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Philip Feinsilver
Southern Illinois University, Carbondale, Illinois 62901, USA, phfeins@siu.edu

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SYMMETRIC FUNCTIONS ALGEBRAS I: INTRODUCTION AND BASIC FEATURES

PHILIP FEINSILVER*

Abstract. Starting with a multiplicative map on an algebra, which we take to be a matrix algebra over the reals, we develop, for each element, an algebra, the symmetric functions algebra, which obeys the identities of the standard algebra of symmetric functions in a given number of variables. These will in general be matrix algebras with special structure. Thus, we determine corresponding elementary, homogeneous, power sum, monomial, and Schur functions and study their properties. Definitions and constructions for the general case are detailed. Then, the multiplicative maps appearing in the examples include the determinant, tensor powers, powers, and absolute value on complex numbers. The case of symmetric tensor powers is taken up in subsequent works leading to the construction of algebras with the various bases consisting of nonnegative matrices with constant row sums when the initial element is a stochastic matrix. For symmetric tensor powers, stochastic matrices are mapped to stochastic matrices with corresponding Markov chains arising from a given underlying chain. This paper provides a basis for these studies.

1. Introduction

This study arose from working with induced matrices, symmetric tensor powers, in the study of the structure of Krawtchouk polynomials, [5], with influence from Cooper, et al. [2] and Zhou & Lange [14] on multinomial extensions of Markov chains, which will be studied in our context in SFA II and SFA III. The main idea is the parallel between mapping $I + vX$ by a multiplicative mapping, such as det, and using any multiplicative map, such as tensor powers or symmetric tensor powers, etc. In the case of the determinant, one gets

$$\det(I + vX) = \sum_{\ell} v^{\ell} \mathrm{e}_{\ell}$$

with e_ℓ the elementary symmetric functions in the eigenvalues of X.

Remark 1.1. Unless otherwise specified, we will be working in the context of a finite number of underlying variables. With the variables explicitly given, we say we are working in the "coordinate version". For the algebras defined directly by

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* Corresponding author.

the bases of symmetric functions, with no underlying variables evident, we are working in the "coordinate-free" version. A main feature of the SFA algebras is that they are coordinate-free but typically finitely generated.

In the coordinate-free version of interest here we will start with a generating function, usually a polynomial, for the e_ℓ, treating them as given variables. This is in line with Littlewood [6, p. 99, 6.4;1-3], where there the generating function for the homogeneous functions is taken as basic, as well as in various places in MacDonald [7, p. 27, no. 5ff]. This approach is well-explored in O'Sullivan [8]. The various identities relating the different bases, derived in the coordinate version are taken as given in the coordinate-free version, and in fact follow from the basic definitions of the elementary, homogeneous and power sum functions via series. Of interest here as well will be the monomial functions and Schur functions, S-functions in Littlewood's terminology.

We recommend Egge [4], Prasad [9], especially [10], Sagan [12], and Stanley [13] for related material and resource information for the subject overall. The website, Alexandersson [1], provides a portal to a rather extensive look at the range of topics comprising the current state of the subject.

Remark 1.2. Throughout, we will be working with algebras, typically algebras of matrices, over the reals.

Remark 1.3. For a given X, the (matrix) algebra $\mathrm{SFA}(\phi, X)$ is commutative. Connections with Lie algebras will be indicated, but not examined in detail in this work.

2. Review of Algebra of Symmetric Functions

We begin with establishing notation for partitions and for the classical bases for symmetric functions, namely, monomial, elementary, homogeneous, power sum and Schur functions. Then we continue with the coordinate-free version(s), where one starts with a generating function for, say, the elementary symmetric functions and from there determines the other bases by using identities established in the coordinate version.

Notation. Given a set of variables $\{x_1, \ldots, x_d\}$ we denote the algebra of symmetric functions (polynomials) they generate by $\mathrm{SFA}(x_1, \ldots, x_d)$.

2.1. Partitions and multi-indices.

Partitions typically are denoted by Greek letters λ, μ. Unless otherwise indicated, λ denotes a sequence of positive integers

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_L > 0$$

where $L = L(\lambda)$ denotes the *length* of λ. Note that λ_1 denotes the largest part.

The sum $|\lambda| = \lambda_1 + \cdots + \lambda_L$ is the *weight* of λ. Sometimes, given n we want to consider all partitions of weight n, then the sequence $\{\lambda_i\}$ may be padded with zeros to make it an n-tuple. This will be explicitly indicated if such a context arises. In any case, $|\lambda| = n$ is usefully indicated by the notation $\lambda \vdash n$.

Definition 2.1. We introduce some *partition domains*:

$$\mathcal{D}(u, N) = \{\lambda : \lambda_1 \leq u, L(\lambda) \leq N\}$$

and

$$\mathcal{D}(u;\ell) = \{\lambda : \lambda_1 \le u, L(\lambda) = \ell\}$$

with the notation varied according to context.

Note that the values $|\lambda|$ for $\lambda \in \mathcal{D}(u, N)$ range from 1 to u times N while those in $\mathcal{D}(u;\ell)$ range up to $u\ell$ with λ starting from $\lambda = [1^\ell]$.

Say we are given a sequence of variables $\{x_1, x_2, \ldots\}$. Then, the notation x_λ denotes the product

$$x_\lambda = x_{\lambda_1} x_{\lambda_2} \cdots x_{\lambda_L}$$

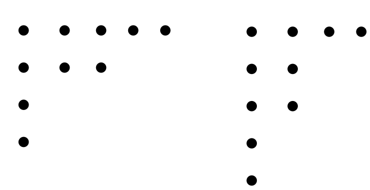
For $\lambda \in \mathcal{D}(u, N)$, we are considering at most u variables and products of at most N at a time.

2.1.1. *Dominance order.* A partial order on partitions of a given integer is *dominance order*, denoted $\lambda \trianglerighteq \mu$, with $\lambda \vdash n$ and $\mu \vdash n$, padded with zeros as necessary, defined by the conditions

$$\begin{gathered}
\lambda_1 \ge \mu_1 \\
\lambda_1 + \lambda_2 \ge \mu_1 + \mu_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
\lambda_1 + \lambda_2 + \cdots + \lambda_i \ge \mu_1 + \mu_2 + \cdots + \mu_i \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
\lambda_1 + \lambda_2 + \cdots + \lambda_n \ge \mu_1 + \mu_2 + \cdots + \mu_n
\end{gathered}$$

noting that λ and μ need not be of the same length.

2.1.2. *Ferrers diagram. Conjugate partition.* We may graphically present a partition by left-justified rows of dots, row i consisting of λ_i dots. Then transposing the diagram yields the diagram for the conjugate partition λ'. For example, if $\lambda = [5311]$, we have



with λ on the left and $\lambda' = [42211]$ on the right.

Some useful observations (here without proof):

Proposition 2.2.
1. $\lambda'_1 = L(\lambda)$ and conversely $\lambda_1 = L(\lambda')$.
2. $\lambda' \trianglelefteq \mu \;\Leftrightarrow\; \mu' \trianglelefteq \lambda$.
3. $\lambda' \trianglelefteq \mu$ implies $\lambda'_1 = L(\lambda) \le \mu_1$.

2.1.3. *Multiplicity notation and multi-indices.* An alternative description of a partition is by indicating the multiplicities of the $\lambda's$. For $\lambda \vdash n$, we denote this by an increasing sequence of the form $(i^{\rho_i})_{1\le i\le n}$:

$$\rho = \rho(\lambda) = (1^{\rho_1} 2^{\rho_2} \dots n^{\rho_n})$$

where some of the multiplicities ρ_i may vanish, with the properties

$$\sum_i \rho_i = L,\ \sum_i i\rho_i = n\ .$$

These serve directly as multi-indices, noting that

$$x_\lambda = x_1^{\rho_1} x_2^{\rho_2} \cdots x_n^{\rho_n} = x^\rho$$

in multi-index notation, the exponent ρ indicating the tuple of exponents. For example, with $\lambda = [4442211]$, we have

$$\rho(\lambda) = (1^2\, 2^2\, 4^3)$$

with

$$x_\lambda = x_1^2 x_2^2 x_4^3 = x^\rho\ .$$

Conjugacy classes of the symmetric group S_n. Note that a conjugacy class of permutations on n symbols is given by the specification of the number of cycles of the various lengths. The ρ notation typically denotes a conjugacy class of elements with ρ_i cycles of length i. The number of elements in the conjugacy class, cf. [6, 5,2;1], is

$$\#\rho = \frac{n!}{1^{\rho_1} 2^{\rho_2} \cdots n^{\rho_n}\, \rho_1!\, \rho_2! \cdots \rho_n!}$$

This is conventionally denoted by $n!/z_\rho$, with the definition

$$z_\rho = 1^{\rho_1} 2^{\rho_2} \cdots n^{\rho_n}\, \rho_1!\, \rho_2! \cdots \rho_n!$$

2.2. Principal bases: coordinate version. We begin with the coordinate version. That is, consider symmetric polynomials in the commuting variables $\{x_1, x_2 \dots, x_d\}$. Classically, the situation of interest arises given the factorization of a polynomial

$$(v - x_1)(v - x_2) \cdots (v - x_d) = v^d - \mathrm{e}_1 v^{d-1} + \mathrm{e}_2 v^{d-2} + \cdots = \sum_{\ell=0}^{d} (-1)^\ell \mathrm{e}_\ell\, v^{d-\ell}$$

with $\mathrm{e}_0 = 1$. And e_ℓ is the ℓ^{th} elementary symmetric function in the variables $\{x_i\}$.

Note that it is convenient to start with the formulation

$$E(v) = (1 + vx_1)(1 + vx_2) \cdots (1 + vx_d) = \prod_{i=1}^{d} (1 + vx_i) = \sum_{\ell=0}^{d} v^\ell \mathrm{e}_\ell \qquad (2.1)$$

2.2.1. *Monomial symmetric functions.* The *monomial symmetric functions*, m_λ, are indexed by partitions, and defined thus

$$\mathrm{m}_\lambda = \text{minimal symmetric polynomial containing the monomial } x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_L^{\lambda_L}$$

cf. [7, p. 18], [10, p. 2]. Note that here the λ's are the multiplicities. We may write, e.g.,

$$\mathrm{e}_4 = \mathrm{m}_{(1^4)}$$

Considering symmetric polynomials in d variables, it is immediate that the set $\{\mathrm{m}_\lambda\}$ is a linear basis. It holds as well that the monomial symmetric functions generate the symmetric functions as an algebra. Note that for $\mathrm{SFA}(x_1, \ldots, x_d)$, we must have $L \leq d$ for nonzero m_λ.

