[Journal of Stochastic Analysis](https://repository.lsu.edu/josa)

Volume 4 | Number 2 Article 1

June 2023

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Journal of Stochastic Analysis
Vol. 4, No. 2 (2023) Article 1 (26 pages)
DOI: 10.31390/josa.4.2.01

SYMMETRIC FUNCTIONS ALGEBRAS (SFA) II: INDUCED MATRICES

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Abstract. We study symmetric functions algebras based on the induced matrix map on the algebra of real $d \times d$ matrices. For fixed integer $N > 0$, the induced matrix map takes a matrix to the symmetric tensor power in degree N. It is determined by the action of the matrix on polynomials in d variables. The symmetric functions algebra has various bases which obey the identities of the standard algebra of symmetric functions. Thus, we determine corresponding elementary, homogeneous, power sum, monomial, and Schur functions and study their properties. For symmetric tensor powers, stochastic matrices are mapped to stochastic matrices with corresponding Markov chains arising from a given underlying chain. This paper details the construction of the multinomial chains and then looks at symmetric functions algebras based on the induced matrix map for a general matrix. A novel proof of the trace formula based on a multidimensional extension of the Mehler kernel formula is provided. Extended versions of the Cayley-Hamilton theorem are given as well.

1. Introduction

We are interested in algebras SFA(ϕ, X) for a given multiplicative map ϕ and a given element (matrix) X. Given ϕ and X we construct matrix analogs of elementary, homogeneous, power sum, monomial, and Schur symmetric functions. Thus the name "symmetric functions algebra". We move from the general considerations of SFA I [4], and focus on SFA(ϕ, X) where ϕ is the induced matrix map explained below. Starting from a stochastic model, we see that the map ϕ, while preserving the property of being stochastic, applies as well to any $d \times d$ matrix, §§2-3. Then we describe the various symmetric functions bases, §4, and calculate traces for the basic elements, §5. The principal tool used for computing traces is the trace formula for the induced matrices. We present a novel formulation of a matrix version of Mehler's kernel and use it to prove the principal trace formula. A specific approach is used as well for the monomial functions, based on connecting the induced matrix of a linear combination of diagonal matrices and the matrix

Received 2023-2-18; Accepted 2023-4-12; Communicated by A. Boukas.
2020 *Mathematics Subject Classification.* Primary 05E05, 11B37, 15A72, 60J10; Secondary 15A69, 22E60.
Key words and phrases. Symmetric functions, transition matrices, multiplicative maps, Lie maps, tensor powers, Markov chains, Kibble-Slepian, Mehler, Cayley-Hamilton.
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with the entries from the diagonal matrices converted into columns. Further properties and a basic example appear in §6. Of special interest, two extensions of the Cayley-Hamilton theorem are shown.

Notation. We will be working with d variables, $\{x_1, x_2, \ldots, x_d\}$. Letters m, n, r, and s will always denote multi-indices, d-tuples. Other indices will be single indices unless otherwise indicated or clear from the context. Multi-index notation will typically be employed, i.e., $x^m = x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d}$, $m! = m_1!\, m_2! \cdots m_d!$, $|m| = \sum_i m_i$, etc.

Remark 1.1. Below, we will use P to denote the transition matrix for a d-state Markov chain. Here P will occur unsubscripted or with double subscripts so there is no confusion with the power sum elements P_i occurring later.

2. Multinomial Extension of a Markov Chain

First we review multinomial distributions, then we introduce the model system described by the multinomial chain and look at the gamma maps.

2.1. Multinomial distribution. For a binomial experiment consisting of N trials with 2 choices per trial, with P(choice i) $= p_i$, $i = 1, 2$, we have the generating function

$$(p_1x_1 + p_2x_2)^N = \sum_{n_1+n_2=N} \frac{N!}{n_1!\, n_2!}\, p_1^{n_1} p_2^{n_2}\, x_1^{n_1} x_2^{n_2}$$

for the probabilities of choosing i n_i times. Similarly, for a multinomial experiment consisting of N trials with d choices per trial, with

$$\text{P(choice i)} = p_i$$

$i = 1, \ldots, d$, we have

$$\begin{aligned}(p_1x_1 + p_2x_2 + \cdots + p_dx_d)^N &= (\sum_i p_ix_i)^N \\ &= \sum_{\sum n_i=N} \frac{N!}{n_1!\, n_2! \cdots n_d!}\, p_1^{n_1} p_2^{n_2} \cdots p_d^{n_d}\, x_1^{n_1} x_2^{n_2} \cdots x_d^{n_d} \\ &= N! \sum_{\sum n_i=N} \left(\prod_j \frac{p_j^{n_j}}{n_j!} \right) x^n \qquad (2.1)\end{aligned}$$

as generating function for multinomial probabilities. With $x_i = 1$, $i = 1, \ldots, d$, we check that the probabilities add to 1.

2.2. Multinomial chain. Consider a homogeneous Markov chain with transition matrix $P = (p_{ij})$ modelled as a system of d slots with $N = 1$ ball. If the ball is in slot i, we say the system is in state i. At each tick of the clock, the ball jumps, moving from i to j with probability p_{ij}. Note that "j" in one transition becomes "i" for the next transition. Now take N balls and put them in slot $i = 1$. For the first step, we perform a multinomial experiment consisting of N trials, with the probability p_{1j} the probability at each stage of assigning a ball to slot j. After all balls are assigned, then the next transition occurs accordingly. After the first step the multinomial chain is in state $n = (n_1, \ldots, n_d)$ with n_i balls in slot i,

$\sum_i n_i = N$. For the next transition, we relabel the state to $m = (m_1, \ldots, m_d)$, $\sum_i m_i = N$. To transition to a new distribution of the balls in slots, we run in each slot i a multinomial experiment consisting of m_i trials with d outcomes per trial, assigning a ball to slot j with probability p_{ij}. After making assignments for all slots, the balls are transferred accordingly and the system is ready for the next step. The transition matrix for the multinomial chain is the *induced matrix*, denoted $\bar{P}$.

Remark 2.1. Note that N is understood fixed.

Remark 2.2. For constructions in this vein, see [3], [16].

The result of the multinomial experiments may be summarized as a $d \times d$ array with n_{ij} denoting the number of balls moving from slot i to slot j satisfying

$$\sum_j n_{ij} = m_i \quad \text{and} \quad \sum_i n_{ij} = n_j \tag{2.2}$$

i.e., a matrix of natural numbers with row sums $\{m_i\}$ and column sums $\{n_j\}$

Now form the generating function, cf. (2.1),

$$\begin{aligned}
\prod_i \left(\sum_j p_{ij} x_j \right)^{m_i} &= \prod_i m_i! \left(\sum_{\sum_j n_{ij} = m_i} \prod_j \frac{p_{ij}^{n_{ij}}}{n_{ij}!} \, x_j^{n_{ij}} \right) \\
&= \sum_{(n_1, \ldots, n_d)} \left(\sum_{\substack{\sum_j n_{ij} = m_i \\ \sum_i n_{ij} = n_j}} \prod_i m_i! \prod_j \frac{p_{ij}^{n_{ij}}}{n_{ij}!} \right) \left(\prod_j x_j^{n_j} \right) \\
&= \sum_n \bar{P}_{mn} x^n
\end{aligned} \tag{2.3}$$

The transition probabilities for the multinomial chain are thus

$$\bar{P}_{mn} = m! \sum_{\substack{\sum_j n_{ij} = m_i \\ \sum_i n_{ij} = n_j}} \prod_{i,j} \frac{p_{ij}^{n_{ij}}}{n_{ij}!} \tag{2.4}$$

We may thus define

Definition 2.3. For any $d \times d$ matrix A we define the *induced matrix*, $\bar{A}$, in degree N, to be the matrix of coefficients in the expansion

$$\prod_i \left(\sum_j A_{ij} x_j \right)^{m_i} = \sum_{|n| = N} \bar{A}_{mn} x^n$$

with $|m| = N$, where the multi-indices are ordered by (reverse) lexicographic order, i.e., x_1^N is first, followed by $x_1^{N-1} x_2$, etc. Note that

$$\bar{A} \text{ is an } \binom{N+d-1}{N} \times \binom{N+d-1}{N} \text{ matrix}$$

Remark 2.4. For terminology and related discussion, see [10, pp. 178-179], [9, p. 122], see also [14].

From now on, we will be interested in SFA(ϕ, A), where $\phi(A) = \bar{A}$, with given, fixed N understood.

Remark 2.5. We will use A as a generic matrix in the context of induced matrices.

2.3. Gamma maps. The gamma maps also have stochastic interpretations, so we will look at them here. We have

$$\overline{I + vA} = \sum_{\ell=0}^{N} v^\ell \, \Gamma_\ell(A)$$

First some notation.

