Communications on Stochastic Analysis

Volume 5 | Number 2 Article 4

6-1-2011

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Communications on Stochastic Analysis
Vol. 5, No. 2 (2011) 309-331

Serials Publications
www.serialspublications.com

ASYMPTOTIC PROPERTIES OF STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS IN HILBERT SPACES DRIVEN BY NON-GAUSSIAN NOISE

V. MANDREKAR* AND LI WANG

Abstract. A class of stochastic evolution equations with additive noise (compensated Poisson random measures) in Hilbert spaces is considered. The existence and uniqueness of a mild solution to the stochastic equation with Lipschitz type coefficients has been studied. We first study the stability and exponential ultimate boundedness properties of the solution by using Lyapunov function technique. We then study the conditions for the existence of invariant measure associated to the solution. Finally, some examples are given to illustrate the theory.

1. Introduction

The study of stochastic partial differential equations driven by Lévy noise has been the subject of recent papers [1], [5], [6], [10], [14], [17], where Ito integral studied by [18] is used. This is done by embedding the PDE as an infinite-dimensional equation. All the works mentioned above study SPDE's as SDE's in infinite-dimensional case. In case these equations are driven by Brownian Motion, asymptotic properties are studied by using Lyapunov function methods originally in [2], [3] [9], [12], [13], and [15]. For the detailed exposition, see the recent work [11]. Motivated from this, the generalization of these works to infinite-dimensional SDE's was undertaken in [20]. The purpose of our work is to generalize and complete the work in [20]. This is done by systematically studying, stability in probability, moments and existence of invariant measure.

The paper is arranged as follows. Section 2 contains preliminaries followed by approximation result by strong solutions to mild solution in Section 3. This is needed to use Ito Formula. In Section 4, we study the stability of zero solution in probability. In Section 5, we study exponential ultimate boundedness. Both of these are studied for non-linear equations. Finally, in Section 6, we show that the ultimate boundedness gives us the existence of invariant measure. We end the paper by giving existence of invariant measures for some examples.

Received 2010-7-6; Communicated by the editors.
2000 *Mathematics Subject Classification.* Primary 60H15; Secondary 35R60.
Key words and phrases. SPDE, Feller, Markov, Ito formula, Lyapunov function, exponential stability in m.s.s., exponential ultimate boundedness in m.s.s.
* This research is supported by NSF DMS grant 0803360.

2. Preliminaries

Let A be an unbounded operator (with domain $\mathcal{D}(A)$) which generates a semigroup $\{S_t, t \geq 0\}$ continuous at zero. It is called a C_0-semigroup. It is known that $\|S_t\| \leq Me^{\alpha t}, \alpha \in \mathbb{R})$, $M > 0$ finite and real. In case $M = 1$ we call $\{S_t, t \geq 0\}$ a pseudocontraction semigroup. Let $(\mathcal{X}, \mathcal{S}, \mu)$ be a measure space and $\{N(A), A \in \mathcal{S}, \mu(A) < \infty\}$ be a Poisson random field with $EN(A) = \mu(A)$. We call $q(A) = N(A) - \mu(A)$, a compensated Poisson random measure and note that $Eq^2(A) = \mu(A)$. Let $(H\backslash\{0\}, \mathcal{B}(H\backslash\{0\}), \beta)$ be a σ-finite measurable space with $\mathcal{B}(H\backslash\{0\})$ denoting the Borel sets of $H\backslash\{0\}$ and β a measure on $\mathcal{B}(H\backslash\{0\})$ with

$$\int_{H\backslash\{0\}} (\|u\|_H^2 \wedge 1)\,\beta(du) < \infty.$$

We refer to this as a Lévy measure on $H\backslash\{0\}$.

Consider with $\mathbb{R}_+ = [0, \infty]$, $\mathcal{X} = (H\backslash\{0\}) \times \mathbb{R}_+$, $\mathcal{S} = \mathcal{B}(H\backslash\{0\}) \times \mathcal{B}(\mathbb{R}_+)$ and $\mu(A \times B) = \beta(A)\lambda(B)$ a product measure on $\mathcal{S}$. We shall denote in this case the compensated Poisson random measure q by $q(A \times B)$, $A \in \mathcal{B}(H\backslash\{0\})$ and $B \in \mathcal{B}(\mathbb{R}_+)$.

Let $\mathcal{H}^T = \{\varphi(x, t, \omega) : (H\backslash\{0\}) \times [0, T] \times \Omega \to H$, such that φ is jointly measurable w.r.t. $\mathcal{B}(H\backslash\{0\}) \times \mathcal{B}([0, T]) \times \mathcal{F}_T/\mathcal{B}(H)$ and for all x, $\varphi(x, t, \cdot)$ is $\mathcal{F}_t$ -measurable $\}$. Denote by $\mathcal{H}_2^T = \{\varphi \in \mathcal{H}^T,\ E\int_0^T \int_{H\backslash\{0\}} \|\varphi(x, t, \omega)\|_H^2\,\beta(dx)dt < \infty\}$. Stochastic integrals with respect to compensated Poisson random measure q of φ $in \mathcal{H}_2^T$ and their properties are given in [18]. We denote this integral by $\int_0^T \int_{H\backslash\{0\}} \varphi(x, t, \cdot)\,q(dxdt)$.

We shall be studying the asymptotic properties of the solutions of stochastic partial differential equation

$$d\,Z_t = AZ_t\,dt + a(Z_t)\,dt + \int_{H\backslash\{0\}} f(v, Z_t)\,q(dv\,dt),\ Z_0 = x \in H, \tag{2.1}$$

where H is a real separable Hilbert space, and i) A generates a pseudocontraction semigroup $\{S_t, t \geq 0\}$, (i.e., $\|S_t\| \leq e^{\alpha t}, \alpha \in \mathbb{R}$), ii) q is a compensated Poisson random measure (cPrm), iii) $a : H \to H$ and $f : H \times H \to H$ satisfying

(A1) a, f are continuous,

(A2) there exists a constant l, such that for all $x \in H$,

$$\|a(x)\|_H^2 + \int_{H\backslash\{0\}} \|f(v, x)\|_H^2\,\beta(dv) \leq l(1 + \|x\|_H^2),$$

(A3) for all $x, y \in H$, there exists a constant k, such that

$$\|a(x) - a(y)\|_H^2 + \int_{H\backslash\{0\}} \|f(v, x) - f(v, y)\|_H^2\,\beta(dv) \leq k\|x - y\|_H^2,$$

with $E(q(A \times B))^2 = \beta(A)\lambda(B)$, where $A \in \mathcal{B}(H\backslash\{0\})$, $B \in \mathcal{B}(\mathbb{R})$ and β is a Lévy measure and λ Lebesgue measure. We first clarify the meaning of "solution".

Definition 2.1. A stochastic process $\{Z_t, t \geq 0\}$ is called a *mild solution* of (2.1), if for all $t \leq T$,

(i) Z_t is $\mathcal{F}_t$-adapted (where $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P})$ is a probability space on which cPrm is defined),

(ii) $\{Z_t, t \geq 0\}$ is jointly measurable and $\int_0^T E\|Z_t\|_H^2 \, dt < \infty$,

(iii) $Z_t = S_t x + \int_0^t S_{t-s} a(Z_s) ds + \int_0^t \int_{H \setminus \{0\}} S_{t-s} f(v, Z_s) q(dv, ds)$ with probability one.

Definition 2.2. A stochastic process $\{Z_t, t \geq 0\}$ is a *strong solution* of (2.1), if for all $t \leq T$,

(i) Z_t is $\mathcal{F}_t$ adapted,

(ii) Z_t is cádlàg in t with probability one,

(iii) $Z_t \in \mathcal{D}(A)$ a.e. $\mathbb{T} \times \Omega$ and $\int_0^T \|AZ_t\|_H \, dt < \infty$ a.e.,

(iv) $Z_t = x + \int_0^t AZ_s ds + \int_0^t a(Z_s) ds + \int_0^t \int_{H \setminus \{0\}} f(v, Z_s) q(dv, ds)$ a.e..

For the definition and properties of Ito integral with respect to cPrm, see [18]. The existence and uniqueness of mild solution of (2.1) has been proved in various papers [1], [6], [10], [20] under (A1), (A2), (A3). For the reference of the reader, we state the known facts which will be used later. The proofs are in the above works.

Lemma 2.3. *Let q be cPrm as above and $\varphi \in \mathcal{H}_2^T$, then with $\{S_t, t \geq 0\}$ a pseudo-contraction semigroup, and τ a stopping time*

$$E \sup_{0 \leq t \leq T \wedge \tau} \left\| \int_0^t \int_{H \setminus \{0\}} S_{t-s} \varphi(v, s) q(dv, ds) \right\|_H^2 \leq b_1 E \int_0^{T \wedge \tau} \int_{H \setminus \{0\}} \|\varphi(v, s)\|_H^2 \, \beta(dv) ds,$$

where b_1 depends only on α, T. Here $\mathcal{H}_2^T := \{ \varphi(x, t, \omega) : (H \setminus \{0\}) \times [0, T] \times \Omega \to H$, such that φ is $\mathcal{F}_T / \mathcal{B}(H)$-measurable and $\varphi(x, t, \omega)$ is $\mathcal{F}_t$ -measurable for all x with $E \int_0^T \int_{H \setminus \{0\}} \|\varphi(x, t, \omega)\|_H^2 \, \beta(dx) dt < \infty \}$. Let $I(t, \xi(t)) = \int_0^t S_{t-s} a(\xi(s)) \, ds + \int_0^t \int_{H \setminus \{0\}} S_{t-s} f(v, \xi(s)) \, q(dv \, ds)$.