Example 2.3. Interpreting the meaning of λ for monomial symmetric functions is different from the indexing for products. E.g., take $d = 3$, $\lambda = [322]$, $\rho = (2^2 3^1)$. Then

$$\mathrm{m}_\lambda = x_1^3 x_2^2 x_3^2 + x_1^2 x_2^3 x_3^2 + x_1^2 x_2^2 x_3^3 = \mathrm{m}_\rho$$

while

$$\mathrm{e}_\lambda = \mathrm{e}_2^2 \mathrm{e}_3 = \mathrm{e}^\rho$$

2.2.2. *Elementary symmetric functions.* Starting with (2.1), we have the elementary symmetric functions in the variables $\{x_1, \ldots, x_d\}$. The algebra generated by $\{\mathrm{e}_\ell\}_{1 \leq \ell \leq d}$ is indeed $\mathrm{SFA}(x_1, \ldots, x_d)$. For a linear basis, form the products

$$\mathrm{e}_\lambda = \mathrm{e}_{\lambda_1} \mathrm{e}_{\lambda_2} \cdots \mathrm{e}_{\lambda_L} = \mathrm{e}_1^{\rho_1} \cdots \mathrm{e}_n^{\rho_n} = \mathrm{e}^\rho$$

with $\lambda \vdash n$. Note that $d \geq \lambda_1$ as $\mathrm{e}_\ell = 0$ if $\ell > d$.

We have the change-of-basis, [7, p. 102, 6.7 (i)], [10, Th. 2.3], expressing e_λ in terms of the monomial symmetric functions,

Proposition 2.4. *For given partitions λ, μ, let $T_{\lambda\mu}$ denote the number of* 0-1 *matrices with row sums λ_i and column sums μ_j, respectively. Then we have*

$$\mathrm{e}_\lambda = \sum_\mu T_{\lambda\mu} \mathrm{m}_\mu \ .$$

Note that the *transition matrix* $T_{\lambda\mu}$ is symmetric, [7, 6.6 (iii)].

2.2.3. *Homogeneous symmetric functions.* The (complete) homogeneous symmetric functions are defined as the sum of all monomials of a given homogeneous degree. For $n > 0$,

$$\mathrm{h}_n = \sum_{\lambda \vdash n} \mathrm{m}_\lambda$$

with $\mathrm{h}_0 = 1$. They generate the algebra $\mathrm{SFA}(x_1, \ldots, x_d)$. As in the case of the elementary symmetric functions, one forms the basis

$$\mathrm{h}_\lambda = \mathrm{h}_{\lambda_1} \mathrm{h}_{\lambda_2} \cdots \mathrm{h}_{\lambda_L} = \mathrm{h}_1^{\rho_1} \cdots \mathrm{h}_m^{\rho_m} = \mathrm{h}^\rho$$

with $\lambda \vdash m$. And we have an analogous result to (2.4), namely

Proposition 2.5. *For given partitions λ, μ, let $S_{\lambda\mu}$ denote the number of nonnegative integer matrices with row sums λ_i and column sums μ_j, respectively. Then we have*

$$\mathrm{h}_\lambda = \sum_\mu S_{\lambda\mu}\mathrm{m}_\mu$$

noting that the transition matrix $S_{\lambda\mu}$ is symmetric, [7, 6.6 (iv)].

Observe the generating function

$$\begin{aligned} H(v) &= \frac{1}{(1-vx_1)(1-vx_2)\cdots(1-vx_d)} = 1 + v\mathrm{h}_1 + v^2\mathrm{h}_2 + \cdots = \sum_{\ell\geq 0} v^\ell \mathrm{h}_\ell \\ &= 1/E(-v) \end{aligned} \tag{2.2}$$

cf. (2.1).

2.2.4. *Power sum symmetric functions.* The power sum symmetric functions are defined just as their name implies, namely, in $\mathrm{SFA}(x_1,\ldots,x_d)$, for $k>0$,

$$\mathrm{p}_k = x_1^k + x_2^k + \cdots + x_d^k = \mathrm{m}_{(k)}$$

Remark 2.6. Note that the monomial symmetric functions "interpolate" between the power sums and the elementary symmetric functions:

$$\mathrm{e}_n = \mathrm{m}_{(1^n)}, \quad \mathrm{p}_n = \mathrm{m}_{(n)} \tag{2.3}$$

The power sum functions generate the algebra $\mathrm{SFA}(x_1,\ldots,x_d)$ as well, and one forms the linear basis as for the elementary and homogeneous classes:

$$\mathrm{p}_\lambda = \mathrm{p}_{\lambda_1}\mathrm{p}_{\lambda_2}\cdots\mathrm{p}_{\lambda_L} = \mathrm{p}_1^{\rho_1}\cdots\mathrm{p}_n^{\rho_n} = \mathrm{p}^\rho$$

with $\lambda \vdash n$.

Remark 2.7. We will often be interested in expansions in terms of the p's.

Now we have

$$\log E(v) = \sum_{i=1}^d \log(1+vx_i) = \sum_i\sum_{k\geq 1}\frac{(-1)^{k-1}v^k}{k}x_i^k = \sum_{k\geq 1}\frac{(-1)^{k-1}v^k}{k}\,\mathrm{p}_k \tag{2.4}$$

which may be interpreted in terms of formal power series. Similarly

$$\log H(v) = -\sum_{i=1}^d \log(1-vx_i) = \sum_i\sum_{k\geq 1}\frac{v^k}{k}x_i^k = \sum_{k\geq 1}\frac{v^k}{k}\,\mathrm{p}_k\ . \tag{2.5}$$

2.2.5. *Schur functions.* For the coordinate version, we will denote S-functions thus: s_λ. In subsequent sections, we will use Littlewood's notation $\{\lambda\}$.

As we are working in d variables, we will consider partitions λ with $L\leq d$ and if $L<d$, then we pad with zeros accordingly. Using notation from [10, §3], also see [7, I,3], here using d instead of n, set $\delta = (d-1, d-2, \ldots, 0)$ and for any λ of length at most d,

$$a_{\lambda+\delta} = \det(x_i^{\lambda_j+d-j})$$

recalling that a_δ is the Vandermonde determinant

$$a_\delta = \prod_{i<j}(x_i - x_j) \tag{2.6}$$

Note that these are alternating functions of the variables $\{x_1, \ldots, x_d\}$.

An S-function may be defined as the ratio of alternants determined by λ:

$$\mathrm{s}_\lambda = \frac{a_{\lambda+\delta}}{a_\delta} \tag{2.7}$$

hence symmetric.

Running through partitions of a given weight, say k, these functions interpolate between e_k and h_k according to the evaluations

$$\mathrm{s}_{(1^k)} = \mathrm{e}_k, \quad \text{and } \mathrm{s}_{(k)} = \mathrm{h}_k \tag{2.8}$$

The set $\{\mathrm{s}_\lambda\}_{L(\lambda)\leq d}$, are a linear basis for $\mathrm{SFA}(x_1, \ldots, x_d)$.

We have, e.g., [7, 5.13,6.4], [4, Def. 4.16, 4.6],

Proposition 2.8. *The S-functions are expressed in terms of monomial symmetric functions, the coefficients called* Kostka numbers*,*

$$\mathrm{s}_\lambda = \sum_{\mu\vdash|\lambda|} K_{\lambda\,\mu}\mathrm{m}_\mu$$

Remark 2.9. Although tableaux (see, e.g., [4, §4.1]) will not play a significant rôle in the present work, there are numerous connections relating S-functions and tableaux.

2.3. Principal bases: coordinate-free version. We may take the e's to be the fundamental variables, with the other bases determined accordingly. The homogeneous functions will be determined by the series defining the e's. Alternatively it is convenient to express everything in terms of power sums. From either point of view, we will express S-functions using well-known formulas. We will look as well as to how the monomial symmetric functions may be determined, as there are no explicit underlying variables.

Remark 2.10. Note that we will [except for one example] be working with a finite number of nonzero elementary symmetric functions, corresponding to a finite number of underlying variables.

2.3.1. *Elementary and homogeneous symmetric functions.* We start with a polynomial of degree $N > 0$, which serves as the generating function for the elementary functions:

$$E(v) = 1 + v\,\mathrm{e}_1 + v^2\mathrm{e}_2 + \cdots + v^N\mathrm{e}_N \tag{2.9}$$

with $\mathrm{e}_0 = 1$. Then we set, with $\mathrm{h}_0 = 1$,

$$H(v) = 1/E(-v) = \sum_{\ell\geq 0} v^\ell \mathrm{h}_\ell \tag{2.10}$$

And the product of the series' provides the identities, for $n > 0$,

$$\sum_{j=0}^{n} (-1)^j \mathrm{e}_{n-j}\mathrm{h}_j = 0 \tag{2.11}$$

which recursively expresses h's in terms of the e's, cf. [7, p. 21, 2.6′].

2.3.2. *Power sum symmetric functions.* Recalling (2.4) and (2.5), we may consider the power sum variables as basic and use these formulas to express the elementary and homogeneous functions, [7, 2.13, 2.14′]. Taking our cue from (2.4) and (2.5), write

$$P(v) = \sum_{k\geq 1} \frac{v^k}{k}\,\mathrm{p}_k \tag{2.12}$$

so that

$$H(v) = \exp(P(v)) \quad \text{and} \quad E(v) = \exp(-P(-v)) \tag{2.13}$$

Accordingly,

Proposition 2.11. *We have the expansions*

$$\mathrm{h}_n = \sum_{\lambda\vdash n} \frac{1}{z_\lambda}\,\mathrm{p}_\lambda = \sum_{\rho\vdash n} \frac{1}{z_\rho}\,\mathrm{p}^\rho$$

$$\mathrm{e}_n = \sum_{\lambda\vdash n} \frac{(-1)^{n-L(\lambda)}}{z_\lambda}\,\mathrm{p}_\lambda = \sum_{\rho\vdash n} \frac{(-1)^{\sum_j (1+j)\,\rho_j}}{z_\rho}\,\mathrm{p}^\rho$$

with $z_\lambda = z_{\rho(\lambda)} = 1^{\rho_1}2^{\rho_2}\cdots n^{\rho_n}\,\rho_1!\,\rho_2!\cdots\rho_n!$.

2.3.3. *Schur functions.* S-functions may be expressed in a variety of ways. From here on we will use the notation $\{\lambda\}$ for s_λ.

