Communications on Stochastic Analysis

Volume 5 | Number 2 Article 5

6-1-2011

An extension of bifractional Brownian motion

Xavier Bardina

Khalifa Es-Sebaiy

Follow this and additional works at: https://repository.lsu.edu/cosa

Part of the Analysis Commons, and the Other Mathematics Commons

Recommended Citation

Bardina, Xavier and Es-Sebaiy, Khalifa (2011) "An extension of bifractional Brownian motion," *Communications on Stochastic Analysis*: Vol. 5: No. 2, Article 5.
DOI: 10.31390/cosa.5.2.05
Available at: https://repository.lsu.edu/cosa/vol5/iss2/5

Communications on Stochastic Analysis
Vol. 5, No. 2 (2011) 333-340

Serials Publications
www.serialspublications.com

AN EXTENSION OF BIFRACTIONAL BROWNIAN MOTION

XAVIER BARDINA* AND KHALIFA ES-SEBAIY

Abstract. In this paper we introduce and study a self-similar Gaussian process that is the bifractional Brownian motion $B^{H,K}$ with parameters $H \in (0,1)$ and $K \in (1,2)$ such that $HK \in (0,1)$. A remarkable difference between the case $K \in (0,1)$ and our situation is that this process is a semimartingale when $2HK = 1$.

1. Introduction

Houdré and Villa in [7] gave the first introduction to the bifractional Brownian motion (bifBm) $B^{H,K} = \left(B_t^{H,K}; t \geq 0\right)$ with parameters $H \in (0,1)$ and $K \in (0,1]$ which is defined as a centered Gaussian process, with covariance function

$$R^{H,K}(t,s) = E\left(B_t^{H,K} B_s^{H,K}\right) = \frac{1}{2^K}\left(\left(t^{2H} + s^{2H}\right)^K - |t-s|^{2HK}\right),$$

for every $s,\ t \geq 0$. The case $K = 1$ corresponds to the fractional Brownian motion (fBm) with Hurst parameter H. Some properties of the bifractional Brownian motion have been studied by Russo and Tudor in [12]. In fact, in [12] it is shown that the bifractional Brownian motion behaves as a fractional Brownian motion with Hurst parameter HK. A stochastic calculus with respect to this process has been recently developed by Kruk, Russo and Tudor [9] and Es-Sebaiy and Tudor [6].

In this paper we prove that, with $H \in (0,1)$ and $HK \in (0,1)$, the process $B^{H,K}$ can be extended for $1 < K < 2$. The case $H = \frac{1}{2}$ and $1 < K < 2$ plays a role to give an extension of sub-fractional Brownian motion (subfBm) (see [4]). The subfBm $(\xi_t^h, t \geq 0)$ with parameter $0 < h \leq 2$ is a centered Gaussian process with covariance:

$$E\left(\xi_t^h \xi_s^h\right) = C_h\left(t^{2h} + s^{2h} - \frac{1}{2}\left((t+s)^{2h} + |t-s|^{2h}\right)\right); \quad s,\ t \geq 0,$$

where $C_h = 1$ if $0 < h < 1$ and $C_h = 2(1-h)$ if $1 < h \leq 2$.

Received 2010-10-6; Communicated by L. G. Gorostiza.
2000 *Mathematics Subject Classification.* Primary 60G15; Secondary 60G18.
Key words and phrases. Bifractional Brownian motion, self-similar processes, long-range dependence.
* This author is supported by the grant MTM2009-08869 from the Ministerio de Ciencia e Innovación.

2. Definition of Bifractional Brownian Motion

For any $K \in (0,2)$, let $X^K = (X_t^K, t \geq 0)$ be a Gaussian process defined by

$$X_t^K = \int_0^\infty (1 - e^{-rt}) r^{-\frac{1+K}{2}} dW_r, \quad t \geq 0, \tag{2.1}$$

where $(W_t, t \geq 0)$ is a standard Brownian motion.