Lemma 2.4. *With the above notation, if a and f satisfy (A1), (A2), then for any stopping time τ*

$$E \left(\sup_{0 \leq s \leq t \wedge \tau} \|I(s, \xi(s))\|_H^2 \right) \leq b_2 \left(t + \int_0^t E \sup_{0 \leq u \leq s \wedge \tau} \|\xi(u)\|_H^2 \, ds \right),$$

where b_2 depends on α, T and l.

Lemma 2.5. *Let conditions (A1) and (A3) be satisfied and $\{S_t, t \geq 0\}$ be a pseudo-contraction semigroup. Then*

$$E \sup_{0 \leq s \leq t} \|I(s, \xi_1(s)) - I(s, \xi_2(s))\|^2 \leq b_3 \int_0^t E \sup_{0 \leq u \leq s} \|\xi_1(u) - \xi_2(u)\|^2 ds$$

with b_3 depending on α, T and k.

From these results, one gets

Theorem 2.6. *Let coefficients a, f satisfy $(A1), (A2)$ and $(A3)$, and $\{S_t, t \geq 0\}$ be a pseudo-contraction semigroup generated by A. Then equation (2.1) has unique mild solution $Z_t \in (\mathcal{D}[0,T], H)$ satisfying $E \sup_{0\leq s\leq T} \|Z_s\|_H^2 < \infty$ and is continuous with respect to x.*

Let $\psi \in C_b^{2,loc}(H)$, the space of twice continuously Fréchet differentiable functions which are locally bounded with locally bounded derivatives, and $y \in \mathcal{D}(A)$, define

$$\begin{aligned} (\mathcal{L}\psi)(y) &= (\psi'(y), Ay + a(y)) \\ &\quad + \int_{H\setminus\{0\}} [\psi(y + f(x,y)) - \psi(y) - \psi'(y, f(x,y))]\, \beta(dx). \end{aligned}$$

Then it is known [1] that the solution $\{Z_t^x, t \geq 0\}$ is homogeneous Markov process with infinitesimal generator (i.g.) $\mathcal{L}$ and is continuous in x. We shall need the Ito formula for $G(t, Z_t)$ where $G : [0,T] \times H \to \mathbf{R}$, $\frac{\partial}{\partial t} G(t,x) = \partial_t G$ exists and is continuous and $G(t, \cdot) \in C_b^{2,loc}(H)$,

$$\begin{aligned} G(t, Z_t) - G(0, x) &= \int_0^t [\partial_s G(s, Z_s) + \mathcal{L}G(s, Z_s)]\, ds \\ &\quad + \int_0^t \int_{H\setminus\{0\}}^0 [G(s, Z_s + f(x, Z_s)) - G(s, Z_s)]\, q(dx, ds). \end{aligned}$$

3. Approximation

We shall begin by first considering conditions sufficient for the mild solution of of (2.1), which exists under (A1), (A2), (A3) to be a strong solution. We use ideas of Ichikawa [7] who studied the problem for Brownian motion case. We need for this the following Fubini type theorem.

Theorem 3.1. *Let T be finite and let $B : [0,T] \times [0,T] \times H \times \Omega \to H$ be measurable, and $B(s,t,v)$ is $\mathcal{F}_t$-measurable for each s, and*

$$\int_0^T \int_0^T \int_{H\setminus\{0\}} E\|B(s,t,v)\|_H^2 \beta(dv) dt ds < \infty.$$

Then

$$\int_0^T \int_0^T \int_{H\setminus\{0\}} B(s,t,v) q(dv,dt) ds = \int_0^T \int_{H\setminus\{0\}} \int_0^T B(s,t,v) ds q(dv,dt).$$

Proof. (Sketch) Approximating B by B_n of the form [14], one has

$$B_n(s,t,x,\omega) = \sum_{j=1}^{p-1} \sum_{k=1}^{n-1} \sum_{l=1}^{m} 1_{A_{jkl}}(x) 1_{F_{jkl}}(\omega) 1_{(t_{jk},\, t_{jk+1}]}(t) 1_{(s_j,\, s_{j+1}]}(s) a_{jkl},$$

where $A_{jkl} \in \mathcal{B}(H\setminus\{0\})$, $(0 \notin \overline{A}_{jkl}), t_{jk} \in (0,T], t_{jk} < t_{jk+1,} s_j \in (0,T], s_j < s_{j+1,} F_{jkl} \in \mathcal{F}_{t_{jk}}, a_{jkl} \in H$. One can easily verify the conclusion for B_n for all n.

Furthermore, using inequality

$$E\left\|\int_0^T\int_{H\backslash\{0\}}\int_0^T B(s,t,v)ds\,q(dv,dt)\right\|_H^2$$
$$\leq T\int_0^T\int_0^T\int_{H\backslash\{0\}} E\|B(s,t,v)\|_H^2 ds\beta(dv)dt,$$

we get the desired result by taking limit as $n\to\infty$ using the definition of Ito Integral and Lebesgue DCT (dominate convergence theorem). □

Theorem 3.2. *Suppose*

(a) $x\in\mathcal{D}(A)$, $S_{t-r}a(y)\in\mathcal{D}(A), S_{t-r}f(v,y)\in\mathcal{D}(A)$, *for* $r<t$, $y\in H$ *and* $v\in H\backslash\{0\}$,
(b) $\|AS_{t-r}a(y)\|_H\leq g_1(t-r)(1+\|y\|_H), g_1\in L_1(0,T)$,
(c) $\int_{H\backslash\{0\}}\|AS_{t-r}f(v,y)\|^2\beta(dv)\leq g_2(t-r)(1+\|y\|_H^2), g_2\in L_1(0,T)$.

Then any mild solution of equation (2.1) (if it exists) is a strong solution.

Proof. By the above conditions, we have $\int_0^T\int_0^t\|AS_{t-r}a(Z_r)\|drdt<\infty$ with probability one and

$$\int_0^T\int_0^t\int_{H\backslash\{0\}} E\|AS_{t-r}f(v,Z_r)\|^2\beta(dv)drdt<\infty.$$

Thus by Fubini theorem and integration by parts,

$$\begin{aligned}\int_0^t\int_0^s AS_{s-r}a(Z_r)drds &= \int_0^t\int_r^t AS_{s-r}a(Z_r)dsdr\\ &= \int_0^t S_{t-r}a(Z_r)dr-\int_0^t a(Z_r)dr.\end{aligned}$$

Similarly using Theorem 3.1,

$$\begin{aligned}&\int_0^t\int_0^s\int_{H\backslash\{0\}} AS_{s-r}f(v,Z_r)q(dv,dr)ds\\ &= \int_0^t\int_{H\backslash\{0\}} S_{t-r}f(v,Z_r)q(dv,dr)-\int_0^t\int_{H\backslash\{0\}} f(v,Z_r)q(dv,dr).\end{aligned}$$

Hence AZ_t is integrable with probability one and

$$\begin{aligned}\int_0^t AZ_s ds &= S_tx-x+\int_0^t S_{t-r}a(Z_r)dr-\int_0^t a(Z_r)dr\\ &\quad+\int_0^t\int_{H\backslash\{0\}} S_{t-r}f(v,Z_r)dr-\int_0^t\int_{H\backslash\{0\}} f(v,Z_r)q(dv,dr)\\ &= Z_t-x-\int_0^t a(Z_r)dr-\int_0^t\int_{H\backslash\{0\}} f(v,Z_r)q(dv,dr).\end{aligned}$$

Thus by definition 2.2, $\{Z_t,t\geq 0\}$ is a strong solution of equation (2.1). □

Consider the approximating system

$$dZ_t = AZ_t\,dt + R(n)a(Z_t)\,dt + \int_{H\setminus\{0\}} R(n)f(v,Z_t)a(dv\,dt),\ Z_0 = x \in \mathcal{D}(A), \quad (3.1)$$

where i) $R(n) = nR(n,A)$ with $R(n,A) = (nI-A)^{-1}$ the resolvent of A evaluated at n, ii)a, f satisfy (A1)-(A3) and (iii) A generates a pseudo-contraction semigroup. Then we have that solution of (3.1) exists and is unique. In addition, by Theorem 3.2, the solution $\{Z_t^n, t \geq 0\}$ is the strong solution of (3.1). Also by Theorem 3.1, it lies in $C\left([0,T]; L_2\left(\Omega, \mathcal{F}, \mathcal{P}, H\right)\right)$ for T finite.

