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OPTIMAL CONTROL PROBLEMS FOR STOCHASTIC PROCESSES WITH ABSORBING REGIME

YAACOV KOPELIOVICH*

ABSTRACT. In this paper we formulate and solve an optimal problem for Stochastic process with a regime absorbing state. The solution for this problem is obtained through a system of partial differential equations. The method is applied to obtain an explicit solution for the Merton portfolio problem when an asset has a default probability in case of a log utility.

1. Introduction

The Bellman principle has been established for almost 60 years. For continuous control problems this leads to the HJB equation for the value function which is a candidate for a solution of optimal control problems. The stochastic analogue for Brownian motions is well known and at least heuristically a simple application of Itô's lemma. In this paper we consider a problem in which the stochastic process is a regime switching process between 2 regimes and the second regime is an absorbing one. Such regimes are ubiquitous in finance and economy, examples include:

- Bankruptcy - the state of bankruptcy is final and absorbing
- Bond convertibility - When bonds are converted into stocks.
- The state of the economy - For portfolio optimization one can approximate recessions as an absorbing state for short enough time horizons

As an industrial application we can consider linear quadratic control problem under the scenario that there is a finite probability that the factory will be upgraded and thus a new regime of cost structure and production will be initiated. The new state will be an absorbing state for a certain period of time.

Optimal control problems with regime switching were considered before. For example in [7] the authors consider Merton's portfolio problem under regime switching models that are characterized by volatility regimes for multiple stocks. They develop a theory of nearly optimal solutions to Merton problem. However the setup in their paper is less constrained than we consider in this paper and hence the theory that is developed is more involved from the set-up we obtain in this paper. One of the authors [8] has also considered a multi-state problem analogous to Markowitz setup in Finance and for this case has obtained an analytical solution

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for the allocation problem. (The setup in the latter paper was somewhat simpler as the authors were looking to minimize risk in a multi-stage Markowitz problem.)

A problem that included the Merton problem for corporate bonds with a bankruptcy option stock and cash was considered by [1], however the authors hasn't written the HJB equation and considered the final solution only.

The regime switching model we consider in this paper is less general than [7]. It's simplified nature enables a straightforward treatment, which is analogous to the usual HJB for a diffusion process (no regime switching.). We develop the method in the first section of our paper.

Using Bellman's principal we break the problem into sub-problems:

- (1) Optimal problem after in the absorbing state of the world
- (2) Optimal problem prior to the jump to the absorbing state.

First we solve the optimal control problem for the value function in the absorbing state. Then we apply Ito's lemma to write the HJB for the J^{pre} value function prior to the absorbing state. The resulting system of PDE's is similar to the usual Bellman equation for a regular diffusion process without jumps. We comment that while this method is straightforward we haven't presented a formal argument showing that our approach solves the optimal problem with regime changes. In the second section we apply the framework developed in the first section to obtain an analytical solution for the Merton's portfolio problem for a *log* utility function in a presence of bankruptcy default rate of h . While the bankruptcy problem was considered before, the solution we propose haven't appeared in the literature.

The last section summarizes our results and indicates possible directions for future generalizations.

2. HJB Equation - Recap

We consider the following process:

$$\min_u \mathbb{E} \left\{ \int_0^T C(t, X_t, u_t) dt + D(X_T) \right\} \quad (2.1)$$

With stochastic process to $(X_t)_{t \in [0, T]}$ to optimize and $(u_t)_{t \in [0, T]}$ is the optimal vector we need to find. Assuming that we have the usual Stochastic process (No jumps) the HJB equation looks like:

$$\min_u \{ \mathcal{A}V(x, t) + C(t, x, u) \} = 0 \quad (2.2)$$

and \mathcal{A} is the Stochastic differentiation operator for the process X_t . Consider now the problem above for these kind of processes:

$$X_t = 1_{t \leq \tau} (X_1)_t + 1_{t > \tau} (X_2)_t + h dt 1_{d\tau} [(X_1)_\tau + (X_2)_\tau - (X_1)_\tau] \quad (2.3)$$

In words : $X(t)$ satisfies the following:

- The initial state is given by the stochastic process X_1
- The terminal state if it occurs given by the stochastic process X_2
- At time τ the probability of moving from X_1 to X_2 is given by $h dt$ conditional on the fact that the transition hadn't occurred earlier.

