_2\sigma_2\varsigma_2 y_s^{\theta_2}) dw_s.$$

Taking expectations of both sides implies

$$\mathbb{E}V(x_t,y_t) + \mathbb{E}\int_0^t V(x_s,y_s) ds \le V(x_0,y_0) + K_2 t.$$

Therefore, $\mathbb{E}\int_0^t V(x_s,y_s)ds \le V(x_0,y_0) + K_2 t$, and

$$\limsup_{t\to\infty} \frac{1}{t}\mathbb{E}\left[\int_0^t (\varsigma_1 x_s^{\theta_1} + \varsigma_2 y_s^{\theta_2}) ds\right] \le K_2.$$

□

4. Upper Rate Estimation

In this section, we show the long-time behavior of solutions of system (1.3). Putting $x_t = \exp\{\xi_t\}, y_t = \exp\{\eta_t\}$, and substituting this transformation into system (1.3) we obtain

$$\begin{cases} d\xi_t = \left[\vartheta_1 - b_1\exp\{\xi_t\} - \frac{c_1\exp\{\eta_t\}}{\alpha+\beta\exp\{\xi_t\}+\gamma\exp\{\eta_t\}}\right] dt + \sigma_1 dw_t, \\ d\eta_t = \left[-\vartheta_2 + \frac{c_2\exp\{\xi_t\}}{\alpha+\beta\exp\{\xi_t\}+\gamma\exp\{\eta_t\}} - b_2\exp\{\eta_t\}\right] dt + \sigma_2 dw_t, \end{cases} \tag{4.1}$$

or equivalently

$$\begin{cases} d\xi_t = \left[\vartheta_1 - b_1 x_t - \frac{c_1 y_t}{\alpha+\beta x_t+\gamma y_t}\right] dt + \sigma_1 dw_t, \\ d\eta_t = \left[-\vartheta_2 + \frac{c_2 x_t}{\alpha+\beta x_t+\gamma y_t} - b_2 y_t\right] dt + \sigma_2 dw_t, \end{cases}$$

where $\vartheta_1 = a_1 - \frac{\sigma_1^2}{2}, \vartheta_2 = a_2 + \frac{\sigma_2^2}{2}$. Denoting

$$f_3(x,y) = \vartheta_1 - b_1 \exp\{x\} - \frac{c_1 \exp\{y\}}{\alpha + \beta \exp\{x\} + \gamma \exp\{y\}},$$
$$f_4(x,y) = -\vartheta_2 + \frac{c_2 \exp\{x\}}{\alpha + \beta \exp\{x\} + \gamma \exp\{y\}} - b_2 \exp\{y\},$$

we can rewrite (4.1) under the form

$$\begin{cases} d\xi_t = f_3(\xi_1, \eta_t)dt + \sigma_1 dw_t, \\ d\eta_t = f_4(\xi_1, \eta_t)dt + \sigma_2 dw_t. \end{cases}$$

The infinitesimal operator of the equation (4.1) is

$$Lv = \frac{1}{2}\sigma_1^2 \frac{\partial^2 v}{\partial x^2} + \frac{1}{2}\sigma_2^2 \frac{\partial^2 v}{\partial y^2} + \sigma_1\sigma_2 \frac{\partial^2 v}{\partial x \partial y} + f_3 \frac{\partial v}{\partial x} + f_4 \frac{\partial v}{\partial y}. \tag{4.2}$$

The density of the random variable (ξ_t, η_t), if it exists and is smooth, can be found from the Fokker - Planck equation:

$$\frac{\partial v}{\partial t} = \frac{1}{2}\sigma_1^2 \frac{\partial^2 v}{\partial x^2} + \frac{1}{2}\sigma_2^2 \frac{\partial^2 v}{\partial y^2} + \sigma_1\sigma_2 \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial (f_3 v)}{\partial x} - \frac{\partial (f_4 v)}{\partial y}.$$

The behavior of two boundary equations can be easily studied. We have the following theorems.