Theorem 3.3. *The stochastic differential equation (3.1) has a unique strong solution* $\{Z_t^{n,x}, t \geq 0\}$ *in* $C(0,T; L_2(\Omega, \mathcal{F}, P, H))$ *(*T *finite) and*

$$E \sup_{s \leq T} \|Z_t^{n,x} - Z_t^x\|^2 \to 0,$$

as $n \to \infty$*, where* $\{Z_t^x, t \geq 0\}$ *is the solution of (2.1).*

Proof. Consider

$$\begin{aligned}
& Z_t^x - Z_t^{n,x} \\
= & \left[\int_0^t S_{t-r}R(n)\left[a\left(Z_r^x\right) - a\left(Z_r^{n,x}\right)\right]\right] \\
& + \left[\int_0^t \int_{H\setminus\{0\}} S_{t-r}R(n)\left[f(v,Z_r^x) - f(v,Z_r^{n,x})\right]q(dv,dr)\right] \\
& + \left[\int_0^t S_{t-r}\left(I - R(n)\right)a\left(Z_r^x\right)dr + \right. \\
& \left. \int_0^t \int_{H\setminus\{0\}} S_{t-r}\left(I - R(n)\right)f(v,Z_r^x)q(dv,dr)\right] \\
= & I_1(t) + I_2(t) + I_3(t).
\end{aligned}$$

We get, with obvious notation denoting the square of three bracketing quantities,

$$E \sup_{0\leq s\leq t} \|Z_s^x - Z_s^{n,x}\|_H^2 \leq 3E \sup_{0\leq s\leq t}[I_1\left(s\right) + I_2\left(s\right) + I_3\left(s\right)].$$

Now note that, using (A3)

$$\begin{aligned}
E \sup_{0\leq s\leq t} I_1\left(s\right) &= E \sup_{0\leq s\leq t}\left\|\int_0^s S_{s-r}R(n)\left[a(Z_r^x) - a(Z_r^{n,x})\right]dr\right\|_H^2 \\
&\leq 4ke^{2\alpha t}t\int_0^t E \sup_{0\leq r\leq s}\|Z_r^x - Z_r^{n,x}\|_H^2\,ds.
\end{aligned}$$

Using arguments as in Lemma's 2.3-2.5, one has

$$\begin{aligned} & E \sup_{0\le s\le t} I_2(s) \\ = \ & E \sup_{0\le s\le t} \left\| \int_0^s \int_{H\setminus\{0\}} S_{s-r} R(n) \left[f(v, Z_r^x) - f(v, Z_r^{n,x}) \right] q(dv, dr) \right\|_H^2 \\ \le \ & 4kb_1 \int_0^t E \sup_{0\le r\le s} \|Z_r^x - Z_r^{n,x}\|_H^2 \, ds. \end{aligned}$$

Consider now $I_3(s)$, note that

$$\begin{aligned} & E \sup_{0\le s\le t} I_3(s) \\ = \ & E \sup_{0\le s\le t} \left\| \int_0^s S_{s-r} \left[I - R(n) \right] a(Z_r^x) \, dr \right. \\ & \left. + \int_0^s \int_{H\setminus\{0\}} S_{s-r} \left[I - R(n) \right] f(v, Z_r^x) q(dv, dr) \right\|_H^2 \\ \le \ & 2E \sup_{0\le s\le t} (I_{31}(s) + I_{32}(s)), \end{aligned}$$

Therefore, by (A2),

$$\begin{aligned} E \sup_{0\le s\le t} I_{31}(s) &= E \sup_{0\le s\le t} \| \int_0^s S_{s-r}[I - R(n)] a(Z_r^x) \, dr \|_H^2 \\ &\le |R(n) - I|^2 e^{2\alpha t} t l \left(t + \int_0^t E\|Z_r^x\|_H^2 dr \right). \end{aligned}$$

Similarly as in Lemma 2.4,

$$E \sup_{0\le s\le t} I_{32}(s) \le \|R(n) - I\|^2 \left[b_1 l \left(t + \int_0^t E \|Z_r\|^2 \, dr \right) \right].$$

We get, for some constant c_4, c_5 depending on T, α, b_1, k, a,

$$E \sup_{0\le s\le t} \|Z_s^x - Z_s^{n,x}\|_H^2 \le c_4 \int_0^t E \sup_{0\le r\le s} \|Z_r^x - Z_r^{n,x}\|_H^2 dr + c_5 |R(n) - I|^2.$$

By Gronwall's Lemma,

$$E \sup_{0\le s\le t} \|Z_s^x - Z_s^{n,x}\|_H^2 \le \|R(n) - I\|^2 c_5 e^{c_4 t} \to 0, \forall t \le T, \text{ as } n \to \infty.$$

□

We call $\{Z_t^{n,x}\}$ in Theorem 3.3 the Yosida approximation of the mild solution of (2.1).

We now study the asymptotic properties of the solution of (2.1) as $t \to \infty$ using Lyapunov function as in [9] for Brownian motion noise.

4. Exponential Stability in m.s.s.

Definition 4.1. We say that mild solution $\{Z_t^x, t \geq 0\}$ of (2.1) under $(A1)-(A3)$ is *exponentially stable in mean square sense (for short, m.s.s.)* if there exist positive finite real constants c, θ, such that for all $t > 0$ and $x \in H$, $E\|Z_t^x\|_H^2 \leq ce^{-\theta t}\|x\|_H^2$.

Let us recall $\mathcal{L}$ is infinitesimal generater (i.g.) of the solution process $\{Z_t^x, t \geq 0\}$.

We shall need condition (B) for a function $\psi \in C_b^{2,loc}(H)$.

(B) $\|\psi'(x)\|_H \leq c_4\|x\|_H$ and $\|\psi''(x)\|_{L(H,H)} \leq c_5, c_4, c_5 > 0,$ finite.

Theorem 4.2. *Suppose there is a function $\psi \in C_b^{2,loc}(H)$ satisfying condition (B) and positive finite constants $c_i, i = 1, 2, 3$ such that*

$$c_1\|x\|_H^2 \leq \psi(x) \leq c_3\|x\|_H^2, \text{ for all } x \in H \tag{4.1}$$

$$\mathcal{L}\psi(x) \leq -c_2\psi(x), \text{ for } x \in \mathcal{D}(A). \tag{4.2}$$

Then the mild solution of (2.1) is exponentially stable in m.s.s..

Proof. Let $\{Z_t^x, t \geq 0\}$ be the mild solution of (2.1) and $\{Z_t^{n,x}, t \geq 0\}$ be the approximation given by the strong solution of (3.1). Denote by

$$\begin{aligned} \mathcal{L}_n\psi(x) &= \langle\psi'(x), Ax + R(n)a(x)\rangle_H \\ &\quad + \int_{H\backslash\{0\}} [\psi(x + R(n)f(v,x)) - \psi(x) - \langle\psi'(x), f(v,x)\rangle_H]\, \beta(dv). \end{aligned}$$

Using Ito Formula with $G(t,x) = e^{c_2 t}\psi(x)$ and taking expectation, we get

$$e^{c_2 t}E\psi(Z_t^{n,x}) - \psi(x) = E\int_0^t e^{c_2 s}(c_2 + \mathcal{L}_n)\psi(Z_s^{n,x})\, ds.$$

By (4.2), $c_2\psi(x) + \mathcal{L}_n\psi(x) \leq -\mathcal{L}\psi(x) + \mathcal{L}_n\psi(x)$. Using definition of $\mathcal{L}_n$ and $\mathcal{L}$, one has,

$$\begin{aligned} &-\mathcal{L}\psi(x) + \mathcal{L}_n\psi(x) = \langle\psi'(x), (R(n) - I)a(x)\rangle \\ &+ \int_{H\backslash\{0\}} \left[\begin{array}{c} [\psi(x + R(n)f(v,x)) - \psi(x) - \langle\psi'(x), R(n)f(v,x)\rangle] \\ -[\psi(x + f(v,x)) - \psi(x) - \langle\psi'(x), f(v,x)\rangle] \end{array} \right] \beta(dv). \end{aligned}$$

Hence, one has,

$$e^{c_2 t}E\psi(Z_t^{n,x}) - \psi(x) \leq E\int_0^t e^{c_2 s}(I_1(Z_s^{n,x}) + I_2(Z_s^{n,x}))\, ds, \tag{4.3}$$

where

$$\begin{aligned} I_1(h) &= \langle\psi'(h), (R(n) - I)a(h)\rangle, \\ I_2(h) &= \int_{H\backslash\{0\}} \left[\begin{array}{c} [\psi(h + R(n)f(v,h)) - \psi(h + f(v,h)) \\ +[\langle\psi'(h), (I - R(n))f(v,h)\rangle] \end{array} \right] \beta(dv). \end{aligned}$$

Using Theorem 3.3, condition (B) and Lebesgue DCT, one has $e^{c_2 t}E\psi(Z_t^x) \leq \psi(x)$ for all $x \in \mathcal{D}(A)$. Hence by (4.1), one has $E\|Z_t^x\|_H^2 \leq ce^{-\theta t}\|x\|_H^2$ for some $c, \theta > 0$, and $x \in \mathcal{D}(A)$. Since $\{Z_t^x, t \geq 0\}$ is continuous in x, we get the above inequality for $x \in H$. $\square$

The function ψ above satisfying (4.1) and (4.2) is called Lyapunov function. Now we want to show that in linear case $a(x) = 0$ and $f(v, x) = f_0(v)x$, for which clearly 0 is a solution, we can construct a Lyapunov function. We consider solution of the equation

$$\begin{cases} dZ_t = AZ_t\,dt + \int_{H\backslash\{0\}} f_0(v)Z_t q(dt, dv) \\ Z_0 = x. \end{cases}, \tag{4.4}$$

where $f_0(v) : H\backslash\{0\} \to \mathbb{R}$, and $\int_{H\backslash\{0\}} |f_0(v)|^2\beta(dv) < \infty$. Hence

$$\int_{H\backslash\{0\}} \|f_0(v)y\|_H^2\,\beta(dv) \le d\,\|y\|_H^2 \text{ for } y \in H.$$