Remark 2.1. In equivalent terms we can consider a regime switching absorbing process among two states given by stochastic processes X_i and with the Markov transition matrix of the form¹ :

$$\begin{bmatrix} 1 - hdt & hdt \\ 0 & 0 \end{bmatrix}$$

We consider an optimal control problem under this process. We will see that this has a clear economic motivation in the next section. Introduce the value function $V(x, t)$ in two pieces:

- $V^{pre}(x, t)$ - the function before the jump
- $V^{after}(x, t)$ - the function after the jump

To solve the problem we need to find $V^{pre}(x, t)$ and $V^{after}(x, t)$. Consider first $V^{after}(x, t)$ in this case the jump already occurred so we are in a regular optimization problem for stochastic process X_2 but without jumps. In this case we can apply the usual Ito's lemma to obtain the following differential equation we need to solve:

$$\min_u \{ \mathcal{A}_2 V^{after}(x, t) + C(t, x, u) \} = 0 \quad (2.4)$$

and \mathcal{A}_2 is the diffusion operator for the stochastic process X_2 . Now we consider the corresponding equation for dV^{pre} . We need to incorporate the regime change event into Ito's lemma. Conditional on the fact that regime change has not happened until time t we consider 2 outcomes:

- Regime change **has not happened** in the interval $[t, t + dt]$ with probability $1 - hdt$
- Regime change **happened** in the interval $[t, t + dt]$ with probability hdt

Taking the expectation of both outcomes we arrive to the following modification of Ito's lemma:

$$E_t(dV^{pre}) = (1 - hdt)\mathcal{A}_1 V^{pre}(x, t) + hdt \left[V^{after}(x_t^{after}, t) - V^{pre}(x_t, t) \right] \quad (2.5)$$

x_t is the value of the process before the jump while x_t^{after} is the value after the jump. Using the fact that, $hdt\mathcal{A}_1 = O(dt^2)$ we obtain the following stochastic differential equation for V^{pre}

$$\min_u \left\{ \mathcal{A}_1 V(x, t) + h \left[V^{after}(x_t^{after}, t) - V^{pre}(x_t, t) \right] + C(t, x, u) \right\} = 0 \quad (2.6)$$

We summarize the previous discussion in the following theorem:

Theorem 2.2. *Let*

$$X_t = 1_{t \leq \tau}(X_1)_t + 1_{t > \tau}(X_2)_t + hdt 1_{d\tau} [(X_1)_\tau + (X_2)_\tau - (X_1)_\tau] \quad (2.7)$$

be a diffusion process with an absorbing regime X_2 with probability of h . Let $V(x, t)$ be the value function optimizing:

$$\min_u \mathbb{E} \left\{ \int_0^T C(t, X_t, u_t) dt + D(X_T) \right\} \quad (2.8)$$

¹I thank George Yin for pointing this fact out

Then the solution $V(x, t)$ can be broken into two separate functions $V^{after}(x, t)$ and $V^{pre}(x, t)$ such that each satisfies the following system of partial differential equations:

$$\min_u \{ \mathcal{A}_2 V^{after}(x, t) + C(t, x, u) \} = 0 \quad (2.9)$$

and

$$\min_u \left\{ \mathcal{A}_1 V(x, t) + h \left[V^{after}(x_t^{after}, t) - V^{pre}(x_t, t) \right] + C(t, x, u) \right\} = 0 \quad (2.10)$$

where $\mathcal{A}_1, \mathcal{A}_2$ are the diffusion operators for the processes X_1, X_2

In the next section we apply these formulas to solve a portfolio Merton problem with bankruptcy in a new way.

3. Example - Stock With a Bankruptcy

As an example of the outline in the previous section we consider an optimal stock allocation with a probability of bankruptcy h for the time horizon $[0, T]$. For other works in this direction see [4] and [1] who appears to be closest to our approach.

If r is the risk free rate the dynamics of stock is given by:

$$S_t = \tilde{S}_t 1_{t < \tau} \quad (3.1)$$

and

$$\frac{d\tilde{S}}{\tilde{S}} = (\mu dt + \sigma dB)$$

The probability of the stock being bankrupt in the interval $[\tau, \tau + dt]$ CONDITIONAL on the fact that no bankruptcy event occurred until time τ is $h dt$. Now we like to use the method outlined to find an optimal allocation to the stock with S_t . We show the following lemma:

Lemma 3.1. π_1 the optimal stock weight allocation is given by:

$$\pi_1 = \frac{\mu - r - h}{\sigma^2} \quad (3.2)$$

Proof. According to the method outlined, define the value function in 2 parts. J^{after} and J^{pre} . After bankruptcy only cash is present with risk free rate r . Assume the utility of wealth is:

$$U(W) = \log W.$$

If bankruptcy occurred prior to the maturity T we can invest only in cash and hence if bankruptcy event occurs in time T the value of our investment is:

$$W \exp(r(t - T))$$

and therefore:

$$J^{after}(W, t) = r(t - T) + \log W \quad (3.3)$$

At time t we have that the probability to go bankrupt in the interval $[t, t + dt]$ is h conditional that bankruptcy hasn't occurred before. We have:

$$dJ^{pre} = J_W^{pre} dW + J_{WW}^{pre} (dW)^2 + J_t^{pre} dt + h dt [J^{after}(W \times \exp(-\pi_1), t) - J^{pre}(W, t)] \quad (3.4)$$