Notation. For multi-indices define the binomial coefficient as the product of component binomial coefficients, i.e,

$$\binom{m}{r} = \prod_{i=1}^{d} \binom{m_i}{r_i}$$

for $m = (m_1, \ldots, m_d),\ r = (r_1, \ldots, r_d)$.

Now set

$$y_i = \sum_j A_{ij} x_j \ .$$

We have the expansion, using a superscript on $\bar{A}$ to indicate the degree explicitly,

$$\begin{aligned}
\prod_i (x_i + v y_i)^{m_i} &= \prod_i \sum \binom{m_i}{r_i} x_i^{m_i - r_i} v^{r_i} y_i^{r_i} \\
&= \sum x^{m-r} \binom{m}{r} v^{|r|} y^r \\
&= \sum \binom{m}{r} x^{m-r} v^\ell \sum_{|r|=|s|=\ell} \bar{A}^{(\ell)}_{rs} x^s \\
&= \sum v^\ell \sum_{|r|=|s|=\ell} \bar{A}^{(\ell)}_{rs} x^{m-r+s} \\
&= \sum v^\ell (\Gamma_\ell(A))_{mn} \, x^n
\end{aligned} \tag{2.5}$$

where

$$(\Gamma_\ell(A))_{mn} = \sum_{\substack{|r|=|s|=\ell \\ n=m-r+s \\ r\le m,\, s\le n}} \binom{m}{r} \bar{A}^{(\ell)}_{rs} \tag{2.6}$$

where $r \le m$, e.g., means $r_i \le m_i$, $1 \le i \le d$. In particular,

$$\Gamma(A)_{mn} = \sum_{\substack{|r|=|s|=1 \\ n=m-r+s \\ r\le m,\, s\le n}} \binom{m}{r} A_{rs}$$

Introducing ε_i as the standard basis for $\mathbb{R}^d$, we can write

$$\Gamma(A)_{mn} = \sum_{n=m-\varepsilon_i+\varepsilon_j} m_i \, A_{ij}$$

corresponding to a choice of one of m_i balls to be moved from slot i. Observe that the diagonal entries are

$$\Gamma(A)_{mm} = \sum_i m_i A_{ii}$$

And we see that, generally, Γ_ℓ gives the transition probabilities when ℓ of the N balls are chosen to be moved. Note that all of the gamma matrices are nonnegative, but are not normalized. In fact, for stochastic A, $x_i = 1$, $1 \le i \le d$, we have $y_i = 1$ for each i, and

$$\prod_i (1+v)^{m_i} = (1+v)^N = \sum_\ell \binom{N}{\ell} v^\ell$$

That is, the row sums of $\Gamma_\ell(A)$ equal $\binom{N}{\ell}$ for stochastic A.

3. Induced Matrices

Now we look at the basic properties of the induced matrix map, including another look at gamma maps from a different perspective.

3.1. Properties of the induced matrix map.

3.1.1. *Scaling property.* Directly from Definition 2.3, for a scalar c, we have, with $|m| = N$,

$$\sum_n \overline{(cA)}_{mn} x^n = \prod_i (\sum_j cA_{ij}x_j)^{m_i} = c^{|m|} \sum_n \bar{A}_{mn} x^n = c^N \sum_n \bar{A}_{mn} x^n$$

i.e.,

$$\overline{cA} = c^N \bar{A}$$

And for the gamma maps,

$$\overline{I + v(cA)} = \sum_\ell v^\ell \Gamma_\ell(cA) = \sum_\ell v^\ell (c^\ell \, \Gamma_\ell(A))$$

so that

$$\Gamma_\ell(cA) = c^\ell \Gamma_\ell(A) \ .$$

And we have

Proposition 3.1. *In degree N,*

$$\Gamma(I) = N \, \bar{I}$$

where on the right-hand side $\bar{I}$ is the $\binom{N+d-1}{N} \times \binom{N+d-1}{N}$ identity.

Proof. Using the exponential definition of the gamma map and the scaling property we have

$$\overline{e^{tI}} = \overline{e^t I} = e^{Nt} \bar{I} = e^{t(N\bar{I})} = e^{t\,\Gamma(I)}$$

as required. ☐

Remark 3.2. Usually, by abuse of notation, we will simply write I for the induced identity matrix, with the size understood according to context.

3.1.2. *Homomorphism property.* We check that $\phi(A) = \bar{A}$ is a multiplicative map. Write $y = A_2 x$, i.e., $y_i = \sum_j (A_2)_{ij} x_j$. Then

$$\begin{aligned}
\sum_n (\overline{A_1 A_2})_{mn} x^n &= \prod_i (\sum_j (A_1 A_2)_{ij} x_j)^{m_i} = \prod_i (\sum_j (A_1)_{ij} y_j)^{m_i} \\
&= \sum_r (\bar{A}_1)_{mr}\, y^r \\
&= \sum_r (\bar{A}_1)_{mr} \sum_n (\bar{A}_2)_{rn}\, x^n \\
&= \sum_n (\bar{A}_1 \bar{A}_2)_{mn}\, x^n
\end{aligned}$$

So

$$\overline{A_1 A_2} = \bar{A}_1 \bar{A}_2$$

accordingly. In particular, if A is diagonalized by W, with columns eigenvectors of A, and eigenvalues D_{ii}, for diagonal D, the relation

$$AW = WD \quad \Rightarrow \quad \bar{A}\bar{W} = \bar{W}\bar{D}$$

And similarly, $AW = WD$ implies $(I + vA)W = W(I + vD)$ so that

$$AW = WD \quad \Rightarrow \quad \Gamma_\ell(A)\bar{W} = \bar{W}\Gamma_\ell(D)$$

diagonalizing the gamma maps as well.

Remark 3.3. Analogous relationships hold for W having rows that are left eigenvectors, with W multiplying on the left.

Remark 3.4. For SFA$(^-, A)$, we see that $\bar{W}$ provides common eigenvectors for the entire algebra, via the power sums basis with $P_i = \Gamma(A^i)$ all diagonalized.

3.1.3. *Diagonal matrices.* So it is of interest to inspect the properties of diagonal matrices under the induced matrix map. Introduce variables $\{\alpha_1, \alpha_2, \ldots, \alpha_d\}$ as the diagonal entries of the diagonal matrix D, which we may interpret as the eigenvalues of a diagonalizable matrix A, i.e,

$$D = \begin{pmatrix} \alpha_1 & 0 & \cdots & \cdots \\ 0 & \alpha_2 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ & & \ddots & \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & 0 & \alpha_d \end{pmatrix}$$

which is conveniently denoted as $\mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_d)$. We have

$$\sum_{|n|=N} \bar{D}_{mn} x^n = \prod_i (\sum_j D_{ij} x_j)^{m_i} = \prod_i (\alpha_i x_i)^{m_i} = \alpha^m x^n\, \delta_{mn}$$

so that $\bar{D}$ is diagonal with entries

$$(\bar{D})_{mm} = \alpha^m = \alpha_1^{m_1} \alpha_2^{m_2} \cdots \alpha_d^{m_d}$$

In other words, the entries are just the corresponding monomials in the variables α_i. For the gamma maps, we have for $\overline{I+vD}$,

$$\prod_i (x_i + v\alpha_i x_i)^{m_i} = x^m \prod_i (1+v\alpha_i)^{m_i}$$
$$= x^m \sum_\ell v^\ell \mathrm{e}_\ell(\alpha_1,\cdots,\alpha_1,\alpha_2,\cdots,\alpha_2,\ldots,\alpha_d,\cdots,\alpha_d)$$

with α_i repeated m_i times. That is,

Proposition 3.5. *For a diagonal matrix D, $\Gamma_\ell(D)$ is diagonal and the (mm) entry on the diagonal is the ℓ^{th} elementary symmetric function in the multiset of variables $\{\alpha_i\}$ where the variable α_i is repeated m_i times.*

Thus

Corollary 3.6. *For a diagonal matrix an analogous statement holds for all of the classes of symmetric functions of* SFA($^-$, A).