Note that (A1)-(A3) are satisfied, so (4.4) has a unique solution. Let us consider Yosida approximation [1],

$$dZ_t = A_n Z_t dt + \int_{H\backslash\{0\}} f_0(v)Z_t q(dt, dv), \tag{4.5}$$

where A_n is Yosida approximation of A. Let us denote by $\{Z_t^{n,x}, t \ge 0\}$ the strong solution of (4.5). Denote by

$$\mathcal{L}_0\psi(x) = \langle\psi'(x), Ax\rangle + \int_{H\backslash\{0\}} [\psi(x + f_0(v)x) - \psi(x) - \langle\psi'(x), f_0(v)x\rangle]\,\beta(dv).$$

Let $t \in \mathbb{R}^+$ and $n \in \mathbb{N}$ arbitrary. Then $(x, y) \to \int_0^t E\left[(Z_s^{n,x}, Z_s^{n,y})_H\right] ds$ defines a bounded bilinear form on $H \times H$, by the linearity of $\{Z_t^{n,x}\}$ in x. Hence there exists a unique $T_n(t) \in \mathcal{L}(H)$ such that, $(T_n(t)x, y) = \int_0^t E\,(Z_s^{n,x}, Z_s^{n,y})\,ds$. Set $\psi_n(t)(x) = (T_n(t)x, x) = \int_0^t E\,\|Z_s^{n,x}\|^2\,ds$. Let $\{P_t\}_{t\ge0}$ be the Markov semigroup of Z, $(P_t\phi)(x) = E\phi(Z_t^x)$, $x \in H$. Note that $E\left(\phi(Z_t^x)|\mathcal{F}_s^{Z^x}\right) = (P_{t-s}\phi)(Z_s^x)$. Set $\phi(h) = \|h\|_H^2$. Using the Markov property, we have

$$\begin{aligned} E\left[\psi_n(t)(Z_s^{n,x})\right] &= E\left[\int_0^t E\left(\phi(Z_t^y)|y = Z_s^{n,x}\right)\right] \\ &= E\left[\int_0^t (P_u\phi)(Z_s^{n,x})du\right] = \int_0^t E\left[\phi(Z_{u+s}^{n,x})|\mathcal{F}^{Z_s^{n,x}}\right] du \\ &= \int_0^t E\left\|Z_{u+s)}^{n,x}\right\|^2 du = \psi_n(t+s)(x) - \psi_n(s)(x). \end{aligned} \tag{4.6}$$

Note that $\psi_n(t) \in C_n^{2,loc}(H)$. By Ito Formula,

$$E\left[\psi_n(t)(Z_s^{n,x})\right] = \psi_n(t)(x) + \int_0^t E\left[(\mathcal{L}\psi_n(t))(Z_s^{n,x})\right] ds. \tag{4.7}$$

Now

$$\begin{aligned} \mathcal{L}\psi_n(t)(x) &= 2\,(T_n(t)x, A_n x)_H \\ &\quad + \int_{H\backslash\{0\}} (T_n(t)(x + f_0(v)x), (x + f_0(v)x))_H \\ &\quad - (T_n(t)x, x) - 2\,(T_n(t)x, f_0(x))\,\beta(dt). \end{aligned}$$

From (4.6) and (4.7), one has

$$\psi_n(t+s)(x) - \psi_n(s)(x) = \int_0^s E\left[(\mathcal{L}_n\psi_n)(t)(Z_u^{n,x})\right] du + \psi_n(t)(x).$$

Since $\lim_{s\to 0} \frac{\psi_n(s)(x)}{s} = \|x\|^2$, we get

$$\frac{d}{dt}\psi_n(t)(x) = \mathcal{L}_n\psi_n(t)(x) + \|x\|^2 . \tag{4.8}$$

Introduce $\psi(t)(x) = \int_0^t E\left\|Z_u^x\right\|^2 du$, then

$$\frac{d}{dt}\psi(t)(x) = E\left\|Z_t^x\right\|^2 = \lim_{n\to\infty} E\left\|Z_t^{n,x}\right\|^2 = \lim_{n\to\infty} \frac{d}{dt}\psi_n(t)(x).$$

For $x \in \mathcal{D}(\mathcal{L}_0)$, $\lim_{n\to\infty} \mathcal{L}_n\left(T_n(t)x, x\right) = \mathcal{L}_0\left(T(t)x, x\right)$, where $\psi(t)(x) = (T(t)x, x)$ similar to $\psi_n(t)(x)$. So $\lim_{n\to\infty} \mathcal{L}_n\psi_n(x) = \mathcal{L}_0\psi(t)(x)$. From (4.8), $\frac{d}{dt}\psi(t)(x) = \mathcal{L}_0\psi(t)(x) + \|x\|^2$ for $x \in \mathcal{D}(\mathcal{A})$. By exponential stability, $\frac{d}{dt}\psi(t)(x) = E\left\|Z_t^x\right\|^2 \to 0$. For $x \in \mathcal{D}(\mathcal{A})$, we have

$$\begin{aligned}\mathcal{L}_0\left(T(t)x, x\right) &= 2\left(T(t)x, Ax\right)_H \\ &\quad + \int_{H\setminus\{0\}} \left(T(t)(x + f_0(v)x), (x + f_0(v)x)\right)_H \\ &\quad - \left(T(t)x, x\right) - 2\left(T(t)x, f_0(v)x\right)\beta(dv).\end{aligned}$$

Define $T \in \mathcal{L}(H, H)$ under exponential stability, $(Tx, x) = \int_0^\infty E\left\|Z_u^x\right\|^2 du$. Then

$$\mathcal{L}_0\left(Tx, x\right) = \lim_{t\to\infty} \mathcal{L}_0\left(T(t)x, x\right) = \lim_{t\to\infty}\left(\frac{d}{dt}\psi(t)(x) - \|x\|^2\right) = -\|x\|^2 .$$

Hence we have the following theorem.

Theorem 4.3. *If the solution of equation (4.4) is exponentially stable in m.s.s., then there exists a Lyapunov function Λ_0 satisfying condition (B).*

Proof. Define $\Lambda_0 : H \to \mathbb{R}$, $\Lambda_0(x) = (Tx, x) + w\|x\|_H^2$, for w to be chosen. Clearly $\Lambda_0 \in C_b^{2,loc}(H)$ and satisfies (4.1). Since S_t is pseudo-contraction semigroup by a theorem of Lumer-Pillips, there exists a $\lambda \geq 0$, such that $(Ax, x) \leq \lambda\|x\|_H^2$ for $x \in \mathcal{D}(A)$, and

$$\mathcal{L}_0\|x\|_H^2 \leq 2(x, Ax) + \int_{H\setminus\{0\}} \|f_0(v)x\|^2 \beta(dv) \leq (2\lambda + d)\|x\|^2 .$$

Hence

$$\mathcal{L}_0\Lambda_0(x) \leq -\|x\|_H^2 + w(2\lambda + d)\|x\|^2 , \tag{4.9}$$

by choosing w so that $(2\lambda + d)w < 1$. ∎

The reason we consider the linear case is that in general we do not know if $\psi(x) = E\int_0^\infty \|Z_t^x\|^2 dt$ is in $C_b^{2,loc}$. However using work in [1], we can give conditions on a,f to assure this. Of course, we need to know 0 is a solution to (2.1), for which we assume $a(0) = 0$ and $f(0, v) = 0$. Suppose ψ above is in $C_b^{2,loc}$. Then under exponentially stability in m.s.s., $\psi(x) < \infty$ and clearly $\psi(x) \leq \frac{c}{\theta}\|x\|_H^2$. Then $\Lambda(x) = \psi(x) + w\|x\|_H^2$ satisfies (4.1). To get (4.2), we can follow the proof with $\psi_n(t)(x)$ as before using approximation and conclude using Markov property $\mathcal{L}\psi(x) = -\|x\|_H^2$. This gives

$$\begin{aligned}\mathcal{L}\Lambda(x) &= \mathcal{L}\psi(x) + w\mathcal{L}\|x\|_H^2 \\ &= -\|x\|_H^2 + w\left\{2(x, Ax) + 2(x, a(x)) + \int_{H\setminus\{0\}} \|f(v,x)\|_H^2 \,\beta(dv)\right\}.\end{aligned}$$

By condition (A3) on a, f and $a(0) = 0$, $f(v, 0) = 0$, we get

$$\mathcal{L}\Lambda(x) \le -\|x\|_H^2 + w(2\lambda + 2\sqrt{k} + k)\|x\|_H^2.$$

Thus with a choice of w we get (4.2).

We can also get Λ under exponential stability if we assume condition on the first order approximation.

Theorem 4.4. *Suppose the solution of equation (4.4) is exponentially stable in m.s.s. with given θ, c. Assume*

$$2\|x\|_H\|a(x)\|_H + \int_{H\setminus\{0\}} \|f(v,x) - f_0(v)x\|_H\|f(v,x) + f_0(v)x\|_H \,\beta(dv) < \frac{\theta}{c}\|x\|_H^2. \tag{4.10}$$

for all $x \in H$. Then the solution of (2.1) is exponentially stable in m.s.s..