Theorem 4.1. *If the prey is absent, i.e., $x_t = 0$ a.s. for all $t \geq 0$, then the predator dies out with probability 1. Further, the death rate of predator is exponential, i.e.,*

$$\limsup_{t\to\infty} \frac{\ln y_t}{t} \leq -\vartheta_2.$$

Proof. The quantity of predator at time t satisfies the equation

$$dy_t = \left(-a_2 y_t - b_2 y_t^2\right) dt + \sigma_2 y_t dw_t. \tag{4.3}$$

Using the same arguments as in proof of Theorem 2.1, it is easy to see that with any initial $y_0 \in \mathbb{R}_+$, solution of (4.3) is unique, positive and global. By putting $y_t = \exp\{\eta_t\}$, we have

$$d\eta_t = (-\vartheta_2 - b_2 e^{\eta_t})dt + \sigma_2 dw_t \leq -\vartheta_2 dt + \sigma_2 dw_t.$$

Thus, $\eta_t \leq \eta_0 - \vartheta_2 t + \sigma_2 w_t$. Using the law of iterated logarithm which implies that $\lim_{t\to\infty} \frac{w_t}{t} = 0$ and noting that $\vartheta_2 > 0$, we have $\limsup_{t\to\infty} \frac{\eta_t}{t} \leq -\vartheta_2$. Therefore, $\limsup_{t\to\infty} \frac{\ln y_t}{t} \leq -\vartheta_2$, and $\lim_{t\to\infty} y_t = 0$ a.s. □

Theorem 4.2. *If the predator is absent, i.e., $y_t = 0$ a.s. for all $t \geq 0$, then*

(i) *If $\vartheta_1 \leq 0$ then $\lim_{t\to\infty} x_t = 0$ a.s. Further, if $\vartheta_1 < 0$, the death rate of prey is exponential, i.e., $\limsup_{t\to\infty} \frac{\ln x_t}{t} \leq \vartheta_1$.*

(ii) *If $\vartheta_1 > 0$ then the quantity of prey oscillates between 0 and ∞. Further, the distribution of the process $\ln x_t$ converges in distribution to a stationary distribution which has the density*

$$f_*(x) = \kappa_1 \exp\left\{\frac{2\vartheta_1}{\sigma_1^2} x - \frac{2b_1}{\sigma_1^2} \exp\{x\}\right\}$$

where

$$\kappa_1 = \left[\int_{-\infty}^{\infty} \exp\left\{ \frac{2\vartheta_1}{\sigma_1^2} x - \frac{2b_1}{\sigma_1^2} \exp\{x\} \right\} dx \right]^{-1}.$$

Proof. Similar to Theorem 4.1, the quantity $x_t = \exp\{\xi_t\}$ of prey at time t satisfies the equation

$$d\xi_t = (\vartheta_1 - b_1 \exp\{\xi_t\})dt + \sigma_1 dw_t. \tag{4.4}$$

Let

$$\begin{aligned} s(x) &= \int_0^x \exp\left\{ -\int_0^y \frac{2(\vartheta_1 - b_1 \exp\{u\})}{\sigma_1^2} du \right\} dy \\ &= \int_0^x \exp\left\{ \frac{-2(b_1 + \vartheta_1 y - b_1 \exp\{y\})}{\sigma_1^2} \right\} dy. \end{aligned}$$

For Case (i), we see easily that $\lim_{x\to\infty} s(x) = \infty$ and $\lim_{x\to-\infty} s(x) > -\infty$. Then by [14, Theorem 3.1], $\lim_{t\to\infty} \xi_t = -\infty$, or equivalently $\lim_{t\to\infty} x_t = 0$ a.s. Further, if $\vartheta_1 < 0$, we have $d\xi_t \le \vartheta_1 dt + \sigma_1 dw_t$. Using the same arguments as in Theorem 4.1 yields $\limsup_{t\to\infty} \frac{\ln x_t}{t} \le \vartheta_1$. Then, $\lim_{t\to\infty} x_t = 0$ a.s. We now consider Case (ii). It follows from $\lim_{x\to\infty} s(x) = \infty$ and $\lim_{x\to-\infty} s(x) = -\infty$ that $\limsup_{t\to\infty} \xi_t = \infty, \liminf_{t\to\infty} \xi_t = -\infty$, i.e., $\limsup_{t\to\infty} x_t = \infty, \liminf_{t\to\infty} x_t = 0$ a.s. This means that without the predator, the quantity of prey oscillates between 0 and ∞. Further, the equation (4.4) has a unique stationary distribution which has a density $f_*(x)$ satisfying the Fokker - Planck equation

$$\frac{1}{2}\sigma_1^2 \frac{d^2 f_*(x)}{dx^2} - \frac{d}{dx}\left[(\vartheta_1 - b_1 \exp\{x\}) f_*(x)\right] = 0. \tag{4.5}$$