3.1.4. *Basic inequality.* For continuity of the induced matrix map, we look at the definition using variables $\zeta_j = e^{\sqrt{-1}\,\theta_j}$, to avoid confusion between i and $\sqrt{-1}$. We have

$$G_m(\theta) = G_m(\theta_1,\ldots,\theta_d) = \prod_i (\sum_j A_{ij}\zeta_j) = \sum_n \bar{A}_{mn}\zeta^n$$

so that

$$\bar{A}_{mn} = (2\pi)^{-d}\int_0^{2\pi}\int_0^{2\pi}\cdots\int_0^{2\pi} G_m(\theta)\,\zeta_1^{-n_1}\cdots\zeta_d^{-n_d}\,d\theta_1\cdots d\theta_d$$

We recall the identity for any sets of numbers $\{u_i\}_{1\le i\le\ell}$, $\{v_i\}_{1\le i\le\ell}$, cf. [1],

$$\prod_{1\le i\le l} u_i - \prod_{1\le i\le \ell} v_i = \sum_{k=1}^{\ell} (\prod_{j<k} u_j)(u_k - v_k)(\prod_{i>k} v_i) \tag{3.1}$$

(empty products equal to 1) observing that this is a telescoping sum. For a matrix A, define

$$\|A\| = \sup_i \sum_j |A_{ij}|$$

so that each term of the product for $G(\theta)$ is bounded by $\|A\|$. Now consider $G^{(1)}(\theta)$, $G^{(2)}(\theta)$, corresponding to matrices A_1 and A_2 respectively. We have

$$(\bar{A}_1)_{mn} - (\bar{A}_2)_{mn}$$
$$= \frac{1}{(2\pi)^d}\int_0^{2\pi}\cdots\int_0^{2\pi}\left(G_m^{(1)}(\theta) - G_m^{(2)}(\theta)\right)\zeta_1^{-n_1}\cdots\zeta_d^{-n_d}\,d\theta_1\cdots d\theta_d$$

which gives the bound

$$|(\bar{A}_1)_{mn} - (\bar{A}_2)_{mn}| \le (2\pi)^{-d}\int_0^{2\pi}\cdots\int_0^{2\pi}\left|G_m^{(1)}(\theta) - G_m^{(2)}(\theta)\right| d\theta_1\cdots d\theta_d$$

Now apply (3.1) to get the bound, using $\|A\|' = \max(1,\|A\|)$,

$$|G_m^{(1)}(\theta) - G_m^{(2)}(\theta)| \le N\,(\|A_1\|')^N\,(\|A_2\|')^N \sum_{i,j} |(A_1)_{ij} - (A_2)_{ij}|$$

which arises by differencing term-by-term in the products noting that the differences for row i are repeated m_i times, giving an overall upper bound of N. Thus,

Proposition 3.7. *Given matrices A_1, A_2, we have for the induced matrices in degree N*

$$\sup_{m,n} |(\bar{A_1})_{mn} - (\bar{A_2})_{mn}| \le N\,(\|A_1\|')^N\,(\|A_2\|')^N \sum_{i,j} |(A_1)_{ij} - (A_2)_{ij}|$$

Observe that for a stochastic matrix P, we have $\|P\| = 1$, so that

Corollary 3.8. *For stochastic matrices P_1 and P_2 we have for induced matrices in degree N the bound*

$$\sup_{m,n} |(\bar{P_1})_{mn} - (\bar{P_2})_{mn}| \le N \sum_{i,j} |(P_1)_{ij} - (P_2)_{ij}|$$

3.2. Generating function. Gamma maps revisited. We can derive an interesting generating function for the matrix elements of the induced maps in all degrees. Let $\xi = (\xi_1, \dots, \xi_d)$ and form the sum, noting that we define the induced map in degree 0 to be the number 1,

$$\mathcal{G} = \sum_{\substack{|m|=N \\ N\ge 0}} \frac{\xi^m}{m!} \prod_i (\sum_j A_{ij} x_j)^{m_i} = \sum_{m,n} \frac{\xi^m}{m!} \bar{A}_{mn} x^n = e^{\langle Ax, \xi\rangle} \tag{3.2}$$

with $\langle y, \xi\rangle = \sum\limits_{i=1}^{d} y_i \xi_i$ the usual inner product of vectors. Now for the gamma maps, replace $A \to I + vA$ and expand in powers of v:

$$e^{\langle (I+vA)x, \xi\rangle} = \sum_{m,n} \frac{\xi^m}{m!} (\overline{I + vA})_{mn} x^n$$

so that

$$\frac{1}{\ell!} \left(\frac{\mathrm{d}}{\mathrm{d}v}\right)^{\ell} \bigg|_{v=0} e^{\langle (I+vA)x, \xi\rangle} = \sum_{m,n} \frac{\xi^m}{m!} (\Gamma_\ell(A)_{mn}) x^n = \frac{1}{\ell!} \langle Ax, \xi\rangle^{\ell} e^{\langle x, \xi\rangle}$$

which provides an alternative perspective on the gamma maps.

4. Bases

Now we look at the construction of the various bases for SFA$(^-, A)$. After introducing the bases in this section, we will continue to study the traces in the next.

4.1. Elementary and power sum functions. Homogeneous functions. Recall the dual rôle played by the map Γ. For the one-parameter group generated by A, we have

$$\overline{e^{tA}} = e^{t\Gamma(A)}$$

as well as

$$\Gamma(A) = \frac{\mathrm{d}}{\mathrm{d}v}\bigg|_{v=0} \overline{I + vA}$$

noting that, as a derivative map, it is linear. We start with the expansion

$$\mathcal{E}(v) = \overline{I + vA} = I + \sum_{\ell=1}^{N} v^\ell \Gamma_\ell(A) = I + \sum_{\ell=1}^{N} v^\ell E_\ell$$

i.e.,

$$E_\ell = \Gamma_\ell(A)$$

Recall the standard relation between elementary symmetric functions and power sum functions:

$$E(v) = \prod_i (1 + vx_i) = \sum_\ell v^\ell \mathrm{e}_\ell = \exp\bigl(-\sum_{k\geq 1} \frac{(-1)^k v^k}{k}\, \mathrm{p}_k\bigr)$$

Now observe the expansion

$$I + vA = \exp\Bigl(-\sum_{k\geq 1} \frac{(-1)^k v^k}{k}\, A^k\Bigr)$$

so that

$$\mathcal{E}(v) = \overline{I + vA} = \exp\left(-\sum_{k\geq 1} \frac{(-1)^k v^k}{k}\, \Gamma(A^k)\right)$$

So we have for power sum functions

$$P_k = \Gamma(A^k)$$

And for homogeneous functions, we have the generating function

$$\mathcal{H}(v) = (\mathcal{E}(-v))^{-1} = \overline{(I - vA)^{-1}} = I + \sum_{\ell\geq 1} v^\ell H_\ell \tag{4.1}$$

4.2. Monomial functions. For the monomial functions, since we are working in a coordinate-free context, we use [4], Theorem 3.12, which in this case reads

$$\overline{I + \sum_{1\leq k\leq u} c_k A^k} = \sum_{\mathcal{D}(u,N)} c^\rho M_\rho \tag{4.2}$$

where $\mathcal{D}(u,N)$ is the domain of partitions with maximum size u and length at most N. That is M_ρ is found via the coefficient c^ρ in the expansion of the left-hand side.

We will return to equation (4.2) in our study of traces of the monomial functions.

4.3. S-functions. For S-functions, we may use the Jacobi-Trudi determinants. Expand

$$\{\lambda\} = \det(\mathrm{h}_{\lambda_i - i + j})$$

as a polynomial in the h's and replace h_j by H_j. Alternatively, use the Frobenius expansion in the power sum basis:

$$\{\lambda\} = \sum_{\rho\vdash|\lambda|} \chi^\lambda_\rho \frac{1}{z_\rho} P^\rho$$

with P^ρ the matrix product $P_1^{\rho_1}\cdots P_n^{\rho_n}$ in multiplicity notation $\rho(\lambda)$, with $n = |\lambda|$.

Remark 4.1. We will come back to this formula in the next installment SFA III in the context of essentially stochastic matrices, i.e., nonnegative matrices with positive constant row sums.

5. Traces

We begin with an extension of the classical Mehler formula for the bilinear sum of Hermite polynomials in the vein of Kibble-Slepian formulas [8],[13], also e.g., [6], [7], [11]. The fundamental trace formula will follow directly. With this is hand, we will find traces for the elementary and homogeneous classes. The monomial functions will use a different approach via the Columns Theorem of [5].

5.1. Multidimensional Mehler formula. We recall the classical Mehler formula

$$K(x,\xi) = \frac{1}{\sqrt{1-\sigma^2}}\times$$
$$\exp\bigl(-\frac{1}{2}\,\frac{1}{1-\sigma^2}(\sigma^2\xi^2+\sigma^2x^2-2\sigma x\xi)\bigr) = \sum_{i\ge 0}\frac{H_i(\xi)\,\sigma^i\,H_i(x)}{i!} \tag{5.1}$$

for small enough $\sigma > 0$, where $H_i(x)$ are Hermite polynomials with generating function

$$e^{x\xi-\xi^2/2} = \sum_{i\ge 0}\frac{\xi^i}{i!}H_i(x)$$

We have the orthogonality relations

$$\int_{\mathbb{R}} H_i(x)H_j(x)\,e^{-x^2/2}dx/\sqrt{2\pi} = \delta_{ij}\,j! \tag{5.2}$$

with respect to the standard Gaussian measure. We observe the result of integrating $K(x,x)$:

Proposition 5.1. *If we integrate $K(x,x)$ with respect to standard Gaussian measure, we derive the geometric series*

$$\int_{\mathbb{R}} K(x,x)\,e^{-x^2/2}\,dx/\sqrt{2\pi} = \frac{1}{1-\sigma} = \sum_{i\ge 0}\sigma^i$$