(1) As determinants in terms of h's or e's. These are called the *Jacobi-Trudi identities*, [7, p. 41, 3.4, 3.5], [10, §14]. Given λ, form the $L \times L$ determinants

$$\{\lambda\} = \det(\mathrm{h}_{\lambda_i - i + j}) \tag{2.14}$$

alternatively

$$\{\lambda'\} = \det(\mathrm{e}_{\lambda_i - i + j}) \tag{2.15}$$

λ' denoting the conjugate partition. Zero subscripts follow the convention $\mathrm{e}_0 = \mathrm{h}_0 = 1$, while negative subscripts give zero.

(2) Expressed in terms of the power sum functions, we have, for $|\lambda| = n$,

$$\{\lambda\} = \sum_{\rho\vdash n} \chi^\lambda_\rho\,\frac{\mathrm{p}^\rho}{z_\rho} \tag{2.16}$$

cf., [6, p. 86, 6.2;14], [7, p. 114], where χ^λ_ρ denotes the character table of the symmetric group S_n. And, in fact, [7, 7.8], for each conjugacy class ρ,

$$\mathrm{p}^\rho = \sum_\lambda \chi^\lambda_\rho\,\{\lambda\}$$

2.3.4. *Monomial symmetric functions.* For a coordinate-free description of the monomial symmetric functions, we may use Proposition 2.4 or Proposition 2.5. Referring to [7, §6], especially the discussion around **Table 1** and statements (6.5)-(6.7), we see that, in the notation of Proposition 2.4 above that

$$T_{\lambda\mu} = 0 \quad \text{unless} \quad \lambda \trianglelefteq \mu'$$

while, denoting T^{-1} by $\hat{T}$,

$$\hat{T}_{\lambda\mu} = 0 \quad \text{unless} \quad \lambda \trianglerighteq \mu'$$

Directly from Proposition 2.4, we have

$$\mathrm{m}_\lambda = \sum_{\mu \trianglerighteq \lambda'} \hat{T}_{\lambda\mu} \mathrm{e}_\mu \tag{2.17}$$

We may also develop a generating function for the m_λ's using the basic identity, a part of the *Cauchy identities*, [7, 4.2'],

$$\prod_{i,j} (1 + x_i y_j) = \sum_\lambda \mathrm{m}_\lambda(x)\,\mathrm{e}_\lambda(y) = \sum_\lambda \mathrm{e}_\lambda(x)\,\mathrm{m}_\lambda(y) \tag{2.18}$$

Since we are working with a finite number of e's, we will use a finite version of this identity without involving the x_i variables explicitly.

Theorem 2.12. *Let u and N be positive integers. With $E(v) = \sum_{\ell=0}^{N} v^\ell \mathrm{e}_\ell$, we have*

$$\prod_{j=1}^{u} E(y_j) = \sum_{\lambda \in \mathcal{D}(u,N)} \mathrm{e}_\lambda(y)\,\mathrm{m}_\lambda$$

Proof. We have

$$\prod_{j=1}^{u} E(y_j) = \prod_{j=1}^{u} \sum_{\ell=0}^{N} y_j^\ell\,\mathrm{e}_\ell$$

Consider how a product of terms is formed. Every term e_{λ_i} has an associated factor $y_j^{\lambda_i}$. Build up e_λ in terms of $\rho = \rho(\lambda)$. We have a product like

$$\mathrm{e}_1^{\rho_1} \mathrm{e}_2^{\rho_2} \mathrm{e}_3^{\rho_3} \cdots y_1 \cdots y_{\rho_1} y_{\rho_1+1}^2 \cdots y_{\rho_1+\rho_2}^2 y_{\rho_1+\rho_2+1}^3 \cdots$$

yielding, by symmetry in the y's,

$$\sum_\lambda \mathrm{m}_\lambda(y)\,\mathrm{e}_\lambda$$

What conditions are there on λ's appearing in this sum? We see that $L(\lambda) = \sum \rho_j \le u$. In other words, since the number of y variables is u, $L(\lambda) \le u$. And since $\mathrm{e}_\ell = 0$ if $\ell > N$, we have $\lambda_1 \le N$. I.e., we get

$$\prod_{j=1}^{u} E(y_j) = \sum_{\lambda \in \mathcal{D}(N,u)} \mathrm{m}_\lambda(y)\,\mathrm{e}_\lambda$$

Now we want to switch the rôles of the y's and the virtual x's. Write

$$\mathrm{m}_\lambda(y) = \sum_{\mu' \trianglelefteq \lambda} \hat{T}_{\lambda\mu} \mathrm{e}_\mu(y)$$

which yields

$$\sum_{\lambda\in\mathcal{D}(N,u)}\sum_{\mu'\trianglelefteq\lambda} \mathrm{e}_\mu(y)\hat{T}_{\lambda\mu}\,\mathrm{e}_\lambda$$

We see that $L(\mu)=\mu_1'\le\lambda_1\le N$. Since μ is now indexing the y's, we must have $\mu_1\le u$. Use the symmetry of $\hat{T}$, i.e., $\hat{T}_{\lambda\mu}=\hat{T}_{\mu\lambda}$ and resum over λ, as in (2.17), to get

$$\sum_{\mu\in\mathcal{D}(u,N)} \mathrm{e}_\mu(y)\,\mathrm{m}_\mu$$

as required, modulo a slight change of notation. □

As the products of e's form a linear basis, this determines the monomial functions in coordinate-free fashion.

2.4. Recurrences. Many relations, especially recurrences, do not involve underlying variables explicitly. The most well-known are the *Newton identities*. Start from (2.13) and differentiate to get

$$H'(v)=P'(v)\,H(v)\quad\text{and}\quad E'(v)=P'(-v)E(v)$$

From the relation for $E(v)$, we have

$$E'(v)=\sum_{n\ge1} n\,v^{n-1}\mathrm{e}_n=\big(\sum_{k\ge1}(-1)^{k-1}v^{k-1}\mathrm{p}_k\big)\big(\sum_{\ell\ge0}v^\ell\mathrm{e}_\ell\big)$$

Equating coefficients of v^{n-1} on both sides yields, cf. [7, 2.11, 2.11′],

$$n\mathrm{e}_n=\sum_{i=1}^{n}(-1)^{i-1}\mathrm{e}_{n-i}\,\mathrm{p}_i \tag{2.19}$$

Similarly, from the relation between $H(v)$ and $P(v)$ we have

$$n\mathrm{h}_n=\sum_{i=1}^{n}\mathrm{h}_{n-i}\mathrm{p}_i \tag{2.20}$$

These relations show how e's and h's are determined using the p's as basic variables.

For $E(v)$ polynomial of degree N in v, we have $\mathrm{e}_l=0$ if $\ell>N$. Thus, we get recurrences for the h's and p's with coefficients the e's.

Proposition 2.13. *If* $\deg E(v)=N$*, we have, for* $n>N$*,*

$$\mathrm{p}_n=\sum_{i=1}^{N}(-1)^{i-1}\mathrm{e}_i\,\mathrm{p}_{n-i}$$

$$\mathrm{h}_n=\sum_{j=1}^{N}(-1)^{j-1}\mathrm{e}_j\,\mathrm{h}_{n-j}$$

Proof. For the first line, the left-hand side of (2.19) is zero, so we can multiply through by -1. Now reorder the sum, sending $i\to n-i$, yielding

$$0=(-1)^n\mathrm{p}_n+\sum_{i=1}^{n-1}(-1)^{n-i}\mathrm{p}_{n-i}\mathrm{e}_i$$

taking off the $i = 0$ term. Rearranging and taking care of the signs yields the result. The second line follows from (2.11) by reversing the order of summation, taking off the h_n term, adjusting signs, and noting that the sum over e_j only goes from 1 to N. □

Thus, the p's and h's for indices exceeding N are in fact determined by the initial segments $\{\mathrm{p}_i\}_{1\le i\le N}$, respectively $\{\mathrm{h}_j\}_{1\le j\le N}$. These are thus recurrences satisfied by these sequences. It shows that the h's and p's may be determined sequentially given the e's.

3. SFA(ϕ, X). General properties.

The definition of SFA(ϕ, X) begins with a multiplicative map between algebras. We will be working with matrix algebras over $\mathbb{R}$.

Remark 3.1. In particular, we will consider the complex numbers as an algebra over the reals.

For example, ϕ could be the determinant map on square matrices to numbers.

Remark 3.2. We typically denote elements of the domain of ϕ by capital letters. Note that by abuse of notation, we will denote by I the identity map with its interpretation appropriate to the context. Similarly for 0.

We assume

Properties of ϕ

1. $\phi(XY) = \phi(X)\phi(Y)$.
2. $\phi(I) = I$, $\phi(0) = 0$.
3. The one-parameter group e^{tX} maps to a one-parameter group in the range space.
4. ϕ is smooth.

Note that we do not assume anything about $\phi(cI)$ for scalar c.

In the subsequent sections we will look at various examples illustrating the features associated to the map ϕ and a given element X. First, the definitions and basic properties.

3.1. Gamma maps. The image $\phi(e^{tX})$ of the one-parameter group $\exp(tX)$ is a one-parameter group as well. We define its generator to be $\Gamma(X)$. That is,

$$\phi(e^{tX}) = e^{t\,\Gamma(X)}$$

In other words,

$$\Gamma(X) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 \phi(e^{tX}) = \phi'(I)X$$

by the chain rule, where ϕ' is the jacobian of the map ϕ. Hence we have as well

Proposition 3.3. *The Gamma map determined by the relation*

$$\phi(e^{tX}) = e^{t\,\Gamma(X)}$$

can be computed as

$$\Gamma(X) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 \phi(I + tX)$$

the directional derivative of ϕ at the identity in the direction X.

Theorem 3.4. *Basic properties of the Γ-map*

The first two properties show that Γ is a linear map. The third shows that it is a Lie homomorphism.