The wealth equation dynamics prior to default is :

$$dW = W (\pi_1 \mu dt + \pi_1 \sigma dB + (1 - \pi_1) r dt) \quad (3.5a)$$

$$dW^2 = W^2 \pi_1^2 \sigma^2 dt \quad (3.5b)$$

Substituting into dJ^{pre} we have:

$$\begin{aligned} E_t(dJ^{pre}) &= J^{pre}_W(W \times (\pi_1 \mu dt + (1 - \pi_1) r dt) + \\ &\quad \frac{1}{2} J^{pre}_{WW} W^2 \pi_1^2 \sigma^2 dt + J^{pre}_t dt \\ &\quad + h dt [J^{after}(W \times \exp(-\pi_1), t) - J^{pre}(W, t)] \end{aligned} \quad (3.6)$$

For J^{pre} assume the following form: $J^{pre}(W, t) = f(t) + \log(W)$. Then the condition for π_1 to be optimal is:

$$\frac{\partial}{\partial \pi_1} E_t(dJ^{pre}) = (\mu - r) dt - \pi_1 \sigma^2 dt - h dt = 0 \quad (3.7)$$

Solving for π_1 conclude the formula in the lemma. \square

Assuming $h = 0$ we arrive to the Merton's original solution:

$$\pi_1 = \frac{\mu - r}{\sigma^2} \quad (3.8)$$

3.1. $J^{pre}(W, t)$ Expression. In this section we prove the following expression to $J^{pre}(W, t)$

Theorem 3.2. *We have:*

$$J^{pre}(t, W) = \frac{-g + \exp(h(t - T))(g - 2r) + r(2 + h(t - T))}{h} + \log W \quad (3.9)$$

Proof. Assume $J^{pre}(W, t) = f(t) + \log(W)$ and substitute back: $\pi_1 = \frac{\mu - r - h}{\sigma^2}$ into

$$\begin{aligned} E_t(dJ^{pre}) &= J^{pre}_W(W (\pi_1 \mu dt + (1 - \pi_1) r dt) + \\ &\quad \frac{1}{2} J^{pre}_{WW} W^2 \pi_1^2 \sigma^2 dt + J^{pre}_t dt \\ &\quad + h dt [J^{after}(W \times \exp(-\pi_1), t) - J^{pre}(W, t)] \end{aligned} \quad (3.10)$$

We obtain the following differential equation for $f(t)$:

$$\frac{(\mu - r - h)^2 (2 - \sigma^2)}{2\sigma^2} + hr(t - T) + r + f'(t) - hf(t) = 0 \quad (3.11)$$

To integrate this equation rewrite it as:

$$f'(t) - hf(t) = -\frac{(\mu - r - h)^2 (2 - \sigma^2)}{2\sigma^2} - hr(t - T) - r \quad (3.12)$$

The ODE from last slide will have the form:

$$f'(t) - hf(t) + hr(t - T) = g - r \quad (3.13)$$

The initial condition is $f(T) = 0$ and the the solutions for $f(t)$:

$$f(t) = \frac{-g + \exp(h(t - T))(g - 2r) + r(2 + h(t - T))}{h} \quad (3.14)$$

Substituting the solution for $f(t)$ back into the expression of $J^{pre}(W, t)$ we conclude the result \square

4. Conclusion

we outlined an approach to address dynamical stochastic problems with absorbing processes. Our main idea is to solve this problem recursively applying Bellman principle and obtaining a system of 2 HJB's for each process. This idea that hasn't appeared explicitly in the literature to our knowledge leads to a simple analytical solution of Merton's problem for an allocation for stock and cash under the assumption of log utility. Let us conclude with remarks on our approach that we plan to pursue in subsequent papers:

- (1) While we restrict ourselves to a two state problem this approach can be generalized to more general jump situations.
- (2) For arbitrary jump process we can approximate the optimal problem using 2 state process. For example if X_1 can transition to X_2 and X_2 can transition back to X_1 approximate the optimal problem by ignoring the second transition and solving the stochastic control problem ignoring the transition from X_2 to X_1 .) An important comment to conclude is to observe that our method of solution is not entirely formalized and the development of proper mathematical framework for our method is desirable.

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