Proof. Define Λ_0 as before then Λ_0 satisfied (4.1). We need to show (4.2).

$$\begin{aligned}&\mathcal{L}\Lambda_0(x) - \mathcal{L}_0\Lambda_0(x) \\ &= \langle\Lambda_0'(x), a(x)\rangle + \int_{H\setminus\{0\}} [\Lambda_0(x + f(v,x)) - \Lambda_0(x) - \langle\Lambda_0'(x), f(v,x)\rangle] \,\beta(dv) \\ &\quad - \int_{H\setminus\{0\}} [\Lambda_0(x + f_0(v)x) - \Lambda_0(x) - \langle\Lambda_0'(x), f_0(v)x\rangle] \,\beta(dv).\end{aligned}$$

Write $\Lambda_0(x) = (Tx, x) + w\|x\|_H^2$. Note that T is positive definite and self-adjoint operator. Moreover, for any $x, y \in H$,

$$(x,x)_H - (y,y)_H = (x - y, x + y)_H,$$

and

$$\begin{aligned}\mathcal{L}\Lambda_0(x) - \mathcal{L}_0\Lambda_0(x) &= 2\langle(T + w)x, a(x)\rangle \\ &\quad + \int_{H\setminus\{0\}} (\langle(T + w)f(v,x), f(v,x)\rangle - \langle(T + w)f_0(v)x, f_0(v)x\rangle) \,\beta(dv) \\ &\le 2(\|T\| + w)\|x\|_H\|a(x)\|_H \\ &\quad + (\|T\| + w)\int_{H\setminus\{0\}} \|f(v,x) - f_0(v)x\|_H\|f(v,x) + f_0(v)x\|_H \,\beta(dv) \\ &= (\|T\| + w)(2\|x\|_H\|a(x)\|_H \\ &\quad + \int_{H\setminus\{0\}} \|f(v,x) - f_0(v)x\|_H\|f(v,x) + f_0(v)x\|_H\beta(dv)),\end{aligned}$$

by (4.9) and the assumption (4.10), $\mathcal{L}\Lambda_0(z)$ satisfies (4.2) if we choose w small enough. □

Following [9], we introduce the following

Definition 4.5. We say that the zero solution of (2.1) is *stable in probability* if for each $\varepsilon > 0$

$$\lim_{\|x\|_H \to 0} P\Big(\sup_{t\geq 0} \|Z_t^x\|_H > \varepsilon\Big) = 0$$

Theorem 4.6. *Let $\mathcal{L}$ be the i.g. of the Markov semigroup associated with the solution of (2.1). Assume that there exists a function $\Lambda \in C_b^{2,loc}(H)$ such that:*

(i) $c_1\|x\|_H^2 \leq \Lambda(x) \leq c_2\|x\|_H^2$, *where c_1, c_2 are finite positive constants,*
(ii) $\inf_{\|x\|_H > \varepsilon} \Lambda(x) = \lambda_\varepsilon > 0$ *for each $\varepsilon > 0$,*
(iii) $\mathcal{L}\Lambda(x) \leq 0$, *for $\forall x \in H$.*

Then the zero solution of equation (2.1) is stable in probability.

Proof. We first obtain the inequality $P\Big(\sup_t \|Z_t^x\|_H > \varepsilon\Big) \leq \frac{\Lambda(x)}{\lambda_\varepsilon}$, for $x \in H$. To prove this, let $O_\varepsilon = \{\, x \in H : \|x\|_H < \varepsilon \,\}, T_\varepsilon = \inf\{\, t : \|Z_t^x\|_H > \varepsilon \,\}$. Now consider the process $Z_{t\wedge T_\varepsilon}^x$. Using Ito formula,

$$E\Lambda(Z_{t\wedge T_\varepsilon}^{n,x}) - \Lambda(x) = E\int_0^{t\wedge T_\varepsilon} \mathcal{L}_n\Lambda(Z_{s\wedge T_\varepsilon}^{n,x})\, ds.$$

Taking limit as $n \longrightarrow \infty$, we get using Lebsegue DCT, $E\Lambda(Z_{t\wedge T_\varepsilon}^x) \leq \Lambda(x)$, using $\mathcal{L}\Lambda(x) < 0$. Hence for all t, $\Lambda(x) \geq \lambda_\varepsilon P(T_\varepsilon < t)$. This proves the inequality. Now letting $x \to 0$, we get the assertion. □

Corollary 4.7. *The solution of the linear equation (4.4) is stable in probability if it is exponentially stable in m.s.s..*

Corollary 4.8. *If the solution of the linear equation (4.4) is exponentially stable in m.s.s. and (4.10) holds, then solution of equation (2.1) is stable in probability.*

5. Exponential Ultimate Boundedness in m.s.s.

We note that exponential stability in m.s.s. leads to the convergence of the solution to zero as $t \to \infty$. However, it is more interesting to study the convergence in distribution of the solution to invariant measure. For this, we follow the ideas of [16] and [21] for SDE's driven by Brownian motion and for SPDE's driven by Brownian motion due to [13]. We consider the study of these ideas by starting with the following definition.

Definition 5.1. The solution $\{Z_t^x\}$ of equation (2.1) is *exponentially ultimately bounded in m.s.s.* if there exist positive finite constants c, θ, M such that for $t > 0$

$$E\|Z_t^x\|^2 \leq ce^{-\theta t}\|x\|_H^2 + M, \tag{5.1}$$

for all $x \in H$.

Theorem 5.2. *Suppose there exists a function $\Lambda \in C_b^{2,loc}(H)$ satisfying condition (B) and positive, finite constants c_i, k_i $(i = 1, 2, 3)$ such that,*

$$c_1\|x\|_H^2 - k_1 \leq \Lambda(x) \leq c_3\|x\|_H^2 - k_3, \tag{5.2}$$

$$\mathcal{L}\Lambda(x) \le -c_2\Lambda(x) + k_2, x \in \mathcal{D}(A). \tag{5.3}$$

Then the solution of equation (2.1) with (A1)-(A3) is exponentially ultimately bounded.

Proof. Using Ito Formula and taking expectation, we get

$$e^{c_2 t} E(\Lambda(Z_t^{n,x}) - \Lambda(x) = E\int_0^t e^{c_2 s}(c_2 + \mathcal{L}_n)\Lambda(Z_s^{n,x})\, ds, \tag{5.4}$$

where $\{Z_t^{n,x}, t \ge 0\}$ is the solution of (3.1), a, f satisfying (A1)-(A3). Now for (5.3), note that

$$c_2\Lambda(x) + \mathcal{L}_n\Lambda(x) \le -\mathcal{L}\Lambda(x) + k_2 + \mathcal{L}_n\Lambda(x), x \in \mathcal{D}(A).$$

Using this inequality in the RHS of (5.4) and taking limits as $n \to \infty$ with some arguments as in the proof of Theorem 4.2, we obtain

$$e^{c_2 t} E\Lambda(Z_t^x) \le \Lambda(x) + \int_0^t e^{c_2 s} k_2\, ds = \Lambda(x) + \frac{k_2}{c_2}(e^{c_2 t} - 1).$$

For $x \in \mathcal{D}(A)$, using (5.2), we get

$$\begin{aligned} c_1 E\|Z_t^x\|_H^2 - k_1 \le & E\Lambda(Z_t^x) \le e^{-c_2 t}\Lambda(x) + \frac{k_2}{c_2}(1 - e^{-c_2 t}) \\ \le & e^{-c_2 t}(c_3\|x\|_H^2 - k_3) + \frac{k_2}{c_2}(1 - e^{-c_2 t}) \\ \le & c_3 e^{-c_2 t}\|x\|_H^2 + \frac{k_2}{c_2}(1 - e^{-c_2 t}). \end{aligned}$$

Take $c = \frac{c_3}{c_1}, \theta = c_2$, and $M = \frac{1}{c_1}(k_1 + \frac{k_2}{c_2})$ to get (5.2) for $x \in \mathcal{D}(A)$. Since $E\|Z_t^x\|_H^2$ is continuous in x, we get the inequality for $x \in H$. □

Corollary 5.3. *Under conditions of existence of $\Lambda \in C_b^{2,loc}(H)$ satisfying condition (B) and $c_1\|x\|_H^2 - k_1 \le \Lambda(x)$ with $\mathcal{L}\Lambda(x) \le -c_2\Lambda(x) + k_2$, then $\lim_{t\to\infty} E\|Z^x(t)\|_H^2$ is finite.*

Remark 5.4. This gives a result of Skorokhod ([19], p.70) using Chebyshev's inequality.

Theorem 5.5. *If the solution of the linear equation (4.4) is exponentially ultimately bounded in m.s.s., then there exists a function $\Lambda_0 \in C_b^{2,loc}(H)$ satisfying condition (B) and (5.2) and $\mathcal{L}_0\Lambda_0(x) \le -c_2\Lambda_0(x) + k_2$ for all $x \in \mathcal{D}(A)$.*

Proof. Define $\Lambda_0(x) = \int_0^T E\|Z_s^x\|_H^2\, ds + w\|x\|_H^2$, where T and w are positive constants to be determined later. Let $\psi_0(x) = \int_0^T E\|Z_s^x\|_H^2\, ds$, which is finite for $T < \infty$ by the exponential ultimate boundedness in m.s.s.. In fact,

$$\psi_0(x) \le \int_0^T (ce^{-\theta t}\|x\|_H^2 + M)\, dt \le \frac{c}{\theta}\|x\|_H^2 + MT.$$

For $\|x\|_H = 1$, we have $\psi_0(x) \le \dfrac{c}{\theta} + MT$. Using the fact that $x \to Z_t^x$ is linear we get

$$\psi_0(kx) = \int_0^T E\|Z_s^{kx}\|_H^2\, ds = k^2\int_0^T E\|Z_s^x\|_H^2\, ds = k^2\psi_0(x).$$

Let $c' = \dfrac{c}{\theta} + MT$, then

$$\psi_0(x) = \|x\|_H^2 \varphi_0\left(\frac{x}{\|x\|_H}\right) \leq \left(\frac{c}{\theta} + MT\right)\|x\|_H^2 = c'\|x\|_H^2 \text{ for } x \in H.$$