The general solution to the equation (4.5) is

$$\begin{aligned} f_*(x) = \exp&\left\{ \frac{2\vartheta_1}{\sigma_1^2} x - \frac{2b_1}{\sigma_1^2} \exp\{x\} \right\} \\ &\times \left[\kappa_1 + \kappa_2 \int_0^x \exp\left\{ \frac{-2\vartheta_1}{\sigma_1^2} u + \frac{2b_1}{\sigma_1^2} \exp\{u\} \right\} du \right], \end{aligned}$$

where κ_1, κ_2 are two constants. It is easy to follow from the conditions $f_*(x) \ge 0$ and

$$\int_{-\infty}^{\infty} f_*(x)\, dx = 1$$

that $\kappa_2 = 0$ and

$$\kappa_1 = \left[\int_{-\infty}^{\infty} \exp\left\{ \frac{2\vartheta_1}{\sigma_1^2} x - \frac{2b_1}{\sigma_1^2} \exp\{x\} \right\} dx \right]^{-1}.$$

Therefore,

$$f_*(x) = \kappa_1 \exp\left\{ \frac{2\vartheta_1}{\sigma_1^2} x - \frac{2b_1}{\sigma_1^2} \exp\{x\} \right\}. \tag{4.6}$$

By the existence of a stationary distribution of a stochastic differential equation (4.4), ξ_t converges in distribution to the stationary solution which has the density f_* as $t \to \infty$ (see [23, Theorems 16 and 17], [21, Lemma 7]). □

We now come back the model of two species. We have the following theorem.

Theorem 4.3. *Let (x_t, y_t) be the solution of (1.3) with initial condition $(x_0, y_0) \in \mathbb{R}^2_+$, or equivalently (ξ_t, η_t) be the solution of (4.1) with $(\xi_0, \eta_0) \in \mathbb{R}^2$. Then,*

(i) *If $\vartheta_1 \le 0$, then $\lim_{t\to\infty} x_t = \lim_{t\to\infty} y_t = 0$ a.s. Further, if $\vartheta_1 < 0$, the death rates of species are exponential.*

(ii) *If $\vartheta_1 > 0$ and $\frac{c_2}{\beta} - \vartheta_2 < 0$, then $\lim_{t\to\infty} y_t = 0$ a.s. Further, the death rate of predator is exponential and the conclusions in Case (ii) of Theorem 4.2 hold.*

Proof. Consider Part (i). From the first equation of (4.1), we have

$$d\xi_t \le (\vartheta_1 - b_1 \exp\{\xi_t\})dt + \sigma_1 dw_t.$$

By comparison theorem [14, Theorem 1.1, p.352]), if $\bar{\xi}_t$ is a solution to (4.4) with $\bar{\xi}_0 = \xi_0$ then $\xi_t \le \bar{\xi}_t$ for all $t \ge 0$. Then from $\vartheta_1 \le 0$ and Theorem 4.2, we have $\lim_{t\to\infty} \xi_t = \lim_{t\to\infty} \bar{\xi}_t = -\infty$ a.s., i.e., $\lim_{t\to\infty} x_t = 0$ a.s. Further, if $\vartheta_1 < 0$, by Theorem 4.2, the death rate of prey is exponential. Now, from the second equation of (4.1) we have $d\eta_t \le \left(-\vartheta_2 + \frac{c_2 \exp\{\xi_t\}}{\alpha}\right) dt + \sigma_2 dw_t$. Then,

$$\eta_t \le \eta_0 - \vartheta_2 t + \frac{c_2}{\alpha} \int_0^t \exp\{\xi_s\} ds + \sigma_2 w_t, \text{ a.s.}$$