Proof. Apply (5.2) term-by-term to (5.1) for the right-hand side. For the left-hand side, setting $x=\xi$, we have, after some simplification,

$$\begin{aligned}\frac{1}{\sqrt{1-\sigma^2}}\int_{\mathbb{R}}\exp\left(-\frac{x^2}{2}(\frac{-2\sigma}{1+\sigma})\right)\,e^{-x^2/2}dx/\sqrt{2\pi}\\ &= \frac{1}{\sqrt{1-\sigma^2}}\int_{\mathbb{R}}\exp\left(-\frac{x^2}{2}(\frac{1-\sigma}{1+\sigma})\right)\,dx/\sqrt{2\pi}\\ &= \sqrt{\frac{1+\sigma}{1-\sigma}}\,\frac{1}{\sqrt{1-\sigma^2}} = \frac{1}{1-\sigma}\end{aligned}$$

using the integral of the Gaussian measure with variance $\dfrac{1+\sigma}{1-\sigma}$. □

This is the path we will use to derive the trace formula. cf. [14, p. 99], [15, pp. 51-52],

Theorem 5.2. Trace Formula

$$\sum_{N\geq 0}\sigma^N\sum_{|m|=N}\bar{A}_{mm}=\sum_{N\geq 0}\sigma^N\,\mathrm{tr}\,\bar{A}^{(N)}=\frac{1}{\det(I-\sigma A)}$$

the superscript on $\bar{A}$ indicating the degree. Thus,

$$\mathrm{tr}\,\bar{A}^{(N)}=\mathrm{h}_N$$

the N^{th} homogeneous symmetric function in the eigenvalues of A.

Remark 5.3. Multiplying A by a diagonal matrix and using the homorphism property yields the Master Theorem of MacMahon as a version of the Trace Formula, cf. [2].

Noting the operational formula

$$e^{-(d/dx)^2/2}e^{x\xi}=e^{-(d/dx)^2/2}\sum_{i\geq 0}\frac{x^i\xi^i}{i!}=e^{x\xi-\xi^2/2}=\sum_{i\geq 0}\frac{\xi^i}{i!}H_i(x)$$

we see that the bilinear sum can be expressed as

$$e^{-(d/dx)^2/2}\sum_{i\geq 0}\frac{\sigma^i x^i}{i!}H_i(\xi)=e^{-(d/dx)^2/2}e^{\sigma x\xi-\sigma^2x^2/2} \tag{5.3}$$

For d variables x_i, we will denote the Laplacian by

$$\Delta_x=\sum_{i=1}^d\frac{\partial^2}{\partial x_i^2}$$

So a multidimensional version of (5.3) is

$$\sum_{m\geq 0}\frac{H_m(\xi)\,\sigma^{|m|}\,H_m(x)}{m!}=e^{-\frac{1}{2}\,\Delta_x}\sum_{m\geq 0}\frac{\sigma^{|m|}x^m}{m!}H_m(\xi)$$
$$=e^{-\frac{1}{2}\Delta_x}e^{\langle\sigma x,\xi\rangle-\langle\sigma x,\sigma x\rangle/2}$$

with the multivariate Hermite polynomials

$$H_m(x)=H_{m_1}(x_1)H_{m_2}(x_2)\cdots H_{m_d}(x_d)\ .$$

We need some formulas for the multivariate normal distribution.

Proposition 5.4. *Recall for the multivariate normal distribution with variance-covariance matrix R:*

$$\int_{\mathbb{R}^d}e^{-\frac{1}{2}\langle R^{-1}x,x\rangle}\,dx=\sqrt{(2\pi)^d\,\det R} \tag{5.4}$$

and the Fourier transform (characteristic function)

$$e^{-\frac{1}{2}\langle Rx,x\rangle}=\int_{\mathbb{R}^d}e^{i\langle x,y\rangle}e^{-\frac{1}{2}\langle R^{-1}y,y\rangle}\,\frac{dy}{\sqrt{(2\pi)^d\,\det R}} \tag{5.5}$$

Now start with (3.2)

$$e^{\langle Ax,\xi\rangle}=\sum_{m,n}\frac{\xi^m}{m!}\bar{A}_{mn}x^n$$

and

$$\begin{aligned}e^{-\frac{1}{2}\,\Delta_\xi}e^{\langle Ax,\xi\rangle}&=e^{\langle Ax,\xi\rangle}e^{-\frac{1}{2}\langle Ax,Ax\rangle}\\&=e^{-\frac{1}{2}\,\Delta_\xi}\sum_{m,n}\frac{\xi^m}{m!}\bar{A}_{mn}x^n=\sum_{m,n}\frac{H_m(\xi)}{m!}\bar{A}_{mn}x^n\end{aligned}$$

and we have a bilinear sum, call it $\mathcal{S}$,

$$\begin{aligned}\mathcal{S}=\sum_{m,n}\frac{H_m(\xi)}{m!}\bar{A}_{mn}H_n(x)&=e^{-\frac{1}{2}\,\Delta_x}\sum_{m,n}\frac{H_m(\xi)}{m!}\bar{A}_{mn}x^n\\&=e^{-\frac{1}{2}\,\Delta_x}e^{\langle Ax,\xi\rangle-\frac{1}{2}\langle Ax,Ax\rangle}\end{aligned}$$

We will assume A small so that the matrices appearing in the following are positive definite, symmetric. After the computation, we will replace A by σA to guarantee existence/convergence of the series' for any given A by choosing σ sufficiently small. Apply (5.5) with $R=A^\top A$, where we will be using $^\top$ to denote transpose:

$$\begin{aligned}\mathcal{S}&=e^{-\frac{1}{2}\,\Delta_x}e^{\langle Ax,\xi\rangle}\int_{\mathbb{R}^d}e^{i\langle y,x\rangle}e^{-\frac{1}{2}\langle (A^\top A)^{-1}y,y\rangle}\,\frac{dy}{\sqrt{(2\pi)^d\,\det A^\top A}}\\&=e^{-\frac{1}{2}\,\Delta_x}\int_{\mathbb{R}^d}e^{\langle x,A^T\xi+iy\rangle}e^{-\frac{1}{2}\langle (A^\top A)^{-1}y,y\rangle}\,\frac{dy}{\sqrt{(2\pi)^d\,\det A^\top A}}\\&=\int_{\mathbb{R}^d}e^{\langle Ax,\xi\rangle+i\langle y,x\rangle}e^{-\frac{1}{2}\langle A^\top\xi,A^\top\xi\rangle-i\langle A^\top\xi,y\rangle+\frac{1}{2}\langle y,y\rangle}\,\frac{e^{-\frac{1}{2}\langle (A^\top A)^{-1}y,y\rangle}}{\sqrt{(2\pi)^d\,\det A^\top A}}\,dy\\&=e^{\langle Ax,\xi\rangle-\frac{1}{2}\langle A^\top\xi,A^\top\xi\rangle}\int_{\mathbb{R}^d}e^{i\langle y,x-A^\top\xi\rangle}e^{-\frac{1}{2}\langle ((A^\top A)^{-1}-I)y,y\rangle}\,\frac{dy}{\sqrt{(2\pi)^d\,\det A^\top A}}\end{aligned}$$

We want to apply (5.5) now with $R=((A^\top A)^{-1}-I)^{-1}$. We have a simple lemma

Lemma 5.5. *With appropriate existence of inverses, we have*

$$((A^\top A)^{-1}-I)^{-1}=(A^\top A)(I-A^\top A)^{-1}$$

Proof. Let $X=((A^\top A)^{-1}-I)^{-1}$, so that $X((A^\top A)^{-1}-I)=I$ so, multiplying through by $A^\top A$,

$$X(I-A^\top A)=A^\top A\;\Rightarrow\;X=(A^\top A)(I-A^\top A)^{-1}$$

as required. □

Now (5.5) yields, appropriately cancelling the determinant factors and the factors of 2π,

$$\mathcal{S}=\sqrt{\frac{1}{\det(I-A^\top A)}}\,e^{\langle Ax,\xi\rangle-\frac{1}{2}\langle A^\top\xi,A^\top\xi\rangle}e^{-\frac{1}{2}\langle x-A^\top\xi,(A^\top A)(I-A^\top A)^{-1}(x-A^\top\xi)\rangle}$$

Now look at terms in the exponent, expand the quadratic form in $x - A^\top\xi$ and combine terms. First, combining with the cross terms,

$$\begin{aligned}&\langle x, A^\top\xi\rangle + \langle x, (A^\top A)(I - A^\top A)^{-1}A^\top\xi\rangle\\ &= \langle x, (I - A^\top A)(I - A^\top A)^{-1}A^\top\xi\rangle + \langle x, (A^\top A)(I - A^\top A)^{-1}A^\top\xi\rangle\\ &= \langle x, (I - A^\top A)^{-1}A^\top\xi\rangle\end{aligned}$$

and, with a factor of $-1/2$,

$$\begin{aligned}\langle A^\top\xi, A^\top\xi\rangle + \langle A^\top\xi, (A^\top A)(I - A^\top A)^{-1}A^\top\xi\rangle&\\ = \langle A^\top\xi, (I - A^\top A)(I - A^\top A)^{-1}A^\top\xi\rangle&\\ + \langle A^\top\xi, (A^\top A)(I - A^\top A)^{-1}A^\top\xi\rangle&\\ = \langle A^\top\xi, (I - A^\top A)^{-1}A^\top\xi\rangle&\end{aligned}$$