1. Homogeneity*: For scalar c, $\Gamma(cX) = c\Gamma(X)$.*

2. Additivity*: $\Gamma(X+Y) = \Gamma(X) + \Gamma(Y)$.*

3. With $[X,Y] = XY - YX$ denoting the commutator of X and Y, we have

$$\Gamma([X,Y]) = [\Gamma(X), \Gamma(Y)] \ .$$

Proof. For Property #1 start from the definition:

$$\phi(e^{tcX}) = e^{ct\Gamma(X)} = e^{t\Gamma(cX)}$$

and differentiating with respect to t at 0 gives #1. For #2,

$$\phi(e^{t(X+Y)}) = e^{t\Gamma(X+Y)}$$

by definition. Now we use multiplicativity and the Trotter product formula as follows:

$$\begin{aligned}\phi(e^{t(X+Y)}) &= \phi\left(\lim_{n\to\infty}\left(e^{(t/n)X}e^{(t/n)Y}\right)^n\right) \\ &= \lim_{n\to\infty}\left(\phi(e^{(t/n)X})\ \phi(e^{(t/n)Y})\right)^n \\ &= \lim_{n\to\infty}\left(e^{(t/n)\Gamma(X)}\ e^{(t/n)\Gamma(Y)}\right)^n \\ &= e^{t(\Gamma(X)+\Gamma(Y))}\end{aligned}$$

and differentiating with respect to t at 0 yields the result. We will show property #3 below. □

Definition 3.5. Define the *gamma maps*, Γ_ℓ, by the expansion

$$\phi(I+vX) = I + v\,\Gamma(X) + v^2\,\Gamma_2(X) + \cdots + v^\ell\,\Gamma_\ell(X) + \cdots = \sum_{\ell\ge 0} v^\ell\,\Gamma_\ell(X)$$

with $\Gamma_0(X) = I$, $\Gamma_1(X) = \Gamma(X)$, for all X. If the sum is finite, running from 0 to N for a positive integer N, we call this the *polynomial case* or case-N.

Remark 3.6. This definition is inspired by the expansion of $\det(I+vX)$ for matrices as well as the study of induced matrices in the context of Markov chains [14].

Lemma 3.7. *Scaling properties of Γ_ℓ. We have, for scalar c,*

$$\Gamma_\ell(cX) = c^\ell\,\Gamma_\ell(X)$$

Proof. Expanding yields

$$\phi(I+cvX) = \sum_\ell c^\ell v^\ell \Gamma_\ell(X) = \sum_\ell v^\ell \Gamma_\ell(cX)$$

□

We also note some scaling properties of ϕ.

Theorem 3.8. *Scaling for ϕ. Let $\alpha > 0$ and assume the scaling property $\phi(cX) = c^\alpha \phi(X)$. Then, in case-$N$, i.e., $\phi(I + vX) = \sum_{0\le\ell\le N} v^\ell \Gamma_\ell(X)$, we have*

1. $\alpha = N$.
2. $\Gamma_N(X) = \phi(X)$.

Proof. Start with $\phi(sI + tX) = s^\alpha \phi(I + (t/s)X)$ so that

$$\phi(sI + tX) = \sum_{\ell=0}^{N} s^{\alpha-\ell} t^\ell\, \Gamma_l(X)$$

Thus $\alpha \ge \ell$ for all ℓ, i.e., $\alpha \ge N$. If $\alpha > N$, then setting $s = 0$ yields $\phi(tX) = 0$, so $\alpha = N$. Now in the expression

$$\phi(sI + tX) = \sum_{\ell=0}^{N} s^{N-\ell} t^\ell \Gamma_l(X)$$

setting $s = 0$ yields $\phi(tX) = t^N \phi(X) = t^N \Gamma_N(X)$ as required. □

Now we will derive the property of preserving commutators and discover the map Γ_2. Calculate as follows:

$$\begin{aligned}\phi\big((1 + vX)(1 + vY)\big) &= \phi\,(1 + v(X + Y + vXY))\\ &= (1 + v\Gamma(X) + v^2\Gamma_2(X) + \cdots)(1 + v\Gamma(Y) + v^2\Gamma_2(Y) + \cdots)\\ &= 1 + v\,(\Gamma(X) + \Gamma(Y)) + v^2\,(\Gamma(X)\Gamma(Y) + \Gamma_2(X) + \Gamma_2(Y))\\ &\qquad + \text{terms higher order in } v\end{aligned}$$

from the left-hand side. And on the right we have, using linearity of Γ,

$$\begin{aligned}&I + v\,\Gamma(X + Y + v\,XY) + v^2\Gamma_2(X + Y + vXY) + \cdots\\ &\qquad = I + v\Gamma(X + Y) + v^2\Gamma(XY) + v^2\Gamma_2(X + Y) + \text{higher order}\end{aligned}$$

The first order term is additivity of Γ. Now compare coefficients of v^2 and rearrange to get

$$\Gamma_2(X + Y) - \Gamma_2(X) - \Gamma_2(Y) = \Gamma(X)\Gamma(Y) - \Gamma(XY) \tag{3.1}$$

Note that the left-hand side is symmetric in X and Y. This has two consequences, namely:

Proposition 3.9.
1. Preservation of Lie brackets: $[\Gamma(X), \Gamma(Y)] = \Gamma([X, Y])$.
2. $B(X, Y) = \Gamma_2(X + Y) - \Gamma_2(X) - \Gamma_2(Y) = \Gamma(X)\Gamma(Y) - \Gamma(XY)$ *is a symmetric bilinear map.*

Proof. Line 1 follows by applying the symmetry of the left-hand side of (3.1) to the right-hand side:

$$\Gamma(X)\Gamma(Y) - \Gamma(XY) = \Gamma(Y)\Gamma(X) - \Gamma(YX)$$

with #2 following directly. □

Setting $X = Y$ in (3.1), noting that $\Gamma_2(2X) = 4\Gamma_2(X)$, we find

$$\Gamma_2(X) = \tfrac{1}{2}(\Gamma(X)^2 - \Gamma(X^2)) \tag{3.2}$$

the associated quadratic map.

Now we are prepared to describe/define SFA(ϕ, X).

3.2. Structure and bases. SFA(ϕ, X) is a commutative algebra of matrices with various bases obeying the identities as those of the bases for symmetric functions. Namely, we have e's, h's, p's, m's and s's. For SFA(ϕ, X), we will use capital letters to distinguish from scalar symmetric functions, such as symmetric functions of the eigenvalues of X.

To start, fix X. Let $N > 0$, and set

$$\begin{aligned}\mathcal{E}(v) &= \phi(I + vX) = I + v\,\Gamma(X) + v^2\,\Gamma_2(X) + \cdots \\ &= I + v\,E_1 + v^2\,E_2 + \cdots + v^N E_N = \sum_{\ell=0}^{N} v^\ell\,E_\ell\end{aligned}$$

i.e., Γ_ℓ denotes the *mapping* with E_ℓ the result of Γ_ℓ evaluated at X. Note that we will discuss mainly the case of finitely many nonzero E's.

Definition 3.10. Given the multiplicative mapping ϕ and an element X from the algebra where it is defined, the *symmetric functions algebra*, SFA(ϕ, X), is the algebra generated by $\{I, E_1, E_2, \ldots, E_\ell, \ldots\}$ with $E_\ell = \Gamma_\ell(X)$.

The elements $\{E_\ell\}$ take the rôle of the elementary symmetric functions, with corresponding basis $\{E_\lambda\}$. So the generating function for the homogeneous elements is $\mathcal{H}(v) = \mathcal{E}(-v)^{-1}$. That is,

$$\mathcal{H}(v) = \phi\big((I - vX)^{-1}\big) = I + v\,H_1 + v^2\,H_2 + \cdots$$

and we immediately have the (matrix) relations

$$\sum_{j=0}^{n}(-1)^j E_{n-j} H_j = 0$$

for every $n > 0$.

Now we want to determine the elements P_i via the generating function $\mathcal{E}(v)$. Writing

$$I + vX = \exp\Big(\sum_{k\ge1}(-1)^{k-1}\frac{v^k}{k}\,X^k\Big)$$

we have, by linearity of Γ,

$$\mathcal{E}(v) = \phi(I + vX) = \exp\Big(\sum_{k\ge1}(-1)^{k-1}\frac{v^k}{k}\,\Gamma(X^k)\Big)$$

In other words, the power sum functions are given by

$$P_i = \Gamma(X^i) \tag{3.3}$$

Note that this immediately confirms the commutativity of the algebra, as the P_i commute, since $\{X^i\}$ are a commutative family.

Example 3.11. We interpret the result (3.2) as the relation

$$E_2 = \tfrac{1}{2}\left(P_1^2 - P_2\right)$$

the same as the relation $\mathrm{e}_2 = (\mathrm{p}_1^2 - \mathrm{p}_2)/2$, cf. Prop. 2.11 for $n = 2$. Similarly, the formulas expressing elementary symmetric functions in terms of power sum functions show that the E_ℓ's and in fact all elements of the algebra may be expressed terms of the family $\{\Gamma(X^i)\}$.

For polynomial $\mathcal{E}(v)$, we have from Prop. 2.13, that the P's satisfy the recurrence, with $n > N$,

$$P_n = \sum_{i=1}^{N} (-1)^{i-1} E_i P_{n-i}$$

We have an interesting mixed recurrence.
Recall the Cayley-Hamilton Theorem. A $d \times d$ matrix, X, satisfies its characteristic equation, which may be expressed as

$$X^d = \mathrm{e}_1 X^{d-1} - \mathrm{e}_2 X^{d-2} + \cdots + (-1)^{d-1} \mathrm{e}_d I$$

where now e_ℓ are the elementary symmetric functions in the eigenvalues of X. Multiplying through by X^{n-d}, for $n > d$, we have a recurrence for powers of X. Taking traces we have the analogous relation for the power sums of the eigenvalues. For the SFA(ϕ, X) case, we may apply Γ and find

$$P_d = \mathrm{e}_1 P_{d-1} - \mathrm{e}_2 P_{d-2} + \cdots + (-1)^{d-1} \mathrm{e}_d \Gamma(I)$$

and for $n > d$, the recurrence

$$P_n = \mathrm{e}_1 P_{n-1} - \mathrm{e}_2 P_{n-2} + \cdots + (-1)^{d-1} \mathrm{e}_d P_{n-d}$$

Notice that the coefficients here are the scalar coefficients from the characteristic polynomial of X. So in fact the algebra is generated by $\Gamma(I)$ and $\{P_i\}_{1 \le i < d}$ which form a linear basis for the P's.

3.2.1. *Monomial symmetric functions.* We will derive generating functions for the elements M_λ, the monomial symmetric basis of SFA(ϕ, X). The idea is to use elementary symmetric functions of auxiliary variables $\{y_j\}$ as indeterminates. Then we find an identity in which we can interpret the functions of the y-variables as arbitrary coefficients.

Theorem 3.12. *Let c_k be given scalars. Then, in the N-case, we have the expansion*

$$\phi\big(I + \sum_{k=1}^{u} c_k X^k\big) = \sum_{\lambda \in \mathcal{D}(u,N)} c_\lambda M_\lambda = \sum_{\mathcal{D}(u,N)} c^\rho M_\rho$$

where one may relax the finiteness restrictions accordingly.