This process was introduced in [10] for $K \in (0,1)$ in order to obtain a decomposition of the bifractional Brownian motion with $H \in (0,1)$ and $K \in (0,1)$. More precisely, they prove the following result:

Theorem 2.1 (see [10])**.** *Let $B^{H,K}$ a bifractional Brownian motion with parameters $H \in (0,1)$ and $K \in (0,1)$, B^{HK} be a fractional Brownian motion with Hurst parameter $HK \in (0,1)$ and $W = \{W_t, t \geq 0\}$ a standard Brownian motion. Let X^K be the process given by (2.1). If we suppose that $B^{H,K}$ and W are independents, then the processes $\{Y_t = C_1 X^K_{t^{2H}} + B_t^{H,K}, t \geq 0\}$ and $\{C_2 B_t^{HK}, t \geq 0\}$ have the same distribution, where $C_1 = \sqrt{\frac{2^{-K}K}{\Gamma(1-K)}}$ and $C_2 = 2^{\frac{1-K}{2}}$.*

The process defined in (2.1) has good properties. The following result is proved in [10] for the case $K \in (0,1)$ and extended to the case $K \in (1,2)$ in [2] and [11]:

Proposition 2.2 (see [2],[10] and [11])**.** *The process $X^K = \{X_t^K, t \geq 0\}$ is Gaussian, centered, and its covariance function is:*

$$\mathrm{Cov}(X_t^K, X_s^K) = \begin{cases} \frac{\Gamma(1-K)}{K}\left[t^K + s^K - (t+s)^K\right] & \text{if } K \in (0,1), \\ \frac{\Gamma(2-K)}{K(K-1)}\left[(t+s)^K - t^K - s^K\right] & \text{if } K \in (1,2). \end{cases} \tag{2.2}$$

Moreover, X^K has a version with trajectories which are infinitely differentiable on $(0,\infty)$ and absolutely continuous on $[0,\infty)$.

Using the fact that when $K \in (1,2)$, the covariance function of X^K is given by

$$Cov(X_t^K, X_s^K) = \frac{\Gamma(2-K)}{K(K-1)}\left((t+s)^K - t^K - s^K\right),$$

and considering also the process

$$X_t^{H,K} = X^K_{t^{2H}};\ t \geq 0, \tag{2.3}$$

we can prove the following result:

Theorem 2.3. *Assume $H \in (0,1)$ and $K \in (1,2)$ with $HK \in (0,1)$. Let B^{HK} be a fractional Brownian motion, and $W = \{W_t, t \geq 0\}$ a standard Brownian motion. Let $X^{K,H}$ the process defined in (2.3). If we suppose that B^{HK} and W are independents, then the processes*

$$B_t^{H,K} = aB_t^{HK} + bX_t^{H,K}, \tag{2.4}$$

where $a = \sqrt{2^{1-K}}$ and $b = \sqrt{\frac{K(K-1)}{2^K\Gamma(2-K)}}$ is a centered Gaussian process with covariance function

$$E\left(B_t^{H,K} B_s^{H,K}\right) = \frac{1}{2^K}\left(\left(t^{2H} + s^{2H}\right)^K - |t-s|^{2HK}\right); \quad s,\ t \geq 0.$$

Proof. It is obvious that the process defined in (2.4) is a centered Gaussian process. On the other hand, its covariance functions is given by

$$\begin{aligned} E\left(B_t^{H,K} B_s^{H,K}\right) &= a^2 E\left(B_t^{HK} B_s^{HK}\right) + b^2 E\left(X_t^{H,K} X_s^{H,K}\right) \\ &= \frac{1}{2^K}\left(t^{2HK} + s^{2HK} - |t-s|^{2HK}\right) \\ &\qquad + \frac{1}{2^K}\left(\left(t^{2H}+s^{2H}\right)^K - t^{2HK} - s^{2HK}\right) \\ &= \frac{1}{2^K}\left(\left(t^{2H}+s^{2H}\right)^K - |t-s|^{2HK}\right), \end{aligned}$$

which completes the proof. □

Thus the bifractional Brownian motion $B^{H,K}$ with parameters $H \in (0,1)$ and $K \in (1,2)$ such that $HK \in (0,1)$ is well defined and it has a decomposition as a sum of a fBm B^{HK} and an absolutely continuous process $X^{H,K}$.