And we have

Theorem 5.6. Extended Mehler formula

$$\sum_{\substack{N\geq 0\\ |m|=|n|=N}} \frac{H_m(\xi)\bar{A}_{mn}H_n(x)}{m!} = \sqrt{\frac{1}{\det(I-A^\top A)}} \times$$

$$e^{-\frac{1}{2}\langle x, A^\top A(I-A^\top A)^{-1}x\rangle + \langle x,(I-A^\top A)^{-1}A^\top\xi\rangle - \frac{1}{2}\langle A^\top\xi,(I-A^\top A)^{-1}A^\top\xi\rangle}$$

and

Corollary 5.7. *For symmetric A,*

$$\sum_{\substack{N\geq 0\\ |m|=|n|=N}} \frac{H_m(\xi)\bar{A}_{mn}H_n(x)}{m!} = \sqrt{\frac{1}{\det(I-A^2)}} \times$$

$$e^{-\frac{1}{2}\langle x, A^2(I-A^2)^{-1}x\rangle + \langle x,(I-A^2)^{-1}A\xi\rangle - \frac{1}{2}\langle \xi, A^2(I-A^2)^{-1}\xi\rangle}$$

which yields (5.1) for scalar $A = \sigma$. Now, for the trace formula, we follow the steps as for Proposition 5.1, i.e., we set $x = \xi$ and integrate the expression above with respect to standard Gaussian measure on $\mathbb{R}^d$. Setting $x = \xi$, the terms in the exponent combine to yield, for convenience using fractional notation for matrix inverses,

$$\sum_{\substack{N\geq 0\\ |m|=|n|=N}} \frac{H_m(x)\bar{A}_{mn}H_n(x)}{m!} = \det(I - A^2)^{-1/2}\ \exp\bigl(\langle x, \frac{A}{I+A}x\rangle\bigr)$$

adding in the exponent from the Gaussian measure, $-\frac{1}{2}\langle x, x\rangle$, we have for the integrand, modulo the factor of $(2\pi)^{-d/2}$, which will cancel upon integration,

$$\det(I - A^2)^{-1/2}\ \exp\bigl(\langle x, \frac{I-A}{I+A}x\rangle\bigr)$$

and applying (5.4) yields, for positive definite $(I-A)/(I+A)$,

$$\sum_{N\geq 0} \operatorname{tr} \bar{A}^{(N)} = \sqrt{\frac{1}{\det(I-A^2)}}\sqrt{\det\frac{I+A}{I-A}} = \frac{1}{\det(I-A)}$$

and scaling by σ recovers 5.2. Thus, we have shown the trace formula

$$\operatorname{tr} \bar{A}^{(N)} = \mathrm{h}_N$$

for symmetric A, in particular for diagonal matrices and hence diagonalizable matrices. A density argument implies the result for general A. Alternatively, we observe that the induced matrix map with lexicographic ordering preserves upper triangularity and so using a similarity transformation to upper-triangular form puts the eigenvalues on the diagonal and hence the trace will be the same as for a corresponding diagonal matrix.

Remark 5.8. Knowing the result, one sees directly that the eigenvalues of the induced matrix are the monomials homogeneous of degree N in the eigenvalues of the original matrix, hence the sum over diagonal elements yields the homogeneous symmetric function in the eigenvalues. Cf., [10, pp. 178-179, 182].

Remark 5.9. We note the simplest application of the trace formula. Since $\bar{I}$ is the identity of the appropriate size, $\operatorname{tr}\bar{I}$ will provide exactly that information, i.e., the size of the matrix $\bar{A}$ in degree N for A a $d\times d$ matrix. With the trace formula as follows

$$\frac{1}{\det(I-tA)} = \sum_{N\geq 0} t^N \ \operatorname{tr} A^{(N)}$$

we let $A=I$ to get

$$\sum_{N\geq 0} t^N \ \operatorname{tr} I^{(N)} = (1-t)^{-d} = \sum_{N\geq 0} t^N \binom{N+d-1}{N}$$

the coefficient of t^N indeed the number of monomials of homogeneous degree N in d variables.

5.2. Traces for E_ℓ, Γ and power sum functions. We want the trace, for each $N>0$, of $\mathcal{E}(v)$ which will yield $\operatorname{tr} E_\ell = \operatorname{tr}\Gamma_\ell(A)$, in particular, $\operatorname{tr}\Gamma(A)$. From the trace formula we want the expansion in t and then the expansion in v:

$$\begin{aligned}\frac{1}{\det(I-t(I+vA))} &= \frac{1}{(1-t)^d\det(I-\dfrac{vt}{1-t}A)}\\ &= \frac{1}{(1-t)^d}\sum_\ell \frac{v^\ell t^\ell}{(1-t)^\ell}\,\mathrm{h}_\ell\\ &= \sum v^\ell \sum t^{k+\ell}\binom{\ell+d+k-1}{k}\mathrm{h}_\ell\end{aligned}$$

We want the coefficient of t^N, i.e., $k+\ell=N$ which yields

$$\sum_\ell v^\ell \sum_N t^N \binom{N+d-1}{N-\ell}\,\mathrm{h}_\ell$$

Hence

Proposition 5.10. *For E_ℓ we have*

$$\operatorname{tr} E_\ell = \binom{N+d-1}{N-\ell} \mathrm{h}_\ell$$

where $\mathrm{h}_\ell = \operatorname{tr} \bar{A}^{(\ell)}$ are the homogeneous symmetric functions in the eigenvalues of A.

Noting that $\mathrm{h}_1 = \mathrm{e}_1 = \operatorname{tr} A$,

Corollary 5.11. *In degree N, we have*

$$\operatorname{tr} \Gamma(A) = \binom{N+d-1}{N-1} \operatorname{tr} A$$

And hence for power sum functions,

Corollary 5.12. *In degree N, we have*

$$\operatorname{tr} P_i = \binom{N+d-1}{N-1} \mathrm{p}_i$$

with $\mathrm{p}_i = \operatorname{tr} A^i$, the power sums of the eigenvalues of A.

Proof. For $P_i = \Gamma(A^i)$ we have

$$\operatorname{tr} P_i = \operatorname{tr} \Gamma(A^i) = \binom{N+d-1}{N-1} \operatorname{tr} A^i$$

as required. □

5.3. Traces for H_ℓ. Beginning as in the previous section, we consider

$$\begin{aligned}
\frac{1}{\det(I - t(I-vA)^{-1})} &= \frac{\det(I-vA)}{\det(I-vA-tI)} \\
&= \frac{1}{(1-t)^d} \frac{\det(I-vA)}{\det(I-\frac{v}{1-t}A)} \\
&= \sum_j (-1)^j v^j \mathrm{e}_j \sum_k \frac{v^k}{(1-t)^{d+k}} \mathrm{h}_k \\
&= \sum_{j,k} (-1)^j v^{j+k} \binom{N+d+k-1}{N} t^N \mathrm{e}_j h_k
\end{aligned}$$

So that

Proposition 5.13. *In degree N we have*

$$\operatorname{tr} H_\ell = \sum_{k=0}^{\ell} \binom{N+d+k-1}{N} (-1)^{\ell-k} \mathrm{e}_{\ell-k} \mathrm{h}_k$$

Now we invoke Proposition 3.18 of [4], in the form

$$\mathrm{h}_k\, e_{\ell-k} = \sum_{\lambda \vdash \ell} \binom{L(\lambda)}{\ell-k} \mathrm{m}_\lambda \tag{5.6}$$

We have the following result:

Proposition 5.14. *In degree N we have the trace of H_ℓ in terms of monomial symmetric functions*

$$\operatorname{tr} H_\ell = \sum_{\alpha\ge 1} \sum_{\substack{\lambda\vdash\ell \\ L(\lambda)=\alpha}} \binom{N+\ell+d-\alpha-1}{N-\alpha} \mathrm{m}_\lambda$$

Proof. Combining (5.6) with Proposition 5.13 for $L(\lambda)=\alpha$, we want to show

$$\sum_k (-1)^{\ell-k}\binom{\alpha}{\ell-k}\binom{N+d+k-1}{N} = \binom{N-\alpha+d+\ell-1}{N-\alpha}$$