Consider $\int_0^T E\langle Z_s^x, Z_s^y\rangle\, ds, x, y \in H$. By the previous argument and linearity, it is a bounded bilinear form on H. Hence there exists a $C \in L^1(H)$ such that

$$\langle Cx, y\rangle = \int_0^T E\langle Z_s^x, Z_s^y\rangle\, ds,$$

since $\psi_0(x) = \langle Cx, x\rangle$. So $\psi_0'(x) = 2Cx$ and $\psi_0''(x) = 2C$. Hence $\psi_0 \in C_b^{2,loc}(H)$ and $\Lambda_0(\varphi) \in C_b^{2,loc}(H)$ and satisfies (5.2). We now need to show (5.3). Note that with $\psi_0^n(x) = \int_0^T E\|Z_s^{n,x}\|_H^2\, ds$, we get $\lim_{n\to\infty} \mathcal{L}_{0,n}\psi_0^n(x) = \mathcal{L}_0\psi_0(x)$, where $\{Z_t^{n,x}\}$ is the solution of (3.1), and

$$\mathcal{L}_{0,n}\psi_0^n(x) = \lim_{r\to 0} \frac{E\psi_0^n(Z_r^{n,x}) - \psi_0^n(x)}{r}.$$

Observe that by Markov property,

$$\begin{aligned} E\psi_0^n(Z_r^{n,x}) &= E\int_0^T E(\|Z_u^{n,x}\|_H^2 | x = Z_r^{n,x})du \\ &= E\int_0^T E(\|Z_{u+r}^{n,x}\|_H^2 | \mathcal{F}_r^{Z^{n,x}})du \\ &= \int_0^T E(\|Z_{u+r}^{n,x}\|_H^2)du \\ &= \int_r^{T+r} E(\|Z_0^{n,x}(u)\|_H^2)du, \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{r}\{E(\psi_0^n(Z_r^{n,x} - \varphi_0^n(x)))\} \\ &= \frac{1}{r}\int_r^{r+T} E\|Z_u^{n,x}\|_H^2\, du - \frac{1}{r}\int_0^T E\|Z_u^{n,x}\|_H^2\, du \\ &= -\frac{1}{r}\int_0^r E\|Z_u^{n,x}\|_H^2\, du + \frac{1}{r}\int_0^{r+T} E\|Z_u^{n,x}\|_H^2\, du - \frac{1}{r}\int_0^T E\|Z_u^{n,x}\|_H^2\, du. \end{aligned}$$

Taking $\lim_{r\to 0}$, above is equal to $-\|x\|^2 + E\|Z_T^{n,x}\|_H^2$. Now note that

$$\mathcal{L}_0\psi_0(x) = -\|x\|^2 + E\|Z_T^x\|_H^2 = (-1 + ce^{-\theta T})\|x\|_H^2 + M'.$$

So for $x \in \mathcal{D}(A)$, one has

$$\begin{aligned} \mathcal{L}_0\Lambda_0(x) =& \mathcal{L}_0\psi_0(x) + w\mathcal{L}_0\|x\|_H^2 \\ \leq& (-1 + ce^{-\theta T})\|x\|_H^2 + w(2\lambda + d)\|x\|_H^2 + M'. \end{aligned} \tag{5.5}$$

By Taking $T > \dfrac{\ln c}{\theta}$, and w small enough, one has the desired result. $\square$

Theorem 5.6. *Suppose the solution of equation (4.4) is exponentially ultimately bounded in m.s.s. with* c, M, θ *given and there exists* W *such that*

$$W < \max_{s > \frac{\ln c}{\theta}} \frac{1 - ce^{-\theta s}}{\frac{c}{\theta} + Ms}. \tag{5.6}$$

and

$$\begin{aligned} &2\|x\|_H \|a(x)\|_H + \int_{H\setminus\{0\}} \|f(v,x) - f_0(v)x\|_H \|f(v,x) + f_0(v)x\|_H \,\beta(dv) \\ &< W\|x\|_H^2 + M_1 \end{aligned} \tag{5.7}$$

for all $x \in H$*. Then equation (2.1) has the solution which is exponentially ultimate bounded in m.s.s.*

Proof. Let $\Lambda_0(x)$ be the Lyapunov function defined in linear case, i.e. $\{Z_{t,0}^x\}$ solution of (4.4) and $\Lambda_0(x) = \int_0^T E\|Z_{t,0}^x\|_H^2\,dt + w\|x\|_H^2$. Then clearly Λ_0 satisfies, $\Lambda_0 \in C_b^{2,loc}(H)$ and (4.1). If we show that $\mathcal{L}\Lambda_0(x) \le -c_2\Lambda_0(x)$, we get the result by Theorem 4.2. Since, with$\|C\| \le \dfrac{c}{\theta} + MT$, $\Lambda_0(x) = (Cx, x) + w\|x\|_H^2$. We get using arguments as in Theorem 4.4,

$$\begin{aligned} &\mathcal{L}\Lambda_0(x) - \mathcal{L}_0\Lambda_0(x) \\ &= \langle \Lambda_0'(x), a(x)\rangle + \int_{H\setminus\{0\}} [\Lambda_0(x + f(v,x)) - \Lambda_0(x) - \langle \Lambda_0'(x), f(v,x)\rangle]\,\beta(dv) \\ &\quad - \int_{H\setminus\{0\}} [\Lambda_0(x + f_0(v)x) - \Lambda_0(x) - \langle \Lambda_0'(x), f_0(v)x)\rangle]\,\beta(dv) \\ &= 2\,\langle (C+w)x, a(x)\rangle \\ &\quad + \int_{H\setminus\{0\}} [\langle (C+w)f(v,x), f(v,x)\rangle - \langle (C+w)f_0(v)x, f_0(v)x\rangle]\,\beta(dv) \\ &\le (\|C\| + w)\,(2\|x\|_H\|a(x)\|_H \\ &\quad + \int_{H\setminus\{0\}} \|f(v,x) - f_0(v)x\|_H \|f(v,x) + f_0(v)x\|_H\,\beta(dv)\Big) \\ &\le \Big(\frac{c}{\theta} + MT + w\Big)\Big(W\|x\|_H^2 + M_1\Big). \end{aligned}$$

Using (5.5), one has

$$\begin{aligned} &\mathcal{L}\Lambda_0(x) \\ &\le (-1 + ce^{-\theta T})\|x\|_H^2 + w(2\lambda + d)\|x\|_H^2 + M' \\ &\quad + \Big(\frac{c}{\theta} + MT + w\Big)\,(W\|x\|_H^2 + M_1) \\ &= \Big(-1 + ce^{-\theta T} + W\Big(\frac{c}{\theta} + MT\Big)\Big)\,\|x\|_H^2 + w(2\lambda + d + W)\|x\|_H^2 \\ &\quad + M' + \Big(\frac{c}{\theta} + MT + w\Big)\,M_1. \end{aligned}$$

Since W satisfies (5.6), $-1 + ce^{-\theta T} + W(\dfrac{c}{\theta} + MT) < 0$, and hence we can choose w small enough such that (5.3) is satisfied. ∎

Let $f: H \to H$, then $f(x) \to 0$ as $\|x\|_H \to \infty$ means that for every $\epsilon > 0$, there exists a K, such that $\|f(x)\|_H < \epsilon$ with $\|x\|_H > K$. We denote as $\|x\|_H \to \infty$, $f(x) = o(\|x\|_H)$ if $\frac{\|f(x)\|_H}{\|x\|_H} \to 0$ as $\|x\|_H \to \infty$.

Corollary 5.7. *Suppose the solution of equation (4.4) is exponentially ultimate bounded in m.s.s. and*

$$\begin{aligned} a(x) =& o(\|x\|_H) \\ \int_{H\setminus\{0\}} \|f(v,x) - f_0(v)x\|_H \|f(v,x) + f_0(v)x\|_H \, \beta(dv) =& o(\|x\|_H^2), \end{aligned}$$

as $\|x\|_H \to \infty$ *Then the solution of equation (2.1) is exponentially ultimately bounded in m.s.s..*

Proof. Let $W > 0$ satisfying (5.6). By the above conditions, there exists $K > 0$, such that

$$2\|x\|_H\|a(x)\|_H + \int_{H\setminus\{0\}} \|f(v,x) - f_0(v)x\|_H \|f(v,x) + f_0(v)x\|_H \, \beta(dv) \le W\|x\|_H^2$$

for $x \in H$, with $\|x\|_H \ge K$. For $x \in H, \|x\|_H \le K$, we obtain by linear growth condition,

$$\begin{aligned} & 2\|x\|_H\|a(x)\|_H + \int_{H\setminus\{0\}} \|f(v,x) - f_0(v)x\|_H \|f(v,x) + f_0(v)x\|_H \, \beta(dv) \\ \le \;& \|x\|_H^2 + \|a(x)\|_H^2 + \int_{H\setminus\{0\}} (\|f(v,x)\|_H + \|f_0(v)x\|_H)^2 \, \beta(dv) \\ \le \;& \|x\|_H^2 + l(1 + \|x\|_H^2) + 2l(1 + \|x\|_H^2) \\ \le \;& K^2 + 3l(1 + K^2). \end{aligned}$$

Hence condition (5.7) is satisfied with $M = K^2 + 3l(1 + K^2$, giving the result by Theorem 5.6. ☐

Corollary 5.8. *Suppose solution of* $dZ_t = AZ_t$ *with* $Z_0 = x$ *is exponentially stable (or even exponentially ultimately bounded). Assume condition (A1)-(A3) are satisfied then equation (2.1) has solution exponentially bounded if* $a(x) = o(\|x\|_H)$ *and* $\int_{H\setminus\{0\}} \|f(v,x)\|_H^2 \, \beta(dv) = o(\|x\|_H^2)$*, as* $\|x\|_H \to \infty$.