Assume that $2HK = 1$. Russo and Tudor [12] proved that if K belong to $(0,1)$, the process $B^{H,K}$ is not a semimartingale. But in the case when $1 < K < 2$, $B^{H,K}$ is a semimartingale because we have a decomposition of this process as a sum of a Brownian motion $B^{\frac{1}{2}}$ and a finite variation process $X^{H,K}$.

The following decomposition is exploited to prove the quasi-helix property (in the sense of J.P. Kahane) of $B^{H,K}$. This result is satisfied for all $K \in (0,2)$.

Proposition 2.4. *Let $H \in (0,1)$ and $K \in (0,2)$ such that $HK \in (0,1)$. Let $(\xi_t^{K/2}, t \geq 0)$ be a sub-fractional Brownian motion with parameter $K/2 \in (0,1)$, independent to $B^{H,K}$ and suppose that $(B_t^{K/2}, t \geq 0)$ and $(B_t^{HK}, t \geq 0)$ are two independent fractional Brownian motions with Hurst parameter $K/2 \in (0,1)$ and $HK \in (0,1)$, respectively. We set $\xi_t^{K,H} = \xi_{t^{2H}}^{K/2}$ and $\widetilde{B}_t^{H,K} = B_{t^{2H}}^{K/2}$, $t \geq 0$. Then it holds that*

$$B^{H,K} + \sqrt{2^{1-K}}\xi^{K,H} \overset{(d)}{=} \sqrt{2^{1-K}}\left(\widetilde{B}^{H,K} + B^{HK}\right), \tag{2.5}$$

where $\overset{d}{=}$ denotes that both processes have the same distribution.

Proof. The result follows easily from the independence and the fact that their corresponding covariance functions satisfy the following equality for all $s,t \geq 0$

$$\begin{aligned} &R^{H,K}(t,s) \\ &= \frac{1}{2^K}\left((t^{2H}+s^{2H})^K - |t-s|^{2HK}\right) \\ &= 2^{1-K}\left[-Cov(\xi_t^{K,H}, \xi_s^{K,H}) + Cov(\widetilde{B}_t^{H,K}, \widetilde{B}_s^{H,K}) + Cov(B_t^{HK}, B_s^{HK})\right]. \end{aligned}$$

□

Proposition 2.5. *Let $H \in (0,1)$ and $K \in (1,2)$ be such that $HK \in (0,1)$. Then for $0 < H \leq 1/2$, we have*

$$2^{1-K}|t-s|^{2HK} \leq E\left(B_t^{H,K} - B_s^{H,K}\right)^2 \leq |t-s|^{2HK}, \quad t,s \geq 0,$$

and for $1/2 < H < 1$, *we have*

$$2^{1-K}|t-s|^{2HK} \leq E\left(B_t^{H,K} - B_s^{H,K}\right)^2 \leq 2^{2-K}|t-s|^{2HK}, \quad t,s \geq 0.$$

Proof. Using the proposition 2.4, we obtain

$$\begin{aligned} & E\left(B_t^{H,K} - B_s^{H,K}\right)^2 \\ = \quad & 2^{1-K}\left(-E\left(\xi_{t^{2H}}^{\frac{K}{2}} - \xi_{s^{2H}}^{\frac{K}{2}}\right)^2 + E\left(B_{t^{2H}}^{\frac{K}{2}} - B_{s^{2H}}^{\frac{K}{2}}\right)^2 + E\left(B_t^{HK} - B_s^{HK}\right)^2\right) \\ = \quad & 2^{1-K}\left(-E\left(\xi_{t^{2H}}^{\frac{K}{2}} - \xi_{s^{2H}}^{\frac{K}{2}}\right)^2 + \left|t^{2H} - s^{2H}\right|^K + |t-s|^{2HK}\right). \end{aligned}$$