Writing out in terms of factorials we have

$$\sum_{\ell-\alpha\le k\le\ell} \frac{(-1)^{\ell-k}\,\alpha!}{(\alpha-\ell+k)!(\ell-k)!}\,\frac{(N+d+k-1)!}{N!(d+k-1)!} = \frac{(N-\alpha+d+\ell-1)!}{(N-\alpha)!(d+\ell-1)!}$$

Rearranging, substituting $k=\ell-u$, and rewriting ratios of factorials in terms of rising factorials, we have, setting $x=d+\ell$, $d+k=x-u$ and

$$\sum_{u=0}^{\alpha}(-1)^u\binom{\alpha}{u}(x-u)_N = \frac{N!}{(N-\alpha)!}\,(x)_{N-\alpha}$$

We give an operational proof of this identity. Noting that

$$e^{\pm D}f(x) = f(x\pm 1)$$

and

$$\begin{aligned}(1-e^{-D})(x)_N = (x)_N-(x-1)_N &= (x)_{N-1}[x+N-1-(x-1)]\\ &= N\,(x)_{N-1}\end{aligned}$$

Iterating, we have,

$$\begin{aligned}\sum_{u=0}^{\alpha}(-1)^u\binom{\alpha}{u}(x-u)_N &= \sum_{u=0}^{\alpha}(-1)^u\binom{\alpha}{u}e^{-uD}\,(x)_N\\ &= (1-e^{-D})^\alpha(x)_N = \frac{N!}{(N-\alpha)!}\,(x)_{N-\alpha}\end{aligned}$$

as required. ☐

Remark 5.15. This is a manifestation of the boson operators $a^\dagger = xe^D$, $a=1-e^{-D}$, satisfying $[a,a^\dagger]=I$, the identity map, with $(a^\dagger)^N=(x)_N\,e^{ND}$ so that, with the constant function 1 as a vacuum state, we have $a1=0$, $a(a^\dagger)^N\,1 = N(a^\dagger)^{N-1}1$, and so on.

5.4. Traces for monomial functions. Here we will focus on diagonal matrices. We will build a matrix so that the corresponding induced matrix has columns that are the diagonal entries for corresponding monomial functions matrices. For a diagonalizable A, this approach is an effective way to find the M_λ's via similarity transformation.

5.4.1. *Generating monomial functions for diagonal matrices.* Recall the Columns Theorem from [4]:

Theorem 5.16. Columns Theorem. *For any matrix A, let Λ_j be the diagonal matrix formed from column j of A. Let*

$$\Lambda = \sum v_j \Lambda_j \ .$$

Then the coefficient of v^n in the level N induced matrix $\bar{\Lambda}$ is a diagonal matrix with entries the n^{th} column of $\bar{A}$.

We will use the idea of this theorem in reverse. That is, we construct a matrix with columns from diagonal matrices and then use it to find the induced matrices of interest. We are interested in the traces of the monomial functions matrices. Our basic formula is

$$\overline{I + \sum_{1\le k\le u} c_k A^k} = \sum_{\mathcal{D}(u,N)} c^\rho M_\rho$$

Assume $A = \operatorname{diag}(\alpha_1, \alpha_2, \dots, \alpha_d)$. Then the ii entry of $S = I + \sum\limits_{1\le k\le u} c_k A^k$ is

$$S_{ii} = 1 + c_1\alpha_i + c_2\alpha_i^2 + \cdots + c_u\alpha_i^u$$

For the induced matrix we are looking at the expansion of

$$(1 + c_1\alpha_1 + c_2\alpha_1^2 + \cdots + c_u\alpha_1^u)^{m_1}(1 + c_1\alpha_2 + c_2\alpha_2^2 + \cdots + c_u\alpha_2^u)^{m_2} \cdots (1 + c_1\alpha_d + c_2\alpha_d^2 + \cdots + c_u\alpha_d^u)^{m_d}$$

Choose $u \ge d-1$ as this will give all of the M_λ's with $\lambda_1 \le u$. If we form the $(u+1)\times(u+1)$ matrix Y with first column 1's in rows 1 through d, entries

$$Y_{ij} = \alpha_i^{j-1} \qquad \text{for } 1 \le i \le d, 1 < j \le u+1$$

and $Y_{ij} = 0$ for $i > d$, then the expression used to compute $\bar{Y}_{mn}$ is, e.g.

$$\prod_{i=1}^{u+1} (Y_{i1}x_1 + Y_{i2}x_2 + \cdots + Y_{i\,u+1}x_{u+1})^{m_i} = \prod_{i=1}^{d} (x_1 + \alpha_i x_2 + \cdots + \alpha_i^u x_{u+1})^{m_i}$$

This is the same expansion as above if we identify $x_1 = 1$, $x_j = c_{j-1}$, $1 < j \le u+1$. The exponents of the zero rows are zeros, giving factors of 1. As long as we interpret the indexing appropriately, the columns of $\bar{Y}$ will be the diagonals of the M_λ's. In the column label n for $\bar{Y}$, we interpret n_1 as the power of 1, so it is ignored. Then we have the partition label

$$\rho = (1^{\rho_1}\, 2^{\rho_2} \cdots u^{\rho_u}) = (1^{n_2}\, 2^{n_3} \cdots u^{n_{u+1}})$$

corresponding to the coefficient monomial $c_1^{n_2} c_2^{n_3} \cdots c_u^{n_{u+1}}$.

Example 5.17. For $d = 3$, $u = 2$, we have

$$Y = \begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 \\ 1 & \alpha_2 & \alpha_2^2 \\ 1 & \alpha_3 & \alpha_3^2 \end{pmatrix}$$

Taking $N = 2$, we have

$$\bar{Y} = \begin{pmatrix} 1 & 2\,\alpha_1 & 2\,\alpha_1{}^2 & \alpha_1{}^2 & 2\,\alpha_1{}^3 & \alpha_1{}^4 \\ 1 & \alpha_1+\alpha_2 & \alpha_1{}^2+\alpha_2{}^2 & \alpha_1\alpha_2 & \alpha_1{}^2\alpha_2+\alpha_1\alpha_2{}^2 & \alpha_1{}^2\alpha_2{}^2 \\ 1 & \alpha_1+\alpha_3 & \alpha_1{}^2+\alpha_3{}^2 & \alpha_1\alpha_3 & \alpha_1{}^2\alpha_3+\alpha_1\alpha_3{}^2 & \alpha_1{}^2\alpha_3{}^2 \\ 1 & 2\,\alpha_2 & 2\,\alpha_2{}^2 & \alpha_2{}^2 & 2\,\alpha_2{}^3 & \alpha_2{}^4 \\ 1 & \alpha_2+\alpha_3 & \alpha_2{}^2+\alpha_3{}^2 & \alpha_2\alpha_3 & \alpha_2{}^2\alpha_3+\alpha_2\alpha_3{}^2 & \alpha_2{}^2\alpha_3{}^2 \\ 1 & 2\,\alpha_3 & 2\,\alpha_3{}^2 & \alpha_3{}^2 & 2\,\alpha_3{}^3 & \alpha_3{}^4 \end{pmatrix}$$

The column labels are

$$(200),\ (110),\ (101),\ (020),\ (011),\ (002)$$

The column labelled (011) gives the elements of the diagonal corresponding to c_1c_2 with corresponding $\rho = (1^1 2^1) = \rho(\lambda)$ with $\lambda = [21]$. Thus

$$M_{(21)} =$$
$$\operatorname{diag}(2\,\alpha_1{}^3, \alpha_1{}^2\alpha_2+\alpha_1\alpha_2{}^2, \alpha_1{}^2\alpha_3+\alpha_1\alpha_3{}^2, 2\,\alpha_2{}^3, \alpha_2{}^2\alpha_3+\alpha_2\alpha_3{}^2, 2\,\alpha_3{}^3)$$

Similarly, the label for column 4 reads as c_1^2, with $\lambda = [11]$, column 6 corresponds to $\lambda = [22]$.

5.4.2. *Traces.* So if we can compute the column sums for $\bar{Y}$, we will have the traces for M_λ's. The taking of induced matrices introduces a slight variation.

Notation. Denote the $d \times d$ all 1's matrix by J and the $\binom{N+d-1}{N} \times \binom{N+d-1}{N}$ all 1's matrix by $\mathfrak{J}$.

Proposition 5.18. *The matrix J satisfies*

$$\bar{J} = \mathfrak{J}\mathcal{B}$$

where $\mathcal{B}$, the diagonal of $\bar{J}$ has entries the multinomial coefficients

$$\mathcal{B}_{nn} = \frac{N!}{n!}$$

with $|n| = N$.