Proof. Interpret $c_k = \mathrm{e}_k(y) = \mathrm{e}_k(y_1, ..., y_u)$ as indeterminates. Then we can factor:

$$\phi\big(I + \sum_{k=1}^{u} c_k X^k\big) = \phi\big(I + \sum_{k=1}^{u} \mathrm{e}_k(y)\, X^k\big) = \phi\big(\prod_{j=1}^{u} (I + y_j X)\big) = \prod_{j=1}^{u} \mathcal{E}(y_j)$$

Now invoke Theorem 2.12 to get

$$\sum_{\mathcal{D}(u,N)} \mathrm{e}_\lambda(y) M_\lambda = \sum_{\mathcal{D}(u,N)} c_\lambda M_\lambda$$

identifying each $\mathrm{e}_k(y)$ with the corresponding coefficient c_k. □

Remark 3.13. Note that one could think of factoring over the complex numbers, say, even keeping the c_k's real numbers, e.g., so the identity holds for all choices of real coefficients and so is an algebraic identity.

Now for some examples to illustrate this identity.

Example 3.14. Let us go back to the basic $\phi(I + vX) = I + v\,E_1 + v^2 E_2 + \cdots$. We have, in Theorem 3.12, $u = 1$, $c_1 = v$. Hence, with λ's restricted to maximum part equal to 1, setting $\rho_1 = \ell$,

$$\phi(I + vX) = \sum_{\mathcal{D}(1,N)} v^\ell M_{(1^\ell)}$$

where the sum is over $\rho(\lambda) = (1^{\rho_1})$. Now recall that, in fact, compare eq. (2.3), $E_\ell = M_{(1^\ell)}$ as required.

Similarly we can check the H's:

Example 3.15. We have $\phi((I - vX)^{-1}) = I + v\,H_1 + v^2 H_2 + \cdots$. We have extended u to infinity, i.e., no upper bound on λ_1. We have

$$(I - vX)^{-1} = 1 + vX(I - vX)^{-1} = 1 + \sum_{k\geq 1} v^k X^k$$

so that $c_k = v^k$. Hence

$$\phi\big((I - vX)^{-1}\big) = \sum_\lambda v^{|\lambda|} M_\lambda = \sum_{n\geq 0} v^n \sum_{\lambda\vdash n} M_\lambda$$

with the identification

$$H_n = \sum_{\lambda\vdash n} M_\lambda$$

as in the classical definition.

And the last example of this group, combining both the E and H sequences.

Example 3.16. Consider $\phi((I + tvX)(I - vX)^{-1})$. On the one hand we have, combining the series',

$$\begin{aligned}\phi\big((I + tvX)(I - vX)^{-1}\big) &= \left(\sum_\ell t^\ell v^\ell\, E_\ell\right)\left(\sum_m v^m\, H_m\right)\\ &= \sum_n v^n \sum_\ell t^\ell H_{n-\ell} E_\ell\end{aligned}$$

Now write

$$(I + tvX)(I - vX)^{-1} = I + (1+t)vX(I - vX)^{-1} = I + (1+t)\sum_{n\geq 1} v^n X^n$$

so that $c_k = (1+t)v^k$. So by the Theorem,

$$\phi\big((I+tvX)(I-vX)^{-1}\big) = \sum_\lambda (1+t)^{L(\lambda)} v^{|\lambda|} M_\lambda = \sum_{\ell,\lambda\vdash n} \binom{L(\lambda)}{\ell} t^\ell v^n M_\lambda$$

Hence the result

Proposition 3.17. *We have*

$$H_{n-\ell}E_\ell = \sum_{\lambda\vdash n} \binom{L(\lambda)}{\ell} M_\lambda \tag{3.4}$$

which will prove useful below and in some calculations in SFA II.

Now we observe a corollary involving the higher-order Γ's.

Corollary 3.18.

$$\Gamma_\ell(\sum_{k=1}^u c_k X^k) = \sum_{\mathcal{D}(u;\ell)} c^\rho M_\rho$$

Proof. We have, by the basic expansion $\phi(I+vX)$ in terms of the Γ maps and by the Theorem,

$$\phi\big(I+v\sum_{k=1}^u c_k X^k\big) = \sum_\ell v^\ell \Gamma_\ell(\sum_{k=1}^u c_k X^k) = \sum_\lambda v^{L(\lambda)} c_\lambda M_\lambda$$

and equating powers of v yields the result. □

Remark 3.19. Note that the number of terms in the sum over $\mathcal{D}(u;\ell)$ equals the number of monomials of degree ℓ in u variables, i.e., $\binom{u+\ell-1}{\ell}$.

This provides many interesting examples. Here is one.

Example 3.20. We have

$$\Gamma_\ell(c_1X + c_2X^2) = \sum_{\mathcal{D}(2;\ell)} c_1^{\rho_1} c_2^{\rho_2}\, M_\rho$$

Taking $\ell = 2$, we have, using the symmetric functions identity $\mathrm{e}_2 = (p_1^2 - p_2)/2$,

$$\Gamma_2(c_1X + c_2X^2) = \tfrac{1}{2}\big((c_1\Gamma(X) + c_2\Gamma(X^2))^2 - \Gamma((c_1X + c_2X^2)^2)\big)$$

Expand, replacing $\Gamma(X^i)$ by P_i, to get

$$\begin{aligned}\Gamma_2(c_1X{+}c_2X^2) = \\ &\tfrac{1}{2}\big(c_1^2P_1^2 + 2c_1c_2P_1P_2 + c_2^2P_2^2 - \big(c_1^2P_2 + 2c_1c_2P_3 + c_2^2P_4\big)\big) \\ &= c_1^2M_{11} + c_1c_2M_{21} + c_2^2M_{22}\end{aligned}$$

corresponding to the identifications

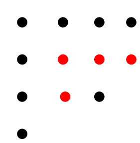
$$\begin{aligned} M_{11} &= \tfrac{1}{2}(P_1^2 - P_2) \\ M_{21} &= P_1P_2 - P_3 \\ M_{22} &= \tfrac{1}{2}(P_2^2 - P_4)\end{aligned}$$

which are easily checked by hand. In general, one finds expansions of the elements M_λ in terms of P's. In outline:

1. Write $f = \sum_{k=1}^{u} c_k X^k$.
2. Express e_ℓ in terms of p's. In turn, replacing p_i by $P_i = \Gamma(f^i)$ in the SFA algebra.
3. Expand f^i interior to the Γ's. Use linearity of Γ to re-express in terms of P_i's.
4. Compare coefficients of c^ρ to find an expression for M_ρ in terms of P's.

Remark 3.21. Complex examples can be done using a computer algebra system. For example, MAPLE has an SF, symmetric functions, package, that is useful.

3.2.2. *Schur functions.* We will focus here on *hook Schur functions*. A *hook partition* consists of a single row and a single column, the diagram resembling a hook. The general form is $\{n\,1^j\}$. The *hook length* of a location in a diagram counts the number of dots below and to the right of a given dot, including the location itself. Here is an example of a hook inside a diagram and a tableau with entries the hook length at each location.



7	5	4	2
6	4	3	1
4	2	1	
1			

(3.5)

By the Jacobi-Trudi identities, (2.14), we have a determinant expression for e_n in terms of the h's. Recalling that $\mathrm{e}_n = \{1^n\}$, we have, e.g.,

$$\mathrm{e}_3 = \det \begin{pmatrix} \mathrm{h}_1 & \mathrm{h}_2 & \mathrm{h}_3 \\ 1 & \mathrm{h}_1 & \mathrm{h}_2 \\ 0 & 1 & \mathrm{h}_1 \end{pmatrix}$$

We use these relations to get an expression for $\{n\,1^j\}$. Again, by (2.14), we have, e.g.,

$$\{n\,1^3\} = \det \begin{pmatrix} \mathrm{h}_n & \mathrm{h}_{n+1} & \mathrm{h}_{n+2} & \mathrm{h}_{n+3} \\ 1 & \mathrm{h}_1 & \mathrm{h}_2 & \mathrm{h}_3 \\ 0 & 1 & \mathrm{h}_1 & \mathrm{h}_2 \\ 0 & 0 & 1 & \mathrm{h}_1 \end{pmatrix}$$

Now expand by cofactors along the top row. Converting determinants with h_1's along the diagonal to corresponding e's we get

$$\{n\,1^3\} = \mathrm{h}_n\mathrm{e}_3 - \mathrm{h}_{n+1}\mathrm{e}_2 + \mathrm{h}_{n+2}\mathrm{e}_1 - \mathrm{h}_{n+3}\mathrm{e}_0$$

with the general result

Proposition 3.22. *We have the expression for hook S-functions*

$$\{n\,1^j\} = \mathrm{h}_n\mathrm{e}_j - \mathrm{h}_{n+1}\mathrm{e}_{j-1} + \cdots = \sum_{k=0}^{j} (-1)^k\, \mathrm{h}_{n+k}\mathrm{e}_{j-k} \tag{3.6}$$

with the corresponding expression holding in $\mathrm{SFA}(\phi, X)$.

We may see this in terms of monomial functions via (3.4). Substituting into (3.6) we have

$$\{n\,1^j\} = \sum_{\lambda\vdash n+j}\sum_{k=0}^{j}(-1)^k\binom{L(\lambda)}{j-k}M_\lambda$$

we recognize the sum over k as the coefficient of v^j in the expansion of

$$\frac{1}{1+v}\,(1+v)^L = (1+v)^{L-1}$$

That is, [7, §6, p. 105],

Proposition 3.23. *We have the expansion*

$$\{n\,1^j\} = \sum_{\lambda\vdash n+j}\binom{L(\lambda)-1}{j}M_\lambda$$

Here is another variation on these relations. In terms of hook S-functions, we have, cf. [10, Ex. 6.5],

$$E_jH_n = \{n\,1^j\} + \{n+1\,1^{j-1}\} \tag{3.7}$$

which follows from (3.6) by pulling off the first term on the right-hand side and adjusting indices accordingly.

Finally, we observe that in case-N, applying Proposition 2.13 to the right-hand side of (3.6),

Proposition 3.24. *In case-N, for $n > N$, we have the recurrence*

$$\{n\,1^j\} = \sum_{k=1}^{N}(-1)^{k-1}\{n-k\,1^j\}\,E_k$$

Application to recurrences.

Remark 3.25. In this part we will use lower case symbols for symmetric functions for convenience.