On the other hand, from [3] we have

$$(2 - 2^{K-1})\left|t^{2H} - s^{2H}\right|^K \leq E\left(\xi_{t^{2H}}^{\frac{K}{2}} - \xi_{s^{2H}}^{\frac{K}{2}}\right)^2 \leq \left|t^{2H} - s^{2H}\right|^K.$$

Thus

$$\begin{aligned} 2^{1-K}|t-s|^{2HK} \quad & \leq \quad E\left(B_t^{H,K} - B_s^{H,K}\right)^2 \\ & \leq \quad 2^{1-K}\left(|t-s|^{2HK} + (2^{K-1}-1)\left|t^{2H} - s^{2H}\right|^K\right). \end{aligned}$$

Then we deduce that for every $H \in (0,1)$, $K \in (1,2)$ with $HK \in (0,1)$

$$2^{1-K}|t-s|^{2HK} \leq E\left(B_t^{H,K} - B_s^{H,K}\right)^2$$

and the other hand for every $H \in (0, \frac{1}{2}]$, $K \in (1,2)$ we have

$$\begin{aligned} E\left(B_t^{H,K} - B_s^{H,K}\right)^2 \quad & \leq \quad 2^{1-K}\left(|t-s|^{2HK} + (2^{K-1}-1)\left|t^{2H} - s^{2H}\right|^K\right) \\ & \leq \quad |t-s|^{2HK}. \end{aligned}$$

The last inequality is satisfied from the fact that $\left|t^{2H} - s^{2H}\right| \leq |t-s|^{2H}$ for $H \in (0, \frac{1}{2}]$.

To complete the proof, it remains to show that for every $H \in (\frac{1}{2}, 1)$, $K \in (1,2)$ with $HK \in (0,1)$ (observe that in this situation we have $HK \in (\frac{1}{2}, 1)$)

$$E\left(B_t^{H,K} - B_s^{H,K}\right)^2 \leq 2^{2-K}|t-s|^{2HK}.$$

Notice that

$$\begin{aligned} E\left(B_t^{H,K} - B_s^{H,K}\right)^2 \quad & = \quad \frac{1}{2^K}\left[(2t^{2H})^K + (2s^{2H})^K\right. \\ & \qquad \left. -2\left((t^{2H} + s^{2H})^K - |t-s|^{2HK}\right)\right] \\ & = \quad \frac{2}{2^K}|t-s|^{2HK} + \left(t^{2HK} + s^{2HK} - \frac{2}{2^K}(t^{2H} + s^{2H})^K\right). \end{aligned}$$

Hence it is enough to prove that

$$t^{2HK} + s^{2HK} - \frac{2}{2^K}(t^{2H} + s^{2H})^K \leq 2^{1-K}|t-s|^{2HK},$$

or equivalently

$$t^{2HK}+s^{2HK}\leq 2^{1-K}\left((t^{2H}+s^{2H})^K+|t-s|^{2HK}\right).$$

From now on we will assume, without loss of generality, that $s\leq t$. Dividing by t^{2HK} we obtain that we have to prove that

$$1+\left(\frac{s}{t}\right)^{2HK}\leq 2^{1-K}\left(\left(1+\left(\frac{s}{t}\right)^{2H}\right)^K+\left(1-\frac{s}{t}\right)^{2HK}\right).$$

Equivalently we have to prove that, for any $u\in(0,1]$, the function

$$f(u):=2^{1-K}\left[(1+u^{2H})^K+(1-u)^{2HK}\right]-u^{2HK}-1$$

is positive.