Proof. For $\bar{J}$ we are looking at expanding

$$\prod_i (x_1 + \cdots + x_d)^{m_i} = (x_1 + \cdots + x_d)^N = \sum_n \frac{N!}{n!} x^n$$

so all rows are the same and factoring out the common factors in each column gives the result. □

In other words, we want to calculate $\overline{JY}$ which will yield

$$\overline{JY} = \mathfrak{J}\mathcal{B}\bar{Y}$$

i.e. we will determine $\operatorname{tr}\mathcal{B}M_\lambda$ rather than $\operatorname{tr} M_\lambda$. We proceed accordingly. Note that every row of JY has the form

$$(d, \mathrm{p}_1, \mathrm{p}_2, \ldots, \mathrm{p}_u)$$

with p_k the k^{th} power sum function of the α's. And $\overline{JY}$ has identical rows with first row of the form

$$(\ldots, \frac{N!}{n!} d^{n_1} \mathrm{p}_1^{n_2} \mathrm{p}_2^{n_3} \cdots \mathrm{p}_u^{n_{u+1}}, \ldots)$$

Thus the result:

Proposition 5.19. *With appropriate interpretation of the indices, we have, for diagonal A,*

$$\operatorname{tr} \mathcal{B} M_\lambda = \frac{N!}{n!} d^{n_1} \mathrm{p}_1^{n_2} \mathrm{p}_2^{n_3} \cdots \mathrm{p}_u^{n_{u+1}}$$

Remark 5.20. For diagonalizable A, we can use a similarity transformation to find the corresponding traces, noting that one must use a similarity transformation on $\mathcal{B}$ as well.

Example 5.21. Continuing Example 5.17, we have the first row of $\overline{JY}$

$$(9, 6\,\alpha_1 + 6\,\alpha_2 + 6\,\alpha_3, 6\,{\alpha_1}^2 + 6\,{\alpha_2}^2 + 6\,{\alpha_3}^2, (\alpha_1 + \alpha_2 + \alpha_3)^2\,,$$
$$2\,\left({\alpha_1}^2 + {\alpha_2}^2 + {\alpha_3}^2\right)(\alpha_1 + \alpha_2 + \alpha_3)\,, \left({\alpha_1}^2 + {\alpha_2}^2 + {\alpha_3}^2\right)^2)$$

which can be compared to the column sums for $\bar{Y}$, noting that multiplying M_λ by $\mathcal{B}$ is the same as multiplying each entry of a column of $\bar{Y}$ by the corresponding multinomial coefficient. In our example, we have

$$\mathcal{B} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

which yields the corresponding values for the traces.

Remark 5.22. We do not yet have general trace formulas for Schur functions.

6. Further Examples and Remarks

6.1. Matrix of coefficients expressing h_λ in terms of m_μ's. Equation(s) (2.2) gives the number of terms in a generic entry of $\bar{P}_{mn}$ according to the number of matrices of natural numbers (n_{ij}) satisfying

$$\sum_j n_{ij} = m_i \quad \text{and} \quad \sum_i n_{ij} = n_j$$

We can identify a multi-index with a corresponding partition by setting the components in (weakly) decreasing order. And we recall, [12, 6.7 (ii)], that the number of such matrices for given partitions λ and μ is the coefficient of m_μ in the expansion of $\mathrm{h}_\lambda = \mathrm{h}_{\lambda_1} \mathrm{h}_{\lambda_2} \cdots \mathrm{h}_{\lambda_L}$. With $P = (p_{ij})$, 4×4, taking $N = 4$ we have, for index (2110)

$$\mathrm{h}_2 \mathrm{h}_1^2 = \mathrm{m}_4 + 3\,\mathrm{m}_{31} + 4\,\mathrm{m}_{22} + 7\,\mathrm{m}_{211} + 12\,\mathrm{m}_{1111}$$

with, e.g.,

$$\bar{P}_{(2110),(2200)} = {p_{11}}^2 p_{22}p_{32} + 2\,p_{32}p_{21}p_{11}p_{12} + 2\,p_{31}p_{11}p_{12}p_{22} + p_{31}{p_{12}}^2 p_{21}$$

the number of terms in $\bar{P}_{(2110),(2110)}$ being 7, the number of terms in $\bar{P}_{(2110),(1111)}$ is 12, etc., each term being of the form $\prod_{i,j} p_{ij}^{n_{ij}}$ with the coefficients in $\bar{P}$ coming from multinomials.

6.2. Further information about Γ_ℓ maps. First, we observe

Proposition 6.1. *We have*

$$\Gamma_N(A) = \bar{A}$$

Proof. Look at the expansion, eq. (2.5),

$$\prod_i (x_i + vy_i)^{m_i} = \sum v^\ell (\Gamma_\ell(A))_{mn}\, x^n$$

with $y_i = \sum_j A_{ij}x_j$. Notice that the only way to get v^N is to take vy_i in the product every time, so that

$$\prod_i (vy_i)^{m_i} = v^N \sum_n (\Gamma_N(A))_{mn}\, x^n = v^N \sum_n \bar{A}_{mn}\, x^n \ .$$

□

Noting that the algebra SFA$(^-, A)$ is generated by $\{E_\ell\}_{0\le\ell\le N}$, with $E_0 = I$, we have the

Corollary 6.2. *$\bar{A}$ itself is an element of* SFA$(^-, A)$.

as it should be. And another interesting observation:

Proposition 6.3. *For invertible A, we have the relations*

$$\Gamma_N(A)\,\Gamma_\ell(A^{-1}) = \Gamma_{N-\ell}(A)$$

Proof. We have

$$\overline{I + vA} = \overline{vA}\ \overline{v^{-1}A^{-1} + I} = (v^N \bar{A}) \sum_\ell v^{-\ell}\Gamma_\ell(A^{-1})$$
$$= \bar{A} \sum_\ell v^{N-\ell}\Gamma_\ell(A^{-1}) = \sum_\ell v^{N-\ell}\Gamma_{N-\ell}(A)$$

and the previous Proposition applies to yield the result. □

We have an extension of the power sums:

Proposition 6.4. *For $\alpha > 0$, $\ell \ge 1$, we have*

$$\Gamma_\ell(A^\alpha) = M_{(\alpha^\ell)}$$

Proof. Start with

$$\overline{I + vA^\alpha} = \sum_\ell v^\ell \Gamma_\ell(A^\alpha)$$

and compare with, where $v = c_\alpha$,

$$\overline{I + c_1 A + \cdots + vA^\alpha} = \sum_{\mathcal{D}(\alpha,N)} c_1^{\rho_1} c_2^{\rho_2} \cdots v^{\rho_\alpha} \, M_\rho$$

So these expressions agree if $c_1 = c_2 = \cdots = c_{\alpha-1} = 0$ leaving

$$I + \sum_{\mathcal{D}(\alpha,N)} v^{\rho_\alpha} \, M_{(\alpha^{\rho_\alpha})}$$

with the sum over $1 \le \rho_\alpha \le N$. Replacing ρ_α by ℓ yields the result. □

For example, extending $P_\alpha = M_{(\alpha)} = \Gamma(A^\alpha)$, we have $\Gamma_2(A^\alpha) = M_{(\alpha\alpha)}$, $\Gamma_3(A^\alpha) = M_{(\alpha\alpha\alpha)}$, etc. This agrees as well with $E_\ell = \Gamma_l(A) = M_{(1^\ell)}$.

6.3. Extended Cayley-Hamilton identities.

Theorem 6.5. Extended polynomial identities

1. If A satisfies a polynomial identity of degree u

$$c_0 I + c_1 A + \cdots + c_u A^u = 0$$

then we have the identity and recurrence, for the power sum functions, with $k > u$,

$$c_0 \, N\bar{I} + c_1 P_1 + c_2 \, P_2 + \cdots + c_u P_u = 0$$
$$c_0 P_{k-u} + c_1 P_{k-u+1} + \cdots + c_u P_k = 0$$

2. If A satisfies a polynomial identity of degree u, with lowest degree term of power $\alpha \ge 0$ having coefficient equal to 1,

$$A^\alpha + c_1 A^{\alpha+1} + c_2 A^{\alpha+2} + \cdots + c_{u-\alpha} A^u = 0$$

then we have the identity for monomial functions M_λ

$$(\bar{A})^\alpha \sum_{\mathcal{D}(u-\alpha,N)} c^\rho \, M_\rho = 0$$

Proof. Given the first identity, applying Γ to each term and converting $\Gamma(A^i) = P_i$ gives the first relation. For the recurrence, first multiply through by $A^{k-u}$and then apply Γ. For the second identity, first factor out A^α and then apply the formula for monomial functions, §4.2. □

We will illustrate these identities in our example below.

6.4. An illustrative example. Here we look at a very simple model that illustrates a variety of the features of the multinomial chains and the SFA algebra. Start with the stochastic matrix

$$P = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix}$$

which is the transition matrix for a 2-state Markov chain. Initially, we are considering $N = 1$, i.e., one ball jumping between the slots. From state 2 the system immediately jumps back to state 1. With N balls, all of the balls in slot 2 will jump back to slot 1, with the states now of the form $(N - m, m)$ with $0 \le m \le N$. So at each step, a binomial experiment is performed, $N - m$ trials with probability $1/2$ that a ball jumps to slot 2, meanwhile slot 2 is emptied out, all balls jumping back to slot 1. Note that the states are effectively labelled by the number of balls in slot 2. We have

$$\bar{P}_{(N-m,m),(N-n,n)} = \binom{N-m}{n} \frac{1}{2^{N-m}} \tag{6.1}$$

for the probability of transition to state $(N - n, n)$.