For every $\omega \in \Omega$, $\lim_{t\to\infty} \xi_t(\omega) = -\infty$, then there exists $t_0 \ge 0$ such that for all $t \ge t_0$, $\frac{c_2}{\alpha} \exp\{\xi_t(\omega)\} < \frac{\vartheta_2}{2}$. Therefore,

$$\eta_t(\omega) \le \eta_0 + \frac{c_2}{\alpha} \int_0^{t_0} \exp\{\xi_s(\omega)\} ds - \frac{\vartheta_2}{2}(t + t_0) + \sigma_2 w_t(\omega).$$

By the law of iterated logarithm, we can conclude that $\limsup_{t\to\infty} \frac{\eta_t(\omega)}{t} \le -\frac{\vartheta_2}{2}$. Thus $\limsup_{t\to\infty} \frac{\ln y_t(\omega)}{t} \le -\frac{\vartheta_2}{2}$, and $\lim_{t\to\infty} y_t = 0$ a.s.

To prove Part (ii), using the inequality $d\eta_t \le (-\vartheta_2 + \frac{c_2}{\beta})dt + \sigma_2 dw_t$ yields $\lim_{t\to\infty} \eta_t = -\infty$ and at an exponential rate, $\lim_{t\to\infty} y_t = 0$ a.s. It remains to show that ξ_t converges in distribution to a stationary solution $\bar{\xi}$ of the following equation

$$\begin{cases} d\bar{\xi}_t = (\vartheta_1 - b_1 \exp\{\bar{\xi}_t\})dt + \sigma_1 dw_t, \\ \bar{\xi}_0 = \xi_0. \end{cases} \tag{4.7}$$

By Theorem 4.2, the distribution of $\bar{\xi}$ has the density $f_*(x)$ defined by (4.6) and

$$\lim_{t\to\infty} \bar{\xi}_t = \bar{\xi}, \qquad \text{in distribution.} \tag{4.8}$$

We shall prove that

$$\lim_{t\to\infty} (\bar{\xi}_t - \xi_t) = 0, \qquad \text{in probability.} \tag{4.9}$$

Given $\varepsilon \in (0, \vartheta_1)$. Consider the following equation

$$\begin{cases} d\bar{\xi}_t(\varepsilon) = \left[(\vartheta_1 - \varepsilon) - b_1 \exp\{\bar{\xi}_t(\varepsilon)\}\right] dt + \sigma_1 dw_t, \\ \bar{\xi}_0(\varepsilon) = \xi_0. \end{cases} \tag{4.10}$$

According to the comparison theorem, we have

$$\bar{\xi}_t \ge \bar{\xi}_t(\varepsilon) \vee \xi_t, \qquad \text{a.s.} \tag{4.11}$$

It follows from $\lim_{t\to\infty} \eta_t = -\infty$ a.s. that for arbitrary small $\bar{\varepsilon} \in (0, \varepsilon)$, there exist t_0 and a set $\Omega_{\bar{\varepsilon}}$ such that $\mathbb{P}(\Omega_{\bar{\varepsilon}}) > 1 - \bar{\varepsilon}$ and

$$\frac{c_1 \exp\{\eta_t(\omega)\}}{\alpha + \beta \exp\{\xi_t(\omega)\} + \gamma \exp\{\eta_t(\omega)\}} < \bar{\varepsilon}$$

for $t \geq t_0$ and $\omega \in \Omega_{\bar{\varepsilon}}$. Thus, $\bar{\xi}_t(\varepsilon) < \xi_t$ on $\Omega_{\bar{\varepsilon}}$ for all $t \geq t_0$. This implies $\liminf_{t\to\infty}[\xi_t - \bar{\xi}_t(\varepsilon] \geq 0$ on $\Omega_{\bar{\varepsilon}}$. Letting $\bar{\varepsilon} \to 0$ yields

$$\liminf_{t\to\infty}[\xi_t - \bar{\xi}_t(\varepsilon] \geq 0, \qquad \text{a.s.} \tag{4.12}$$