6. Invariant Measures

Let $P(t,x,A), t \ge 0, x \in H, A \in \mathcal{B}(H)$ be transition function of the solution of (2.1) assuming (A1)-(A3). Let $B_b(H)$, $C_b(H)$ and $C_w(H)$ denote bounded, bounded continuous and bounded weakly continuous functions on $H \longrightarrow \mathbb{R}$. Define, for $f \in B_b(H)$,

$$(P_t f)(x) = \int_H f(y)p(t,x,dy) = (P_t^x f)(y). \tag{6.1}$$

Definition 6.1. We call $\{P_t\}$ a *Feller* (*w-Feller*) semigroup if

$$P_t(C_b(H)) \subset C_b(H) \;\; (P_t(C_w(H)) \subset C_w(H)).$$

Definition 6.2. A sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of probability measures on a separable metric space X *converges weakly* (*w-weakly*) to a probability measure μ if

$$\lim_{n\to\infty}\int_X f d\mu_n = \int_X f d\mu$$

for all $f \in C_b(X)(C_w(X))$.

Definition 6.3. A set M of probability measures on $\mathcal{B}(H)$ is *weakly (w-weakly) compact*, if from any sequence of probability measures in M, a weakly (w-weakly) convergent subsequence can be extracted.

We note that because of continuity in x, the semigroup defined in (6.1) is Feller. For any fixed $x \in H$, consider family $\{\mu_T^x\}_{T>0}$ of measures

$$\mu_T^x(A) = \frac{1}{T}\int_0^T P(Z_t^x \in A)dt, A \in \mathcal{B}(H).$$

If $\{\mu_T^x, T > 0\}$ is sequentially relatively compact, then every limit point is an invariant measure for $\{Z_t^x, t \geq 0\}$, the solution of (2.1). ([4], Theorem 7, p.240). Here invariant measure is defined as follows.

Definition 6.4. ([4], p.23) A σ-finite measure μ on $(H, \mathcal{B}(H))$ is an invariant measure for $\{P_t\}_{t\geq 0}$ (resp solution of equation (2.1)), if $\int_H (P_t f)d\mu = \int_H f d\mu$.

The following lemma is from [8].

Lemma 6.5. *Let $p > 1$ and g be a nonnegative locally p-integrable function on $[0, +\infty)$. Then for each $\varepsilon > 0$ and real d,*

$$\left(\int_0^t e^{d(t-r)} g(r)\, dr\right)^p \leq C(\varepsilon, p)\int_0^t e^{p(d+\varepsilon)(t-r)} g^p(r)\, dr,$$

for t large enough, where $C(\varepsilon, p) = (1 + q\varepsilon)^{p/q}$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Theorem 6.6. *The set M of probability measures on $\mathcal{B}(H)$ is w-weakly compact if for each $\varepsilon > 0$, there exists a weakly compact set $K \subset H$ such that* $\sup\{\, \mu(H\backslash K); \mu \in M\,\} < \varepsilon$.

Remark 6.7. This is Y. V. Prokhorov's theorem under the weak topology.

Now assume $\lim_{t\to\infty} E\|Z_t^x\|_H^2 < \infty$. Consider a map $J : H \to R^\infty$, $Jh = ((h, e_1), \cdots (h, e_k), \cdots)$, where $\{e_k\}_{k\in N}$ is an orthonormal basis of H. Then

$$J(H) = l_2 = \{x \in R^\infty : \sum |x_i|^2 < \infty\}.$$

Hence we get $\overline{\lim}_{t\to\infty} E\|JZ_t^x\|_{l_2}^2 < \infty$, where $\|x\|_{l_2} = (\sum_{i=1}^\infty |x_i|^2)^{1/2})$. Now for $M > 0$, using Chebyshev inequality

$$\frac{1}{T}\int_0^T P(t, x, \|JZ_t^x\|_{l_2} > M) \leq \frac{1}{T}\frac{1}{M^2}\int_0^T E\|JZ_t^x\|_{l_2}^2 dt.$$

Under $\overline{\lim}_{t\to\infty} E\|JZ_t^x\|_{l_2}^2 < \infty$, we get that given $\epsilon > 0$ there exists a $M_\epsilon < \infty$, such that for any $T > 0$, $\frac{1}{T}\int_0^T P(t, x, \|JZ_t^x\|_{l_2} > M_\epsilon) < \epsilon$. Note that the embedding $i : l_2 \to R^\infty$ is compact operater. Using this and Prokhorov's theorem, we get $\{\frac{1}{T}\int_0^T P_t^x J^{-1}dt\}_{T\geq 0}$ is relatively compact family of probability measures on $(R^\infty, \mathcal{B}(R^\infty))$. Let μ be the limit of a subsequence. We observe that μ is a probability measure on R^∞. Consider $f: R^\infty \to R$ the function

$$f(x) = \begin{cases} \|x\|_{l_2} - \|J^{-1}x\|_H & x \in l^2 \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_n(x) = \begin{cases} \|x\|_{l_2}^2 1_{\{\|x\|_{l_2}^2 \leq n\}} & x \in l^2 \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_n \in L^1(R^\infty, \mathcal{B}(R^\infty), \mu)$. Hence by Ergodic Theorem for Markov processes

$$f_n^*(x) = \lim_{k\longrightarrow\infty} \frac{1}{t_k}\int_0^{t_k} P_t f_n(x)dt$$

exists and $f_n^*: R^\infty \longrightarrow R$ with $E_\mu f_n^*(x) = E_\mu(f_n)$. Now $f_n(J_x) \leq f(J_x)$ for $x \in H$. Using ultimate boundedness for all x,

$$f_n^*(J_x) = \lim_{k\longrightarrow\infty} \frac{1}{t_k}\int_0^{t_k} P_t f_n(J_x)dt \leq \overline{\lim}_{k\longrightarrow\infty} \frac{1}{t_k}\int_0^{t_k} P_t f(J_x)dt \leq K.$$

As $f_n(J_x) \uparrow f(J_x)$ for all $x \in H$, monotone convergence Theorem yields

$$\int_{R^\infty} f(x)d\mu = \lim_n \int_{R^\infty} f_n(x)\mu(dx) = \lim_n \int_{R^\infty} f_n^*(x)d\mu \leq K.$$

Let $I: R^\infty \longrightarrow H$,

$$I(x) = \begin{cases} J^{-1}(x) & x \in l^2 \\ 0 & \text{otherwise.} \end{cases}$$

Define $\nu = \mu \circ I$, then $\int_H \|x\|_H^2 \nu(dx) < \infty$. Hence ν lies on H. We get the following theorem.

Theorem 6.8. *If the solution $\{Z_t^x\}$ of (2.1) satisfying (A1)-(A3) is ultimately bounded in m.s.s, then it has an invariant measure ν and satisfies $\int_H \|x\|_H^2 \nu(dx) < \infty$.*

The following theorem gives conditions for the uniqueness of invariant measure. The proof is similar to [8].

Theorem 6.9. *Suppose Z_t^x is exponentially ultimately bounded and for each $R > 0, \delta > 0$ and $\varepsilon > 0$, there exists $T_0 = T_0(R, \delta, \varepsilon) > 0$ such that*

$$P\{\, \|Z_t^{x_0} - Z_t^{x_1}\|_H > \delta \,\} < \varepsilon \text{ for any } x_0, x_1 \in B_R \text{ whenever } t \geq T_0. \tag{6.2}$$

Here $B_R = \{\, y : \|y\|_H \leq R \,\}$. Then there exists at most one invariant measure.

The following Proposition (6.11) gives a sufficient condition for (6.2) holds.

Remark 6.10. The condition $\langle Ay, y\rangle \leq \alpha\|y\|_H^2$ for $y \in \mathcal{D}(A)$ is equivalent to $|S_t| \leq e^{\alpha t}$, where α is real [7].

Proposition 6.11. *Suppose that $\langle y, Ay\rangle \leq -c_0\|y\|_H^2$, $y \in \mathcal{D}(A)$, and c_0 is the maximum value satisfying the above inequality. Also suppose k is the minimum value satisfy Lipschitz condition. Then if $a = c_0 - 3k > 0$, we have $E\|Z_t^{x_0} - Z_t^{x_1}\|_H^2 \leq e^{-2at}\|x_0 - x_1\|_H^2$, for t large enough.*

Proof. Let $Z_t^{x_1}$ and $Z_t^{x_0}$ be two solutions. Then we have,

$$Z_t^{x_0} - Z_t^{x_1} = S_t(x_0 - x_1) + \int_0^t S_{t-s}[a(Z_s^{x_0}) - a(Z_s^{x_1})]\,ds$$
$$+ \int_0^t \int_{H\setminus\{0\}} S_{t-s}[f(v, Z_s^{x_0}) - f(v, Z_s^{x_1})]\,q(dv\,ds).$$

So

$$\|Z_t^{x_0} - Z_t^{x_1}\|_H^2 \le 3\|S_t(x_0 - x_1)\|_H^2 + 3\Big\| \int_0^t S_{t-s}[a(Z_s^{x_0}) - a(Z_s^{x_1})]\,ds \Big\|_H^2$$
$$+3\Big\| \int_0^t \int_{H\setminus\{0\}} S_{t-s}[f(v, Z_s^{x_0}) - f(v, Z_s^{x_1})]\,q(dv\,ds) \Big\|_H^2.$$