Example 6.6. For N equal to 3 and 4 we have

$$\bar{P}^{(3)} = \begin{pmatrix} \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \bar{P}^{(4)} = \begin{pmatrix} \frac{1}{16} & \frac{1}{4} & \frac{3}{8} & \frac{1}{4} & \frac{1}{16} \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and so on. The trace formula gives

$$\begin{aligned} \frac{1}{1 - \frac{t}{2} - \frac{t^2}{2}} &= 1 + \sum_{N>0} t^N \operatorname{tr} \bar{P}^{(N)} \\ &= 1 + \frac{1}{2}\,t + \frac{3}{4}\,t^2 + \frac{5}{8}\,t^3 + \frac{11}{16}\,t^4 + \frac{21}{32}\,t^5 + \frac{43}{64}\,t^6 + \frac{85}{128}\,t^7 + \cdots \\ &= \sum_{N\ge 0} t^N \,\Big(\sum_{k=0}^{\lfloor N/2 \rfloor} \binom{N-k}{k} \frac{1}{2^{N-k}}\Big) \end{aligned}$$

via (6.1). It is interesting to look at the Γ-matrices, which provide the elementary functions E_ℓ. For $N = 4$, we have, say, $\Gamma_1(P)$ and $\Gamma_3(P)$, rescaled to stochastic matrices,

$$E_1/4 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{3}{8} & \frac{3}{8} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{3}{4} & \frac{1}{8} & \frac{1}{8} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad E_3/4 = \begin{pmatrix} \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & 0 \\ \frac{3}{16} & \frac{13}{32} & \frac{9}{32} & \frac{3}{32} & \frac{1}{32} \\ \frac{1}{4} & \frac{3}{8} & \frac{1}{4} & \frac{1}{8} & 0 \\ \frac{1}{4} & \frac{3}{8} & \frac{3}{8} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

E_1/N corresponds to the associated process where 1 out of the N balls jumps at each step. Similarly, $E_3/\binom{N}{3}$ corresponds to the process where 3 of the 4 balls are chosen for possible change of slot. The states, rows and columns, have labels $(40), (31), (22), (13), (04)$, with (31), e.g., the state with 1 ball in slot 2.

6.4.1. *Diagonalization.* We find a matrix of left eigenvectors and corresponding eigenvalues

$$W = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1/2 \end{pmatrix}$$

so that $P = W^{-1}\Lambda W$. We see immediately that

$$\lim_{k\to\infty} P^k = W^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W = \begin{pmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{pmatrix}$$

with rows being the invariant distribution for the underlying chain. We use the technique for finding the traces to find the M_λ's. Write $\tilde{M}_\lambda$ for diagonalized M_λ. Now form the matrix $X = I + c_1\Lambda + c_2\Lambda^2$, so that

$$X = \begin{pmatrix} 1 + c_1 + c_2 & 0 \\ 0 & 1 - c_1/2 + c_2/4 \end{pmatrix}$$

On the other hand we can take the diagonals of Λ, and Λ^2 and form

$$Y = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1/2 & 1/4 \\ 0 & 0 & 0 \end{pmatrix}$$

as in Example 5.17, here with $d = 2$, $u = 2$. Let us take $N = 2$ for convenience as $N = 4$ would lead to 15×15 matrices. So, with rows and columns labelled,

$$\bar{Y}^{(2)} = \begin{array}{c} \\ (200) \\ (110) \\ (101) \\ (020) \\ (011) \\ (002) \end{array} \begin{array}{c} \begin{array}{cccccc} (200) & (110) & (101) & (020) & (011) & (002) \end{array} \\ \begin{pmatrix} 1 & 2 & 2 & 1 & 2 & 1 \\ 1 & 1/2 & 5/4 & -1/2 & -1/4 & 1/4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1/2 & 1/4 & -1/4 & 1/16 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

The n_1 component corresponds to a power of 1 and is ignored. For $u = 2$, $N = 2$ we are looking at partitions with maximum size 2 and maximum length 2. Interpret (020), e.g., as c_1^2, i.e., $\lambda = [11]$, etc. For $N = 2$ we have 3×3 matrices. We find

$$\begin{aligned} \tilde{M}_{(1)} &= \mathrm{diag}(2, 1/2, -1) \\ \tilde{M}_{(2)} &= \mathrm{diag}(2, 5/4, 1/2) \\ \tilde{M}_{(11)} &= \mathrm{diag}(1, -1/2, 1/4) \\ \tilde{M}_{(21)} &= \mathrm{diag}(2, -1/4, -1/4) \\ \tilde{M}_{(22)} &= \mathrm{diag}(1, 1/4, 1/16) \end{aligned}$$

Now we need $\bar{W}$, with $N = 2$, for the similarity transformation to recover the original M_λ's.

$$\bar{W} = \begin{pmatrix} 4/9 & 4/9 & 1/9 \\ 2/9 & -1/9 & -1/9 \\ 1/9 & -2/9 & 1/9 \end{pmatrix}$$

Using $M_\lambda = \bar{W}^{-1}\tilde{M}_\lambda\bar{W}$, we find

$$M_{(1)} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 2 & 0 \end{pmatrix}, \quad M_{(2)} = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{5}{4} & \frac{1}{4} \\ 0 & 1 & 1 \end{pmatrix}$$

$$M_{(11)} = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad M_{(21)} = \begin{pmatrix} \frac{3}{4} & 1 & \frac{1}{4} \\ 1 & \frac{3}{4} & \frac{1}{4} \\ 1 & 1 & 0 \end{pmatrix}$$

$$M_{(22)} = \begin{pmatrix} \frac{9}{16} & \frac{3}{8} & \frac{1}{16} \\ \frac{3}{8} & \frac{1}{2} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

We can check $M_{(1)} = E_1$, $M_{(2)} = P_2$, $M_{(11)} = E_2 = \bar{P}$, since $N = 2$.

6.4.2. *Cayley-Hamilton identities.* Illustrating Theorem 6.5 for $P = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix}$, we have, via the characteristic polynomial,

$$I + P - 2P^2 = 0 \tag{6.2}$$

Applying Γ yields, here with P_i the power sum functions of SFA$(^-, P)$,

$$NI + P_1 - 2P_2 = 0 \tag{6.3}$$

with I of the appropriate size, and, after rearranging, the recurrence, for $k > 2$,

$$P_k = \tfrac{1}{2}(P_{k-1} + P_{k-2})$$

And from eq. (6.2), as the form $I + c_1P + c_2P^2$, with $c_1 = 1$, $c_2 = -2$, we have

$$0 = \sum_{\mathcal{D}(2,N)} c_1^{\rho_1} c_2^{\rho_2} M_{(1^{\rho_1}2^{\rho_2})} = \sum_{\mathcal{D}(2,N)} (-2)^{\rho_2} M_{(1^{\rho_1}2^{\rho_2})}$$

with the sum over $0 \le \rho_1 + \rho_2 \le N$. In our example, with $N = 2$, we have the sum

$$0 = I + c_1M_{(1)} + c_2M_{(2)} + c_1^2M_{(11)} + c_1c_2M_{(21)} + c_2^2M_{(22)}$$

and converting various M_λ's accordingly, this can be expressed in the form,

$$0 = I + c_1P_1 + c_2P_2 + c_1^2E_2 + c_1c_2M_{(21)} + c_2^2\,\Gamma_2(P^2)$$

that is,

$$0 = I + P_1 - 2P_2 + E_2 - 2M_{(21)} + 4\,\Gamma_2(P^2)$$

Notice that the first three terms add to $-I$ according to the first identity, (6.3).

7. Concluding Remarks

Starting with a multinomial extension of a Markov chain, we have motivated the definition of *induced matrix*, a basic construction in group representation theory. Then we used that as our mapping ϕ to build symmetric functions algebras SFA(ϕ, X) including some specifics for stochastic X. Along the way, we presented a matrix version of Mehler's formula for a bilinear sum of Hermite polynomials and from there derived the fundamental trace formula connecting the traces of the induced matrices with the homogeneous symmetric functions associated to the original matrix. As well, we have derived extensions of the Cayley-Hamilton theorem to power sum functions and monomial functions.

The next part, SFA III, will focus on SFA($^{-}, P$), where P is a stochastic matrix. Immediately one is led to consider *essentially stochastic* matrices, i.e., nonnegative matrices with positive constant row sums, so that any such matrix can be rescaled (by a single factor) to a stochastic matrix. We will see that the elements of the various bases consist of nonnegative matrices, in fact, essentially stochastic matrices. A main feature will be finding the row sums for each of the various systems (E, H, P, M, S).

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