Now, the equations (4.7) and (4.10) can be solved explicitly to give

$$\bar{\xi}_t = \vartheta_1 t + \sigma_1 w_t - \ln\left[e^{-\xi_0} + b_1 \int_0^t e^{\vartheta_1 s + \sigma_1 w_s} ds\right],$$

$$\bar{\xi}_t(\varepsilon) = (\vartheta_1 - \varepsilon)t + \sigma_1 w_t - \ln\left[e^{-\xi_0} + b_1 \int_0^t e^{(\vartheta_1 - \varepsilon)s + \sigma_1 w_s} ds\right].$$

Since $w_s \sim \mathcal{N}(0, s)$, we have easily $\mathbb{E}e^{\sigma_1 w_s} = e^{\frac{\sigma_1^2 s}{2}}$. Then,

$$\begin{aligned}
&\mathbb{E}|\bar{\xi}_t - \bar{\xi}_t(\varepsilon)| = \mathbb{E}[\bar{\xi}_t - \bar{\xi}_t(\varepsilon)] \\
&= \varepsilon t + \ln\left[e^{-\xi_0} + b_1 \int_0^t e^{(\vartheta_1 - \varepsilon)s} \mathbb{E}e^{\sigma_1 w_s} ds\right] - \ln\left[e^{-\xi_0} + b_1 \int_0^t e^{\vartheta_1 s} \mathbb{E}e^{\sigma_1 w_s} ds\right] \\
&\qquad = \varepsilon t + \ln\left[e^{-\xi_0} + b_1 \frac{e^{(\vartheta_1 - \varepsilon + \frac{\sigma_1^2}{2})t} - 1}{\vartheta_1 - \varepsilon + \frac{\sigma_1^2}{2}}\right] - \ln\left[e^{-\xi_0} + b_1 \frac{e^{(\vartheta_1 + \frac{\sigma_1^2}{2})t} - 1}{\vartheta_1 + \frac{\sigma_1^2}{2}}\right].
\end{aligned}$$

From this equality, it is easy to see that

$$\lim_{t\to\infty} \mathbb{E}|\bar{\xi}_t - \bar{\xi}_t(\varepsilon)| = \ln \frac{\vartheta_1 + \frac{\sigma_1^2}{2}}{\vartheta_1 - \varepsilon + \frac{\sigma_1^2}{2}}.$$

Then, $\lim_{\varepsilon\to 0} \lim_{t\to\infty} \mathbb{E}|\bar{\xi}_t - \bar{\xi}_t(\varepsilon)| = 0$. This yields

$$\lim_{\varepsilon\to 0} \lim_{t\to\infty} |\bar{\xi}_t - \bar{\xi}_t(\varepsilon)| = 0, \qquad \text{in probability.} \tag{4.13}$$

Combining (4.11), (4.12) and (4.13) gives (4.9). To prove $\xi_t \to \bar{\xi}$ in distribution as $t \to \infty$, by using (4.8), it suffices to show $\lim_{t\to\infty} \mathbb{E}[f(\bar{\xi}_t) - f(\xi_t)] = 0$ whenever f is bounded and continuous on $\mathbb{R}$. Let such an f be given, and set $M = \sup_{x\in\mathbb{R}} |f(x)| < \infty$. It follows from (4.8) that $\{\bar{\xi}_t\}_{t\geq 0}$ is relatively compact. Then by Prohorov theorem [17, Karatzas, I. and Shreve, S. E.], $\{\bar{\xi}_t\}_{t\geq 0}$ is tight; so for each $\varepsilon > 0$, there exists a compact set $K \subset \mathbb{R}$ such that $\mathbb{P}[\bar{\xi}_t \in K] \geq 1 - \frac{\varepsilon}{6M}$ for any $t \geq 0$. Choose $0 < \delta < 1$ so $|f(x) - f(y)| < \frac{\varepsilon}{3}$ whenever $x \in K$ and $|x - y| < \delta$. Finally, from (4.9) we choose a positive number T such that $\mathbb{P}[|\bar{\xi}_t - \xi_t| \geq \delta] \leq \frac{\varepsilon}{6M}$ for all $t \geq T$. We have

$$\begin{aligned}
\left|\int_\Omega [f(\bar{\xi}_t) - f(\xi_t)] d\mathbb{P}\right| &\leq \frac{\varepsilon}{3}\mathbb{P}[\bar{\xi}_t \in K, |\bar{\xi}_t - \xi_t| < \delta] + 2M\mathbb{P}[\bar{\xi}_t \notin K] \\
&\qquad + 2M\mathbb{P}[|\bar{\xi}_t - \xi_t| \geq \delta] \leq \varepsilon, \qquad \text{for all } t \geq T.
\end{aligned}$$