So

$$\begin{aligned}
&E\|Z_t^{x_0} - Z_t^{x_1}\|_H^2 \le 3e^{-2c_0 t}\|x_0 - x_1\|_H^2 \\
&\quad +3E\Big| \int_0^t \|S_{t-s}[a(Z_s^{x_0}) - a(Z_s^{x_1})]\|_H\,ds \Big|^2 \\
&\quad +3\int_0^t \int_{H\setminus\{0\}} E\|f(v, Z_s^{x_0}) - f(v, Z_s^{x_1})\|_H^2\,\beta(dv)\,ds \\
&\le \;\; 3e^{-2c_0 t}\|x_0 - x_1\|_H^2 + 3kE\Big(\int_0^t e^{-c_0(t-s)}\|Z_s^{x_0} - Z_s^{x_1}\|_H\,ds \Big)^2 \\
&\quad +3\int_0^t kE\|Z_s^{x_0} - Z_s^{x_1}\|_H^2\,ds \\
&\le \;\; 3e^{-2c_0 t}\|x_0 - x_1\|_H^2 \\
&\quad +3k(1+2\epsilon)\int_0^t e^{2(-c_0+\epsilon)(t-s)}E\|Z_s^{x_0} - Z_s^{x_1}\|_H^2\,ds \\
&\quad +3k\int_0^t E\|Z_s^{x_0} - Z_s^{x_1}\|_H^2\,ds \text{ (by Lemma 6.5, } \epsilon \text{ is small positive)} \\
&\le \;\; 3e^{-2c_0 t}\|x_0 - x_1\|_H^2 + 3k\int_0^t E\|Z_s^{x_0} - Z_s^{x_1}\|_H^2\,ds \\
&\quad +3k\int_0^t E\|Z_s^{x_0} - Z_s^{x_1}\|_H^2\,ds.
\end{aligned}$$

Letting $\epsilon \to 0$ and $e^{2(-2c_0+\epsilon)(t-s)} < 1$, we have

$$E\|Z_t^{x_0} - Z_t^{x_1}\|_H^2 \le 3e^{-2c_0 t}\|x_0 - x_1\|_H^2 + 6k\int_0^t E\|Z_s^{x_0} - Z_s^{x_1}\|_H^2\,ds.$$

So By Gronwall's inequality, we have,

$$E\|Z_t^{x_0} - Z_t^{x_1}\|_H^2 \le 3e^{-2c_0 t}\|x_0 - x_1\|_H^2\; e^{6kt} \le 3e^{(-2c_0+6k)t}\|x_0 - x_1\|_H^2. \quad \square$$

Theorem 6.12. *Suppose P_t is w-Feller and that*

$$\frac{1}{t}\int_0^t E\|Z^{y_0}(r)\|_H^2\,dr \le M(1 + \|y_0\|_H^2),\; M > 0 \text{ and for any } t \ge t_0 > 0. \quad (6.3)$$

Then there exists an invariant measure.

Proof. For integers $n \geq t_0$, define $m_n(B) = \dfrac{1}{n}\int_0^n P(r, y_0, B)\,dr,\ B \in \mathcal{B}(H)$. Then m_n is a probability measure and $\int_H \|y\|_H^2\, m_n(dy) \leq M(1+\|y_0\|_H^2)$. Hence for each $\varepsilon > 0$, there exists $R > 0$, such that $m_n(B_R) > 1-\varepsilon, B_R = \{\, y : \|y\|_H \leq R \,\}$.

By Theorem 6.6, $\{\, m_n \,\}$, $n \geq t_0$ is w-weakly compact and there exists a subsequence, again denoted by m_n, which is w-weakly convergent to some probability measure $m_0(\cdot)$. Let $h \in C_w(H)$ be arbitrary. Then

$$\begin{aligned}\int_H [P_t h](y)\, m_0(dy) &= \lim_{n\to\infty} \int_H [P_t h](y)\, m_n(dy)(\text{ since } P_t \text{ is } w\text{-Feller})\\ &= \lim_{n\to\infty} \left(\frac{1}{n}\right) \int_0^n [P_{t+r}h](y_0)\,dr\\ &= \lim_{n\to\infty} \left(\frac{1}{n}\right) \int_t^{t+n} [P_r h](y_0)\,dr\\ &= \lim_{n\to\infty} \left(\frac{1}{n}\right) \int_0^n [P_r h](y_0)\,dr \ (\text{ since } P_t h \text{ is bounded})\\ &= \int_H h(y)\, m_0(dy),\end{aligned}$$

which implies m_0 is an invariant measure of P_t. □

7. Examples

Example 7.1 (Dam storage problem)**.** We consider the semilinear stochastic differential equation

$$\begin{cases} d\eta_t = A\eta_t\,dt + d\xi_t, \quad \text{suppose } d\xi_t = \int_{H\setminus\{0\}} u\, q(du\,dt) \\ \eta_0 = x \in H. \end{cases}$$

Here A is an infinitesimal generator of a pseudo-contraction semigroup $\{\, S_t \,\}$.

Compared to the general case

$$\begin{cases} dZ_t = (AZ_t + a(Z_t))\,dt + \int_{H\setminus\{0\}} f(v, Z_t)\, q(dv\,dt) \\ Z_0 = x \in H, \end{cases} \tag{7.1}$$

we have $a = 0,\ f(v,x) = v$. The growth condition and Lipschitz condition are satisfied, so there exists an unique mild solution η_t^x, such that

$$\eta_t^x = S_t x + \int_0^t \int_{H\setminus\{0\}} S_{t-s} u\, q(du\,ds).$$

For any $h \in H$,

$$\langle \eta_t^{x_1} - \eta_t^{x_2}, h\rangle = \langle S_t(x_1 - x_2), h\rangle = \langle x_1 - x_2, S_t^* h\rangle,$$

so we have,

$$\langle \eta_t^{x_1} - \eta_t^{x_2}, h\rangle \to 0, \text{ for any } h \in H \text{ as } x_2 \to x_1 \text{ weakly}.$$

By Ichikawa [8], we know that η_t^x is w-Feller.

Now if $|S_t| \le e^{-\beta t}, (\beta > 0)$, we have

$$\begin{aligned}
&E\|\eta_t^x\|_H^2 \\
&\le 2e^{-2\beta t}\|x\|_H^2 + 2\Big\| \int_0^t \int_{H\setminus\{0\}} S_{t-s} u\, q(du\, ds)\Big\|_H^2 \\
&\le 2e^{-2\beta t}\|x\|_H^2 + 2\int_0^t \int_{H\setminus\{0\}} \|S_{t-s}u\|_H^2\, \beta(du)\, ds \\
&\le 2e^{-2\beta t}\|x\|_H^2 + 2\int_0^t \int_{H\setminus\{0\}} e^{-2\beta(t-s)}\|u\|_H^2\, \beta(du)\, ds \\
&\le 2e^{-2\beta t}\|x\|_H^2 + 2\int_0^t e^{-2\beta(t-s)}\, ds \int_H \|u\|_H^2\, \beta(du) \\
&\le 2e^{-2\beta t}\|x\|_H^2 + \frac{1}{\beta}(1-e^{-2\beta t}) \int_H \|u\|_H^2\, \beta(du) \\
&\le 2e^{-2\beta t}\|x\|_H^2 + M,
\end{aligned}$$

where M is a positive constant. So η_t^x is exponential ultimate bounded. By Theorem 6.9 and Theorem 6.12, there exists an unique invariant measure.

Example 7.2 (Linear case)**.** Consider in the above example with $a(x) = 0$ and $f(\nu, x) = f_0(\nu)x$. Then we get by Theorem 5.5 and Corollary 5.3, the solution of

$$dZ_t = AZ_t dt + \int_{H\setminus\{0\}} f_0(\nu) Z_t q(dt, d\nu)$$

is exponentially ultimately bounded.

Now let $a(x) = ax$ and $f_0(\nu)x$ in (4.4). Consider the following system,

$$\begin{cases} dZ_t = AZ_t\, dt + aZ_t\, dt + \int_{H\setminus\{0\}} f_0(v) Z_t q(dt, dv) \\ Z_0 = x. \end{cases} , \tag{7.2}$$

It is still linear, similar to the proof of Theorem 5.5, we have

Theorem 7.3. *If the solution of the linear equation (7.2) is exponentially ultimately bounded in m.s.s., then there exists a function $\Lambda_0 \in C_b^{2,loc}(H)$ satisfying (5.2) and $\mathcal{L}_0\Lambda_0(x) \le -c_2\Lambda_0(x) + k_2$ for all $x \in \mathcal{D}(A)$.*

Then we get the above example for linear case.

For the general case as in (7.1), assume condition (A1)-(A3) are satisfied. If the following conditions are satisfied, then

$$a(x) = o(\|x\|_H) \text{ and } \int_{H\setminus\{0\}} \|f(v,x)\|_H^2\, \beta(dv) = o(\|x\|_H^2),$$

as $\|x\|_H \to \infty$. By Corollary 5.8, we can extend the above example to the general nonlinear case. That is, the solution to the equation (7.1) is exponentially ultimately bounded.

Remark 7.4. With appropriate conditions on σ in [5], we can derive from this asymptotic properties of interest rate models for the case. $n = 1$ using Corollary 5.8 as $A = \frac{d}{dx}$ generates pseudo-contraction semigroup.

Acknowledgment. This work is part of the second author's dissertation under the supervision of the first author.

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