The above recurrence, which has the form

$$x_n = \mathrm{e}_1\,x_{n-1} - \mathrm{e}_2\,x_{n-2} + \cdots + (-1)^{N-1}\mathrm{e}_N\,x_{n-N}$$

is generated by the matrix A, with first row $(\mathrm{e}_1, -\mathrm{e}_2, \ldots, (-1)^{N-1}\mathrm{e}_N)$, i.e.,

$$A_{ij} = \begin{cases}(-1)^{j-1}\mathrm{e}_j\,, & \text{if } i=1\\ 1\,, & \text{if } i=j+1\\ 0\,, & \text{otherwise}\end{cases}$$

For example, for $N=3$, we have

$$A = \begin{pmatrix}\mathrm{e}_1 & -\mathrm{e}_2 & \mathrm{e}_3\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}$$

generating the recurrence

$$x_n = \mathrm{e}_1\,x_{n-1} - \mathrm{e}_2\,x_{n-2} + \mathrm{e}_3\,x_{n-3}$$

The columns of A^n are (fundamental) basic solutions to the recurrence with initial conditions reproducing the identity matrix at $n = 0$. We will see that in fact, these entries are given in terms of hook S-functions, namely,

$$(A^n)_{ij} = (-1)^{j-1}\{n - i + 1, 1^{j-1}\} \tag{3.8}$$

consistent with the well-known fact that the $(A^n)_{11}$ entry equals $\mathrm{h}_n = \{n\}$, and, for $n = 1$, the first row is given by

$$(-1)^{j-1}\{1^j\} = (-1)^{j-1}\mathrm{e}_j$$

as noted above. We will extend the initial indices on the recurrence to negatives, whereby an initial vector takes the form

$$(x_0, x_{-1}, \ldots, x_{-(N-1)})^T$$

(^{T} indicating transpose). For a fundamental solution, we match such an initial vector to one of the columns of the identity matrix. Then the columns of A^n will consist of solutions to the recurrence forming a linear basis for all solutions. We may write, for $0 \le j < N$,

$$\xi_n^{(j)} = (-1)^j\{n\,1^j\}$$

denoting the j^{th} fundamental solution, the $(j+1)^{\text{st}}$ column of A^n. Each is determined by N initial conditions

$$\xi_{-j}^{(j)} = 1 \text{ and } \xi_n^{(j)} = 0 \text{ if } n \le 0, n \ne -j$$

remarking that $\xi_n^{(0)} = \mathrm{h}_n$, satisfying $\mathrm{h}_0 = 1$, $\mathrm{h}_n = 0$ if $n < 0$. We verify the initial conditions.

Proposition 3.26. *We have, for* $0 \le j < N$,

$$(-1)^j\{-m\,1^j\} = \delta_{mj}$$

where $0 \le m < N$.

Proof. Note, for $j = 0$, we have $\{-m\} = \mathrm{h}_{-m} = \delta_{0m}$ for $m \ge 0$. Assume $j > 0$ and consider the matrix formulation $\{\lambda\} = \det(h_{\lambda_i - i + j})$, eq. (2.14), with $\lambda = [-m, 1^j]$. Observe that we have a $(j+1) \times (j+1)$ matrix, with top row

$$(\mathrm{h}_{-m}, \mathrm{h}_{-m+1}, \ldots, \mathrm{h}_{j-m})$$

There are three cases:
1. If $j < m$, then the top row has all negative indices, hence all entries are zero.
2. If $j = m$, then the last entry in the first row equals 1, with the cofactor corresponding to a diagonal of all 1's, hence, adjusting the sign, we get the result 1.
3. If $j > m$, then the first m entries in the top row vanish, with the remaining entries of the form $(1, \mathrm{h}_1, \mathrm{h}_2, \ldots)$. That is, the first row matches the row below having m leading zeros. I.e., row 1 and row $m+2$ are identical, so the determinant vanishes. □

Now match (3.8) for $n = 0$:

$$(A^0)_{ij} = (-1)^{j-1}\{-(i-1), 1^{j-1}\} = \delta_{ij}$$

accordingly. The indices are shifted back as we are counting rows and columns starting from 1.

3.3. Diagonalization. If X is diagonalizable with eigenvalues $\{x_1, x_2, \ldots, x_d\}$, we have

$$XW = WD$$

where W is a matrix of eigenvectors and D is diagonal with entries $D_{ii} = x_i$, $1 \leq i \leq d$.

Assume that $\phi(D)$ *is diagonal if* D *is.*

Then we have

$$\phi(X)\phi(W) = \phi(W)\phi(D)$$

so that $\phi(W)$ diagonalizes $\phi(X)$. And $XW = WD$ implies, for $k > 0$,

$$\begin{aligned} X^k W = WD^k &\Rightarrow (I + vX^k)W = W(I + vD^k) \\ &\Rightarrow \phi(I + vX^k)\phi(W) = \phi(W)\phi(I + vD^k) \end{aligned}$$

and taking the coefficient of v on both sides yields

$$\Gamma(X^k)\phi(W) = \phi(W)\Gamma(D^k)$$

with $\Gamma(D^k)$ diagonal. Thus, $P_k \in \mathrm{SFA}(\phi, X)$ are diagonalized by $\phi(W)$, hence the entire algebra is diagonalized. Then, $\mathrm{SFA}(\phi, D)$ provides the spectrum of the algebra $\mathrm{SFA}(\phi, X)$. In general, the entries in the diagonalized algebra do not simply reproduce the corresponding symmetric functions in the eigenvalues of X, while providing essential information as to the structure of the algebra $\mathrm{SFA}(\phi, X)$.

4. Examples

Now we will look at a variety of cases that illustrate many of the features of $\mathrm{SFA}(\phi, X)$ algebras.

4.1. Identity map. The simplest case is just to take $\phi(X) = X$. Then

$$\phi(e^{tX}) = e^{tX} \;\Rightarrow\; \Gamma(X) = X$$

so that $P_i = X^i$, for $i \geq 1$. We immediately have

$$E_0 = I,\; E_1 = X,\; E_\ell = 0,\; \text{for } \ell > 1$$

and expanding $(I - vX)^{-1}$ yields

$$H_\ell = X^\ell,\; \text{for } \ell \geq 0\ .$$

Theorem 3.12 shows

$$\phi(I + \sum_{k=1}^{u} c_k X^k) = I + \sum_{k=1}^{u} c_k X^k = \sum_{\mathcal{D}(u,1)} c_\lambda M_\lambda = \sum_{k=1}^{u} c_k M_{(k)}$$

so that

$$M_{(k)} = P_k = X^k,\; \text{while } M_\lambda = 0 \text{ if } L(\lambda) > 1$$

For S-functions, use the Jacobi-Trudi form (2.15). We see immediately that if $\lambda_1' > 1$, then $\{\lambda\} = 0$. If $\lambda_1' = 1$, then we have $\lambda = [n]$, say, with $\{\lambda\} = H_n = X^n$.

4.2. Contragredient map. An interesting example is provided by the contragredient map $X \to (X^{-1})^T = X^\star$, say. We have $\phi(X) = X^\star$ satisfying $\phi(XY) = \phi(X)\phi(Y)$ as required. We have

$$\phi(e^{tX}) = e^{-tX^T} \;\Rightarrow\; \Gamma(X) = -X^T$$

so that $P_i = \Gamma(X^i) = -(X^T)^i$, $i \ge 1$. Expanding $(I + vX)^\star = (I + vX^T)^{-1}$ yields

$$E_\ell = (-X^T)^\ell = (\Gamma(X))^\ell$$

On the other hand, for H's, we have $((I - vX^T)^{-1})^{-1} = \sum_\ell v^\ell H_\ell$, so that

$$H_1 = -X^T \text{ and } H_\ell = 0 \text{ for } \ell > 1$$

verifying $H_1 = E_1 = P_1 = \Gamma(X)$. For S-functions, by (2.14), we have $\{\lambda\} = 0$ if $\lambda_1 > 1$, thus

$$\{\lambda\} = \{1^n\} = E_n = (-X^T)^n \text{ for } \lambda = [1^n],\ 0 \text{ otherwise}$$

For M_λ, use Prop. 2.5, to write

$$M_\lambda = \sum_\mu (S^{-1})_{\lambda\mu} H_\mu$$

Now, H_λ is nonzero only for $\lambda = [1^n]$, say, and then, $H_\lambda = (-X^T)^n$. This gives the coefficient of $H_{[1^n]}$ for M_λ, the entry in row λ in the last column of S^{-1}. That is

$$M_\lambda = (S^{-1})_{\lambda[1^n]}\,(-X^T)^n$$

Remark 4.1. From [7, §6,Table I], we have $S^{-1} = K^{-1}K^\star$, where K is the matrix of Kostka coefficients. Knowing that K is upper unitriangular, i.e., has 1's along the diagonal and zeros below the diagonal, it follows that the last column of $K^\star$ consists of all zeros except for a 1 as the bottom right corner entry. Thus, the last column of S^{-1} is the same as the last column of K^{-1}.

4.3. Determinant. When ϕ is the determinant map on square matrices, what is interesting is how to express symmetric functions of the eigenvalues of a given matrix in terms of the entries of the matrix. This is another variation on a coordinate-free realization of the symmetric function algebra. With $\phi(X) = \det(X)$, we have

$$\det(I + vX) = 1 + \sum_{\ell=1}^{d} v^\ell \mathrm{e}_\ell$$

with e_ℓ elementary symmetric functions in the eigenvalues. The e's may be calculated as well as the traces of the corresponding exterior powers of X. And

$$\det(I - vX)^{-1} = \sum_{\ell \ge 0} v^\ell\, \mathrm{h}_\ell$$

give the homogeneous symmetric functions in the eigenvalues. As well, $\{h_n\}$ give the fundamental solution to the recurrence

$$x_n = \mathrm{e}_1\, x_{n-1} - \mathrm{e}_2\, x_{n-2} + \cdots + (-1)^{d-1}\mathrm{e}_d\, x_{n-d}$$

with $\mathrm{h}_0 = 1$, $\mathrm{h}_n = 0$ for $n < 0$.
Next, we have

$$\det(e^{tX}) = e^{t\,\mathrm{tr}\,X} = e^{t\Gamma(X)} \;\Rightarrow\; \Gamma(X) = \mathrm{tr}\,X$$

as expected, so that

$$\mathrm{p}_k = \Gamma(X^k) = \mathrm{tr}\,X^k = x_1^k + x_2^k + \cdots + x_d^k$$

accordingly.
For S-functions, the Jacobi-Trudi determinants, eqs. (2.14), (2.15) apply directly. For the monomial symmetric functions, Theorem 3.12 is of interest. We have

$$\det(I + \sum_{k=1}^{u} c_k X^k) = \sum_{\mathcal{D}(u,d)} c^\rho \mathrm{m}_\rho$$

whereby one has generating functions for monomial symmetric functions in terms of the entries of X.