The proof is complete. □

Remark 4.4. Let $\sigma_1 = \sigma_2 = 0$, i.e., the model is deterministic, then $\vartheta_1 = a_1 > 0$. It is easy to see that we never get $\lim_{t\to\infty} x_t = 0$. But from Part (i) in Theorem 4.2 and one in Theorem 4.3, the prey population dies out even if there is no predator. Further, the death rate is very rapid (at an exponential rate). This means that a relatively large stochastic perturbation can cause the extinction of the population.

In the next theorem, we give a pathwise estimation.

Theorem 4.5. *For any initial value* $(x_0, y_0) \in \mathbb{R}^2_+$ *and* $\theta_1, \theta_2 \in \mathbb{R}_+$,

$$\limsup_{t\to\infty} \frac{\ln\left[x_t^{\theta_1} y_t^{\theta_2}\right]}{\ln t} \le \theta_1 + \theta_2, \qquad a.s.$$

Proof. Fix $p > 0$. Applying Itô's formula to $\exp\{pt\}\xi_t$ and $\exp\{pt\}\eta_t$, from (4.1) we have

$$\begin{aligned}\exp\{pt\}\xi_t = \xi_0 &+ \int_0^t \exp\{ps\}\left(\vartheta_1 - b_1 x_s - \frac{c_1 y_s}{\alpha + \beta x_s + \gamma y_s}\right) ds \\ &+ \int_0^t p\exp\{ps\}\xi_s ds + \int_0^t \sigma_1 \exp\{ps\} dw_s, \end{aligned} \tag{4.14}$$

$$\begin{aligned}\exp\{pt\}\eta_t = \eta_0 &+ \int_0^t \exp\{ps\}\left(-\vartheta_2 + \frac{c_2 x_s}{\alpha + \beta x_s + \gamma y_s} - b_2 y_s\right) ds \\ &+ \int_0^t p\exp\{ps\}\eta_s ds + \int_0^t \sigma_2 \exp\{ps\} dw_s. \end{aligned} \tag{4.15}$$

Put $M_{it} = \int_0^t \sigma_i \exp\{ps\} dw_s$ $(i = 1, 2)$. Then, $\{M_{it}\}_{t\ge 0}$ $(i = 1, 2)$ are real valued continuous local martingale vanishing at $t = 0$ with quadratic form

$$< M_i, M_i >_t = \int_0^t \sigma_i^2 \exp\{2ps\} ds = \frac{\sigma_i^2}{2p}[\exp\{2pt\} - 1].$$

Let $\varepsilon \in (0, 1)$ and $\theta > 1$. Using the exponential martingale inequality [18, Theorem 1.7.4] we have for every $k \ge 1$ and $i = 1, 2$,

$$\mathbb{P}\left\{\sup_{0\le t\le k}\left[M_{it} - \frac{\varepsilon}{2}\exp\{-pk\} < M_i, M_i >_t\right] \ge \frac{\theta \exp\{pk\}}{\varepsilon}\ln k\right\} \le \frac{1}{k^\theta}.$$

It then follows from Borel - Cantelli lemma that there exist $\Omega_i \subset \Omega\,(i = 1, 2)$ with $\mathbb{P}(\Omega_i) = 1$ having the following property. For any $\omega \in \Omega_i$, there exist $k_i = k_i(\omega)$ such that for all $k \ge k_i$ and $t \in [0, k]$,

$$\begin{aligned} M_{it} &\le \frac{\varepsilon}{2}\exp\{-pk\} < M_i, M_i >_t + \frac{\theta \exp\{pk\}}{\varepsilon}\ln k \\ &= \frac{\varepsilon\sigma_i^2}{2}\exp\{-pk\}\int_0^t \exp\{2ps\} ds + \frac{\theta\exp\{pk\}}{\varepsilon}\ln k. \end{aligned}$$