Observe that $f(1)=0$, so, it is enough to see that the derivative of this function is negative for $u\in(0,1]$. But,

$$f'(u)=2HK2^{1-K}u^{2HK-1}\left[\left(1+\frac{1}{u^{2H}}\right)^{K-1}-\left(\frac{1}{u}-1\right)^{2HK-1}-2^{K-1}\right].$$

To prove that $f'(u)\leq 0$ for $u\in(0,1]$ it is enough to see that the function

$$h(u):=\left(1+\frac{1}{u^{2H}}\right)^{K-1}-\left(\frac{1}{u}-1\right)^{2HK-1}-2^{K-1},$$

is negative for $u\in(0,1]$. But, since $h(1)=0$, it is enough to prove that its derivative $h'(u)\geq 0$ for $u\in(0,1]$. But,

$$h'(u)=\frac{1}{u^{2HK}}\left(-2(K-1)H(u^{2H}+1)^{K-2}u^{2H-1}+(1-u)^{2HK-2}(2HK-1)\right).$$

Observe that $u^{2H-1}\leq 1$ because $H\in(\frac{1}{2},1)$, $(u^{2H}+1)^{K-2}\leq 1$ and

$$(1-u)^{2HK-2}\geq 1.$$

So,

$$h'(u)\geq\frac{1}{u^{2HK}}\left(-2(K-1)H+2HK-1\right)=\frac{1}{u^{2HK}}(2H-1)\geq 0,$$

because $H\geq\frac{1}{2}$. The proof is now complete. □

Proposition 2.6. *Suppose that $H\in(0,1)$, $K\in(1,2)$, and $HK\in(0,1)$. The bifBm $B^{H,K}$ has the following properties:*

i) *$B^{H,K}$ is a self-similar process with index HK, i.e.*

$$\left(B^{H,K}_{at},t\geq 0\right)\overset{d}{=}\left(a^{HK}B^{H,K}_t,t\geq 0\right),\qquad \textit{for each } a>0.$$

ii) *$B^{H,K}$ has the same long-range property of the fBm B^{HK}, i.e. $B^{H,K}$ has the short-memory for $HK<\frac{1}{2}$ and it has long-memory for $HK>\frac{1}{2}$.*

iii) *$B^{H,K}$ has a $\frac{1}{HK}$-variation equals to $2^{\frac{1-K}{HK}}\lambda t$ with $\lambda=E(|N|^{\frac{1}{HK}})$ and N being a standard normal random variable, i.e.*

$$\sum_{j=1}^{n}\left(B^{H,K}_{t^n_j}-B^{H,K}_{t^n_{j-1}}\right)^{\frac{1}{HK}}\underset{n\to\infty}{\longrightarrow}2^{\frac{1-K}{HK}}\lambda t\quad \textit{in } L^1(\Omega).$$

where $0=t^n_0<\ldots<t^n_n=t$ denotes a partition of $[0,t]$.

iv) $B^{H,K}$ *is not a semimartingale if* $2HK \neq 1$.

The proof of the proposition 2.6 is straightforward from [6] and [12].

3. Space of Integrable Functions with Respect to Bifractional Brownian Motion

Let us consider $\mathcal{E}$ the set of simple functions on $[0,T]$. Generally, if $U := (U_t,\, t \in [0,T])$ is a continuous, centered Gaussian process, we denote by $\mathcal{H}_U$ the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle_{\mathcal{H}} = E\left(U_t U_s\right).$$

In the case of the standard Brownian motion W, the space $\mathcal{H}_W$ is $L^2([0,T])$. On the other hand, for the fractional Brownian motion B^H, the space $\mathcal{H}_{B^H}$ is the set of restrictions to the space of test functions $\mathcal{D}((0,T))$ of the distributions of $W^{\frac{1}{2}-H,2}(\mathbb{R})$ with support contained in $[0,T]$ (see [8]). In the case $H \in (0,\frac{1}{2})$ all the elements of the domain are functions, and the space $\mathcal{H}_{B^H}$ coincides with the fractional Sobolev space $I_{0^+}^{\frac{1}{2}-H}(L^2([0,T]))$ (see for instance [5]), but in the case $H \in (\frac{1}{2},1)$ this space contains distributions which are not given by any function.