Example 4.2. For a small example, the matrix

$$Y = \begin{pmatrix} 2x_1 - x_2 & 2x_1 - 2x_2 \\ -x_1 + x_2 & -x_1 + 2x_2 \end{pmatrix}$$

has eigenvalues x_1 and x_2. We find

$$\begin{aligned} &\det(I + c_1 Y + c_2 Y^2) = \\ &\qquad 1 + (x_1 + x_2)\,c_1 + \left({x_1}^2 + {x_2}^2\right) c_2 \\ &\qquad\quad + {c_1}^2 x_1 x_2 + \left(x_1 {x_2}^2 + x_2 {x_1}^2\right) c_1 c_2 + {c_2}^2 {x_1}^2 {x_2}^2 \end{aligned}$$

with the correspondences c^ρ and m_ρ clearly shown.

Remark 4.3. We observe that the Cauchy identity (2.18) and Theorem 3.12 provide a formula for the E-functions for $X \otimes Y$, i.e., if X, Y have eigenvalues $\{x_i\}$, $\{y_j\}$ respectively, then we see

$$\det(I + v\,X \otimes Y) = \sum_\lambda v^{L(\lambda)} \mathrm{e}_\lambda(y) \mathrm{m}_\lambda(x) = \sum_\lambda v^{L(\lambda)} \mathrm{m}_\lambda(y) \mathrm{e}_\lambda(x)$$

4.4. Tensor powers. Here we consider taking tensor – Kronecker – powers of a matrix for the map ϕ. We look at $N = 3$ in detail which will show the general features. So start with

$$\phi(X) = X \otimes X \otimes X$$

Let

$$\xi_1(X) = X \otimes I \otimes I,\; \xi_2(X) = I \otimes X \otimes I,\; \xi_3(X) = I \otimes I \otimes X$$

with $\phi(X) = \xi_1(X)\xi_2(X)\xi_3(X)$. These are defined similarly for general N, noting that there are N factors in each ξ_j. They satisfy commutation relations $[\xi_i(X), \xi_j(X)] = 0$, i.e., they are mutually commuting. Note that the $\{\xi_j\}$ are multiplicative maps in their own right. In particular, $\xi_j(X^k) = \xi_j(X)^k$ and hence, by linearity,

$$\xi_j(e^{tX}) = e^{t\xi_j(X)}$$

So

$$\phi(e^{tX}) = \prod_j \xi_j(e^{tX}) = \prod_j e^{t\xi_j(X)} = e^{t(\xi_1(X)+\xi_2(X)+\xi_3(X))} \ \Rightarrow\ \Gamma = \sum_j \xi_j$$

Thus,

$$P_i(X) = \sum_j \xi_j(X^i) = \sum_j \xi_j(X)^i$$

In other words, in general,

$$\mathrm{SFA}(\phi, X) = \mathrm{SFA}(\xi_1(X), \xi_2(X), \ldots, \xi_N(X))$$

i.e., $\mathrm{SFA}(\phi, X)$ is equivalent to the coordinate version in the variables $\{\xi_j(X)\}$.

4.5. Absolute value on complex numbers. On the complex numbers, we have the multiplicative map $z \to |z|$, the absolute value mapping. Since it maps to positive numbers for nonzero z, we may consider

$$\phi(z) = |z|^\tau$$

for $\tau \in \mathbb{R}$. We can take τ arbitrary, real, and proceed formally to define Γ and the rest of the algebra. As well, we can work over the domain $z \neq 0$. Writing $z = re^{i\theta}$, polar form, $r > 0$, $\theta \in \mathbb{R}$, we have, for t real,

$$|e^{tz}|^\tau = e^{t\tau r\cos\theta} = e^{t\tau \operatorname{Re} z} \ \Rightarrow\ \Gamma(z) = \tau \operatorname{Re} z$$

so, with $z = x + iy$, $x = \operatorname{Re} z$, we have

$$P_k = \tau \operatorname{Re}(z^k) = r^k \tau \cos k\theta = r^k\, \tau\, T_k(x/r)$$

where T_k denotes the Chebyshev polynomial of the first kind. We see that r can be scaled out, so take $|z| = r = 1$. Also, without loss of generality, we take $\tau > 0$. Then we have, on the unit circle,

$$P_k = \tau\, T_k(x)$$

Now we consider, for v real, $x = \cos\theta$,

$$\phi(1 + vz) = |1 + v\cos\theta + iv\sin\theta|^\tau = (1 + 2vx + v^2)^{\tau/2} = \sum_{\ell=0}^{\infty} v^\ell E_\ell$$

which can be expanded, via the binomial theorem, rearranging to yield

$$E_\ell = \sum_{n=0}^{\lfloor \ell/2 \rfloor} \binom{\tau/2}{\ell - n}\binom{\ell - n}{n}(2x)^{\ell - 2n}$$

while for the homogeneous functions, expanding $(1 - 2vx + v^2)^{-\tau/2}$, yields

$$H_\ell = C_\ell^{(\tau/2)}(x)$$

Gegenbauer polynomials. Note that Proposition 2.11 immediately gives identities expressing Gegenbauer polynomials as well as the corresponding polynomials with negative τ as a sum of products of Chebyshev polynomials, namely,

$$\begin{aligned} H_n &= C_n^{(\tau/2)}(x) = \sum_{\rho\vdash n} \tau^{L(\rho)}\, T_1(x)^{\rho_1} T_2(x)^{\rho_2} \cdots T_n(x)^{\rho_n}/z_\rho \\ E_n &= \sum_{\rho\vdash n} (-1)^{n-L(\rho)}\, \tau^{L(\rho)}\, T_1(x)^{\rho_1} T_2(x)^{\rho_2} \cdots T_n(x)^{\rho_n}/z_\rho \end{aligned}$$

where $L(\rho) = \sum_j \rho_j$.

We look at a simple illustrative example.

Example 4.4. So $\tau = 2$ is the simplest case. We have $\phi(z) = |z|^2$, $\Gamma(z) = 2\,\mathrm{Re}\, z$, and, with $z = x + iy = re^{i\theta}$,

$$\phi(1 + vz) = (1 + vz)(1 + v\bar{z}) = 1 + 2vx + v^2 r^2$$

so $E_1 = 2x$, $E_2 = r^2$, $E_\ell = 0$, for $\ell > 2$. We have the symmetric function algebra $\mathrm{SFA}(z, \bar{z})$. And

$$\phi(1 - vz)^{-1} = (1 - 2rv(x/r) + (rv)^2)^{-1} = \sum_{\ell=0}^{\infty} v^\ell \left(r^\ell U_\ell(x/r)\right)$$

where U_ℓ denote Chebyshev polynomials of the second kind, so $H_\ell = r^\ell U_\ell(x/r)$. The monomial functions are the standard monomial symmetric functions in the variables $(z, \bar{z})$, with at most two parts, we have

$$M_{(n)} = z^n + \bar{z}^n = 2\,\mathrm{Re}\, z^n = 2\, r^n T_n(x/r)$$

and, for $m \neq n$,

$$M_{(mn)} = z^m \bar{z}^n + z^n \bar{z}^m = 2\,\mathrm{Re}\, z^m \bar{z}^n = 2\, r^{m+n} T_{m-n}(x/r)$$

while

$$M_{(nn)} = z^n \bar{z}^n = r^{2n}\ .$$

For the S-functions, we use the alternant definition, (2.7). For $\{\lambda\}$, we must restrict to $L(\lambda) \leq 2$. Above, we found $\{n\} = H_n = r^n\, U_n(x/r)$. So we consider

$$\begin{aligned} \{m, n\} &= \begin{vmatrix} z^{m+1} & z^n \\ \bar{z}^{m+1} & \bar{z}^n \end{vmatrix} /(z - \bar{z}) = r^{2n} \begin{vmatrix} z^{m-n+1} & 1 \\ \bar{z}^{m-n+1} & 1 \end{vmatrix} /(z - \bar{z}) \\ &= r^{2n}\, H_{m-n} = r^{m+n}\, U_{m-n}(x/r) \end{aligned}$$

which compares well with the results for the monomial functions. To conclude this example, take $r = 1$ and consider the expansion Proposition 3.22. We have

$$\{n\,1\} = U_{n-1}(x) = H_n E_1 - H_{n+1} E_0 = 2x\, U_n - U_{n+1}$$

which rearranged we recognize as the recurrence formula for the 2^{nd}-kind Chebyshev polynomials:

$$2x\, U_n = U_{n+1} + U_{n-1} \ \text{or}\ U_{n+1} = 2x\, U_n - U_{n-1}$$

Remark 4.5. We remark that $\tau = 1$ yields the Legendre polynomials for the H's.

In general, the S-functions have expansions in terms of products of Chebyshev polynomials, but explicit formulas have yet to be found. Similarly, formulas for the monomial functions have yet to be determined.