Thus we have from (4.14) and (4.15) that for any $\omega \in \Omega_1 \cap \Omega_2$ and $t \in [0,k]$, $k \geq k_0(\omega) = k_1(\omega) \wedge k_2(\omega)$,

$$\begin{aligned}
&\exp\{pt\}\xi_t \\
&\leq \xi_0 + \int_0^t p\exp\{ps\}\xi_s ds \\
&\quad + \int_0^t \exp\{ps\}\left(\vartheta_1 - b_1 x_s - \frac{c_1 y_s}{\alpha+\beta x_s+\gamma y_s}\right)ds \\
&\quad + \frac{\varepsilon\sigma_1^2}{2}\exp\{-pk\}\int_0^t \exp\{2ps\}ds + \frac{\theta\exp\{pk\}}{\varepsilon}\ln k \\
&= \xi_0 + p\int_0^t \exp\{ps\}\xi_s ds + \frac{\theta\exp\{pk\}}{\varepsilon}\ln k \\
&\quad + \int_0^t \exp\{ps\}\Big[\vartheta_1 + \frac{\varepsilon\sigma_1^2}{2}\exp\{-p(k-s)\} - b_1 x_s - \frac{c_1 y_s}{\alpha+\beta x_s+\gamma y_s}\Big]ds, \quad (4.16)
\end{aligned}$$

and

$$\begin{aligned}
&\exp\{pt\}\eta_t \\
&\leq \eta_0 + \int_0^t p\exp\{ps\}\eta_s ds \\
&\quad + \int_0^t \exp\{ps\}\left(-\vartheta_2 + \frac{c_2 x_s}{\alpha+\beta x_s+\gamma y_s} - b_2 y_s\right)ds \\
&\quad + \frac{\varepsilon\sigma_2^2}{2}\exp\{-pk\}\int_0^t \exp\{2ps\}ds + \frac{\theta\exp\{pk\}}{\varepsilon}\ln k \\
&= \xi_0 + p\int_0^t \exp\{ps\}\eta_s ds + \frac{\theta\exp\{pk\}}{\varepsilon}\ln k \\
&\quad + \int_0^t \exp\{ps\}\Big[-\vartheta_2 + \frac{\varepsilon\sigma_2^2}{2}\exp\{-p(k-s)\} + \frac{c_2 x_s}{\alpha+\beta x_s+\gamma y_s} - b_2 y_s\Big]ds. \quad (4.17)
\end{aligned}$$

From (4.16) and (4.17), we have for any $\omega \in \Omega_1 \cap \Omega_2$ and $t \in [0,k], k \geq k_0(\omega)$,

$$\begin{aligned}
\exp\{pt\}(\theta_1\xi_t + \theta_2\eta_t) \leq (\theta_1\xi_0 + \theta_2\eta_0) &+ \frac{\theta(\theta_1+\theta_2)\exp\{pk\}}{\varepsilon}\ln k \\
&+ \int_0^t \exp\{ps\}\Big[p(\theta_1\xi_s + \theta_2\eta_s) + \theta_1\vartheta_1 - \theta_2\vartheta_2 \\
&\quad + \frac{\varepsilon(\theta_1\sigma_1^2 + \theta_2\sigma_2^2)}{2}\exp\{-p(k-s)\} - b_1\theta_1 x_s \\
&\quad - b_2\theta_2 y_s + \frac{\theta_2 c_2 x_s - \theta_1 c_1 y_s}{\alpha+\beta x_s+\gamma y_s}\Big]ds. \quad (4.18)
\end{aligned}$$

Since $\theta_1, \theta_2 \in \mathbb{R}_+$, there exist $K^* = K^*(p,\theta_1,\theta_2) > 0$ such that for any $(x,y) \in \mathbb{R}_+^2$, we have

$$\Big[p(\theta_1\ln x + \theta_2\ln y) + \theta_1\vartheta_1 - \theta_2\vartheta_2 - b_1\theta_1 x - b_2\theta_2 y + \frac{\theta_2 c_2 x - \theta_1 c_1 y}{\alpha+\beta x+\gamma y}\Big] \leq K^*.$$