As a direct consequence of Theorem 2.3 we have the following relation between $\mathcal{H}_{B^H}$, $\mathcal{H}_{B^{H,K}}$ and $\mathcal{H}_{X^{H,K}}$, where $B^{H,K}$ is the bifractional Brownian motion and $X^{H,K}$ is the process defined in (2.3).

Proposition 3.1. *Let* $H \in (0,1)$ *and* $K \in (1,2)$ *with* $HK \in (0,1)$. *Then it holds that*

$$\mathcal{H}_{X^{H,K}} \cap \mathcal{H}_{B^{HK}} = \mathcal{H}_{B^{H,K}}$$

If we consider the processes appearing in Proposition 2.4 we have also the following result:

Proposition 3.2. *Let* $H \in (0,1)$. *For every* $K \in (0,2)$ *with* $HK \in (0,1)$ *the following equality holds*

$$\mathcal{H}_{\xi^{H,K}} \cap \mathcal{H}_{B^{H,K}} = \mathcal{H}_{\widetilde{B}^{H,K}} \cap \mathcal{H}_{B^{HK}}.$$

Remark 3.3. For the case $K \in (0,1)$ we have the following equality (see [10])

$$\mathcal{H}_{B^{HK}} = \mathcal{H}_{X^{H,K}} \cap \mathcal{H}_{B^{H,K}}.$$

Proof. Both propositions are a direct consequence of the two decompositions into the sum of two independent processes proved in Theorem 2.3 and Proposition 2.4. $\square$

4. Weak Convergence Towards the Bifractional Brownian Motion

Another direct consequence of the decomposition for the bifractional Brownian motion with $H \in (0,1)$, $K \in (1,2)$ and $HK \in (0,1)$ is the following result of convergence in law in the space $\mathcal{C}([0,T])$.

Recall that the fractional Brownian motion of Hurst parameter $H \in (0,1)$ admits an integral representation of the form (see for instance [1])

$$B_t^H = \int_0^t K^H(t,s)\,dW_s,$$

where W is a standard Brownian motion and the kernel K^H is defined on the set $\{0 < s < t\}$ and given by

$$K^H(t,s) = d_H(t-s)^{H-\frac{1}{2}} + d_H(\frac{1}{2}-H)\int_s^t (u-s)^{H-\frac{3}{2}}\left(1-\left(\frac{s}{u}\right)^{\frac{1}{2}-H}\right)du, \quad (4.1)$$

with d_H the following normalizing constant

$$d_H = \left(\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)}\right)^{\frac{1}{2}}.$$

Theorem 4.1. *Let $H \in (0,1)$ and $K \in (1,2)$ with $HK \in (0,1)$. Consider $\theta \in (0,\pi)\cup(\pi,2\pi)$ such that if $HK \in (0,\frac{1}{4}]$ then θ satisfies that $\cos((2i+1)\theta) \neq 1$ for all $i \in \mathbb{N}$ such that $i \leq \frac{1}{4}\left[\frac{1}{H}\right]$. Set $a = \sqrt{2^{1-K}}$ and $b = \sqrt{\frac{K(K-1)}{2^K\Gamma(2-K)}}$. Define the processes,*

$$\begin{aligned} B_\epsilon^{HK} &= \left\{\frac{2}{\epsilon}\int_0^T K^{HK}(t,s)\sin\left(\theta N_{\frac{2s}{\epsilon^2}}\right)ds, \quad t\in[0,T]\right\}, \\ X_\epsilon^{H,K} &= \left\{\frac{2}{\epsilon}\int_0^\infty (1-e^{-st^{2H}})s^{-\frac{1+K}{2}}\cos\left(\theta N_{\frac{2s}{\epsilon^2}}\right)ds, \quad t\in[0,T]\right\}, \end{aligned}$$

where $K^{HK}(t,s)$ is the kernel defined in (4.1). Then

$$\{Y_\epsilon^H(t) = aB_\epsilon^{HK}(t) + bX_\epsilon^{H,K}(t), t\in[0,T]\}$$

weakly converges in $\mathcal{C}([0,T])$ to a bifractional Brownian motion.