4.6. Diagonal matrices. For $d \times d$ diagonal matrices we can take a sequence $m_1, m_2, \ldots, m_d$ of positive integers and map

$$\phi(X) = \begin{pmatrix} x_1^{m_1} & & & \\ & x_2^{m_2} & & \\ & & \ddots & \\ & & & x_d^{m_d} \end{pmatrix}$$

Since each entry along the diagonal transforms independently, we will consider one entry in detail. We can take

$$\phi(x) = x^m$$

for a single variable x. We have

$$\phi(e^{tx}) = e^{t(mx)} \ \Rightarrow\ \Gamma(x) = mx$$

with

$$P_i = \Gamma(x^i) = mx^i$$

For the E's,

$$\phi(1 + vx) = \sum_{\ell=0}^{m} v^\ell \binom{m}{\ell} x^\ell$$

so that

$$E_\ell = \binom{m}{\ell} x^\ell \tag{4.1}$$

with

$$(1 - vx)^{-m} = \sum_{\ell \geq 0} v^\ell \frac{(m)_\ell}{\ell!}\, x^\ell \ \Rightarrow\ H_\ell = \frac{(m)_\ell}{\ell!}\, x^\ell \tag{4.2}$$

where $(m)_\ell$ equals the rising factorial $m(m+1)\cdots(m+\ell-1)$. For the monomial functions, we consider

$$\phi(1 + \sum_{k=1}^{u} c_k x^k) = (1 + \sum_{k=1}^{u} c_k x^k)^m = \sum_{\mathcal{D}(u,m)} c_\lambda M_\lambda \tag{4.3}$$

So, cf. [7, p.26, 1], [13, Prop. 7.8.3],

Proposition 4.6. *For $P_i = mx^i$, $i \geq 1$, we have with, $\rho = \rho(\lambda)$,*

$$M_\lambda = \frac{L(\lambda)!}{\prod_i \rho_i!} \binom{m}{L(\lambda)} x^{|\lambda|}$$

Proof. Note that the first factor is a multinomial coefficient. Use the multinomial expansion for the middle expression of eq. (4.3),

$$(1 + \sum_{k=1}^{u} c_k x^k)^m = \sum_{\mathcal{D}(u,m)} \frac{m!}{n_0!\, \rho_1! \cdots \rho_u!}\, c_1^{\rho_1} c_2^{\rho_2} \cdots c_u^{\rho_u}\, x^{\sum k\,\rho_k}$$

where $n_0 = m - L(\lambda)$, and $\sum k\rho_k = |\lambda|$. Multiplying and dividing by $L(\lambda)!$, we find the desired result. □

4.6.1. *S-functions for ϕ a power function.* In this section, we will take an arbitrary positive number for τ as the S-functions will be polynomials in τ. So $P_i = \tau\, x^i$, $i \geq 1$. The S-functions are known, [6, §7.2], [7, p. 45,4.],

$$\{\lambda\} = \frac{\prod\limits_{1\leq i\leq L} (\tau - i + 1)_{\lambda_i}}{\prod(\text{hook lengths})}\, x^{|\lambda|} \tag{4.4}$$

where the hook lengths are from the tableau given by the shape λ with entries the hook length at each box, as in diagram 3.5. The power of x can be checked with the formula eq. (2.16), where each term in the sum is of degree $|\lambda|$ in x.

Hook S-functions. We will work out the S-functions for hooks first. Consider the diagram for $\{3\,1^3\}$, with hook lengths indicated,

$$\begin{array}{|c|c|c|}\hline 6 & 2 & 1 \\ \hline 3 \\ \cline{1-1} 2 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}$$

For $\{n\,1^j\}$ we see that the corner box has hook length $n+j$, with the rest of the row contributing $(n-1)!$ and the rest of the column contributes $j!$. Formula (4.4) gives,

$$\{n\,1^j\} = \frac{(\tau)_n(\tau-1)(\tau-2)\cdots(\tau-j)}{(n+j)\,(n-1)!\,j!} = \frac{(\tau-j)_{n+j}}{(n+j)\,(n-1)!\,j!} \tag{4.5}$$

Note that terms are homogeneous of the same degree in x, so we are taking $x = 1$. We now derive this formula. The formulas for the E's and H's hold with τ replacing m, with the binomial coefficient taking the form

$$E_\ell = \binom{\tau}{\ell} = \frac{\tau(\tau-1)\cdots(\tau-\ell+1)}{\ell!}$$

The sum (3.6) gives

$$\{n\,1^j\} = \sum_{k=0}^{j} (-1)^k\, \frac{(\tau)_{n+k}}{(n+k)!} \binom{\tau}{j-k}$$

Rearranging via

$$\binom{\tau}{j-k} = \binom{\tau}{j} \frac{(-1)^k(-j)_k}{(\tau-j+1)_k}$$

we have

$$\binom{\tau}{j}\frac{(\tau)_n}{n!} \sum_{k=0}^{j} \frac{(\tau+n)_k(-j)_k(1)_k}{(n+1)_k(\tau-j+1)_k\,k!} = \binom{\tau}{j}\frac{(\tau)_n}{n!}\, {}_3F_2\left(\begin{matrix} -j,\,\tau+n,\,1 \\ n+1,\,\tau-j+1\end{matrix}\,\middle|\, 1\right)$$

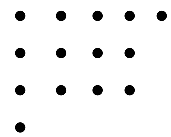
which can be summed by the Pfaff-Saalschütz theorem, [3, 16.4.3],

$${}_3F_2\left(\begin{matrix} -n,\,a,\,b \\ c,\,d\end{matrix}\,\middle|\, 1\right) = \frac{(c-a)_n(c-b)_n}{(c)_n\,(c-a-b)_n}$$

which gives in our case

$$\{n\,1^j\} = \binom{\tau}{j}\frac{(\tau)_n}{n!}\,\frac{(1-\tau)_j(n)_j}{(n+1)_j(-\tau)_j}$$

which simplifies in accordance with (4.5). General S-functions may be computed in terms of hook functions using *Giambelli's formula* [10, §16], [7, Ch. 3, ex. 9]. We will modify the notation slightly to clarify the formula. Given a partition $\{\lambda\}$, we can rewrite it in *Frobenius coordinates*, where, in the Ferrers diagram, going row by row, $\alpha_i = \lambda_i - i$ the number of dots in row i to the right of location (i,i) and, similarly, $\beta_j = \lambda'_j - j$, the number of dots below the box at location (j,j). For example, $\lambda = [5441]$, has Frobenius coordinates $(421|310)$.



We modify the coordinates by augmenting the α's to include the diagonal dots. I.e., we write $a_i = \lambda_i - i + 1$, $b_j = \lambda'_j - j$ and indicate by square brackets

$$\lambda = (421|310) = [532|310]$$

With these coordinates, Giambelli's formula reads

$$\{[a_1 a_2 \ldots a_\ell | b_1 b_2 \ldots b_\ell]\} = \det(\{a_j\, 1^{b_i}\})$$

For example,

$$\{5441\} = \{(421|310)\} = \{[532|310]\} = \begin{vmatrix} \{5\,1^3\} & \{3\,1^3\} & \{2\,1^3\} \\ \{5\,1\} & \{3\,1\} & \{2\,1\} \\ \{5\} & \{3\} & \{2\} \end{vmatrix}$$

In our context, we have, eq. (4.5),

$$\{a_j\, 1^{b_i}\} = \frac{(\tau - b_i)_{a_j + b_i}}{(a_j + b_i)\,(a_j - 1)!\, b_i!}$$

Thus, for $\{[a_1 \ldots a_\ell | b_1 \ldots b_\ell]\}$ we have for numerators the array

$$\begin{bmatrix} (\tau - b_1)_{a_1+b_1} & (\tau - b_1)_{a_2+b_1} & \cdots & (\tau - b_1)_{a_\ell+b_1} \\ (\tau - b_2)_{a_1+b_2} & (\tau - b_2)_{a_2+b_2} & \cdots & (\tau - b_2)_{a_\ell+b_2} \\ \cdots & \cdots & \cdots & \cdots \\ (\tau - b_\ell)_{a_1+b_\ell} & (\tau - b_\ell)_{a_2+b_\ell} & \cdots & (\tau - b_\ell)_{a_\ell+b_\ell} \end{bmatrix}$$

Note that b_ℓ is the minimal b index, so we factor out of each column, $(\tau - b_\ell)_{a_j+b_\ell}$, yielding

$$\left(\prod_j (\tau - b_\ell)_{a_j+b_\ell}\right) \begin{bmatrix} (\tau - b_1)_{b_1-b_\ell} & (\tau - b_1)_{b_1-b_\ell} & \cdots & (\tau - b_1)_{b_1-b_\ell} \\ (\tau - b_2)_{b_2-b_\ell} & (\tau - b_2)_{b_2-b_\ell} & \cdots & (\tau - b_2)_{b_2-b_\ell} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Now every row has the same entry in each column, so these factor out to yield an initial factor of

$$\left(\prod_j (\tau - b_\ell)_{a_j+b_\ell}\right) \left(\prod_i (\tau - b_i)_{b_i-b_\ell}\right)$$

And the factors in the denominator come out yielding

$$\left(\prod_j \frac{1}{(a_j-1)!}\right)\left(\prod_i \frac{1}{b_i!}\right)\det\left(\frac{1}{a_j+b_i}\right)$$

this last being a Cauchy determinant. Referring to [11], for example, we see that

$$\det\left(\frac{1}{a_j+b_i}\right)=\frac{V(a_1,\ldots,a_\ell)\,V(b_1,\ldots,b_\ell)}{\prod_{i,j}(a_j+b_i)}$$

where $V(x_1,\ldots,x_\ell)=\prod_{i<j}(x_i-x_j)$ is the Vandermonde determinant, cf., (2.6), in the corresponding variables. Combining the above results we have

Theorem 4.7. *For constant $P_i=\tau$:*
Writing λ in terms of modified Frobenius notation, we have

$$\{\lambda\}=\frac{\left(\prod_j(\tau-b_\ell)_{a_j+b_\ell}\right)\left(\prod_i(\tau-b_i)_{b_i-b_\ell}\right)}{\prod(\text{hook lengths})}$$

where

$$\prod(\text{hook lengths})=\frac{\left(\prod_j(a_j-1)!\right)\left(\prod_i b_i!\right)\left(\prod_{i,j}(a_j+b_i)\right)}{V(a_1,\ldots,a_\ell)V(b_1,\ldots,b_\ell)}$$

Remark 4.8. Observe that from the expansion of $\{\lambda\}$ in terms of products of power sums, eq. (2.16), we have via (4.4), for $\lambda\vdash n$,

$$\sum_{\rho\vdash n}\chi^\lambda_\rho\,\frac{\tau^{L(\rho)}}{z_\rho}=\frac{\prod\limits_{1\le i\le L}(\tau-i+1)_{\lambda_i}}{\prod(\text{hook lengths})}$$

This concludes our initial set of examples for SFA(ϕ,X) algebras.

4.7. Symmetric tensor powers. This will be the subject of parts SFA II and III. In part III, we will see that if X is nonnegative, then all of the principal bases will consist of nonnegative matrices.

5. Conclusion

Naturally, many results depend on the specific nature of mapping ϕ. The next part, SFA II, deals with the case of induced matrices, symmetric tensor powers. We will construct a multinomial extension of a Markov chain for which the transition matrix of the extended chain is an induced matrix of the original chain. This will provide the mapping ϕ which will then be applied to general matrices as our focus for SFA II.

In SFA III, we look at the case of induced matrices starting from nonnegative matrices, especially stochastic matrices, leading to consideration of nonnegative matrices with constant row sums. In that context, all of the bases for the SFA's will be *essentially stochastic*, i.e., nonnegative matrices with nonzero constant row sums.

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Philip Feinsilver: Department of Mathematics, Southern Illinois University, Carbondale, Illinois 62901, USA

Email address: `phfeins@siu.edu`

URL: `http://chanoir.math.siu.edu`