It then follows from (4.18) that for any $\omega \in \Omega_1 \cap \Omega_2$ and $t \in [0,k], k \geq k_0(\omega)$,

$$\begin{aligned}\exp\{pt\}(\theta_1\xi_t + \theta_2\eta_t) &\leq (\theta_1\xi_0 + \theta_2\eta_0) + \frac{\theta(\theta_1+\theta_2)\exp\{pk\}}{\varepsilon}\ln k \\ &\quad + \int_0^t \exp\{ps\}\Big[K^* + \frac{\varepsilon(\theta_1\sigma_1^2 + \theta_2\sigma_2^2)}{2}\Big]ds \\ &\leq (\theta_1\xi_0 + \theta_2\eta_0) + \frac{\theta(\theta_1+\theta_2)\exp\{pk\}}{\varepsilon}\ln k \\ &\quad + \frac{1}{p}\left[K^* + \frac{\varepsilon(\theta_1\sigma_1^2 + \theta_2\sigma_2^2)}{2}\right](\exp\{pt\}-1),\end{aligned}$$

and then,

$$\begin{aligned}\theta_1\xi_t + \theta_2\eta_t &\leq (\theta_1\xi_0 + \theta_2\eta_0)\exp\{-pt\} + \frac{\theta(\theta_1+\theta_2)\exp\{p(k-t)\}}{\varepsilon}\ln k \\ &\quad + \frac{1}{p}\left[K^* + \frac{\varepsilon(\theta_1\sigma_1^2 + \theta_2\sigma_2^2)}{2}\right](1-\exp\{-pt\}) \\ &\leq (\theta_1\xi_0 + \theta_2\eta_0) + \frac{\theta(\theta_1+\theta_2)\exp\{p(k-t)\}}{\varepsilon}\ln k \\ &\quad + \frac{1}{p}\left[K^* + \frac{\varepsilon(\theta_1\sigma_1^2 + \theta_2\sigma_2^2)}{2}\right]. \end{aligned} \tag{4.19}$$

For any $\omega \in \Omega_1 \cap \Omega_2$, $k \geq k_0(\omega)$ and $t \in [k-1,k]$, from (4.19) we have

$$\begin{aligned}\frac{\theta_1\xi_t + \theta_2\eta_t}{\ln t} \leq \frac{1}{\ln(k-1)}\Big[&(\theta_1\xi_0 + \theta_2\eta_0) + \frac{\theta(\theta_1+\theta_2)\exp\{p\}}{\varepsilon}\ln k \\ &+ \frac{1}{p}\Big\{K^* + \frac{\varepsilon(\theta_1\sigma_1^2 + \theta_2\sigma_2^2)}{2}\Big\}\Big],\end{aligned}$$

which implies

$$\limsup_{t\to\infty}\frac{\theta_1\xi_t + \theta_2\eta_t}{\ln t} \leq \frac{\theta(\theta_1+\theta_2)\exp\{p\}}{\varepsilon}$$

for every $\omega \in \Omega_1 \cap \Omega_2$. Letting $\varepsilon \to 1, \theta \to 1, p \to 0$, and noting that $\mathbb{P}(\Omega_1 \cap \Omega_2) = 1$, we obtain

$$\limsup_{t\to\infty}\frac{\theta_1\xi_t + \theta_2\eta_t}{\ln t} \leq \theta_1 + \theta_2, \qquad \text{a.s.,}$$

i.e.,

$$\limsup_{t\to\infty}\frac{\ln\left[x_t^{\theta_1}y_t^{\theta_2}\right]}{\ln t} \leq \theta_1 + \theta_2, \qquad \text{a.s.}$$

□

5. Conclusion

Our aim in this paper is to discuss the asymptotic properties of the stochastic predator - prey model in populations dynamics. We show the nice property that the solution of the stochastic model will remain in the positive cone with probability one. Making use of this property we have designed various types of Lyapunov functions to discuss the asymptotic behaviour in some detail. Several moments and pathwise asymptotic estimations are obtained and they can be used to better reveal features of the stochastic predator - prey model. It is also shown that a relatively large stochastic perturbation can cause the extinction of the population. This conclusion warns us to have a timely decision to protect species in our ecosystem.

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