Remark 4.2. Obviously we can also obtain the same result interchanging the roles of the sinus and the cosinus functions in the definition of the approximating processes.

Proof. Applying Theorems 3.2 and 3.5 of [2] we know that, respectively, the processes B_ϵ^{HK} and $X_\epsilon^{H,K}$ converge in law in $\mathcal{C}([0,T])$ towards a fBm B^{HK} and to the process $X^{H,K}$. Moreover, applying Theorem 2.1 of [2], we know that the limit laws are independent. Hence, we are under the hypothesis of the decomposition obtained in Theorem 2.3, which proves the stated result. □

Acknowledgment. The authors would like to thank the referees for valuable comments.

References

1. Alòs, E., Mazet, O. and Nualart, D.: Stochastic calculus with respect to Gaussian processes, *Ann. Probab.* **29** (2001) 766–801.
2. Bardina, X. and Bascompte, D.: Weak convergence towards two independent Gaussian processes from a unique Poisson process, *Collect. Math.* **61** (2010) 191–204.
3. Bojdecki, T., Gorostiza, L.G. and Talarczyk, A.: Sub-fractional Brownian motion and its relation to occupation times, *Stat. Prob. Lett.* **69** (2004) 405–419.
4. Bojdecki, T., Gorostiza, L.G. and Talarczyk, A.: Some Extensions of Fractional Brownian Motion and Sub-Fractional Brownian Motion Related to Particle Systems, *Electron. Comm. Probab.* **12** (2007) 161–172.
5. Decreusefond, L. and Üstünel, A.S.: Stochastic analysis of the fractional Brownian motion, *Potential Anal.* **10** (1999) 177–214.

6. Es-Sebaiy, K. and Tudor, C.: Multidimensional bifractional Brownian motion: Itô and Tanaka's formulas, *Stoch. Dyn.* **7** (2007) 365–388.
7. Houdré, C. and Villa, J.: An Example of Infinite Dimensional Quasi-Helix, *Contemp. Math. (Amer. Math. Soc.)* **336** (2003) 195-201.
8. Jolis, M.: On the Wiener integral with respect to the fractional Brownian motion on an interval, *J. Math. Anal. Appl.* **330** (2007) 1115–1127.
9. Kruk, I., Russo, F. and Tudor, C.A.: Wiener integrals, Malliavin calculus and covariance structure measure, *J. Funct. anal.* **249** (2007) 92–142.
10. Lei, P. and Nualart, D.: A decomposition of the bi-fractional Brownian motion and some applications, *Statist. Probab. Lett.* **79** (2009) 619–624.
11. Ruiz de Chávez, J. and Tudor, C.: A decomposition of sub-fractional Brownian motion, *Math. Reports* **61** (2009) 67–74.
12. Russo, F. and Tudor, C.: On the bifractional Brownian motion, *Stoch. Process. Appl.* **116** (2006) 830-856.

Xavier Bardina: Departament de Matemàtiques, Facultat de Ciències, Universitat Autònoma de Barcelona, 08193-Bellaterra, Barcelona, Spain

E-mail address: `bardina@mat.uab.cat`

URL: `http://mat.uab.cat/~bardina`

Khalifa Es-Sebaiy: SAMM, Centre d'Economie de La Sorbonne, Université Paris 1 Panthéon-Sorbonne, 90, rue de Tolbiac, 75634, Paris, France

E-mail address: `Khalifa.Es-Sebaiy@univ-paris1.fr`