

3-1-2011

On zeros and local infima of Brownian motion

Michel Weber

Follow this and additional works at: <https://repository.lsu.edu/cosa>



Part of the [Analysis Commons](#), and the [Other Mathematics Commons](#)

Recommended Citation

Weber, Michel (2011) "On zeros and local infima of Brownian motion," *Communications on Stochastic Analysis*: Vol. 5: No. 1, Article 2.

DOI: 10.31390/cosa.5.1.02

Available at: <https://repository.lsu.edu/cosa/vol5/iss1/2>

ON ZEROS AND LOCAL INFIMA OF BROWNIAN MOTION

MICHEL WEBER

ABSTRACT. Let $W = \{W(t), t \geq 0\}$ be a standard Brownian motion. Consider a partition of \mathbf{R}^+ into bounded intervals (I_N) together with a sequence (η_N) of positive reals. Conditions are given ensuring that with probability one, for any N large enough, either $\inf_{t \in I_N} |W(t)| \geq \eta_N$ or W vanishes on I_N . Prior to the proof, fine estimates of the infima $\inf\{|W(t)|, t \in I\}$, I being a bounded interval, are first established. An application of some of these estimates to level sets in probability is also given.

1. Main Result

Let $W = \{W(t), t \geq 0\}$ be a standard Brownian motion. It is well-known (see [6]) that the zero set Z of W , namely $Z(\omega) = \{t : W(\omega, t) = 0\}$, is an almost sure uncountable closed subset of \mathbf{R}^+ , with no isolated points and of Lebesgue measure zero. We also refer to [6], Chapter 4 where a construction of the zero set of W is made, as well as a reconstruction of W starting from Z , and for some delicate related laws (see Remark 2.5).

In this work, we investigate the asymptotic size of those intervals which are zerofree for W . This naturally leads to consider the functional $\inf\{|W(t)|, t \in I\}$, where I is a bounded interval of \mathbf{R}^+ . More precisely, suppose $W(t) \neq 0$, for all $t \in I$. What can be said about the size of $\inf\{|W(t)|, t \in I\}$? This one only depends on the location of I and of its length. The object of this Note is to prove the following result.

Theorem 1.1. *Let $\vartheta_k \geq 0$ be such that $T_N = \sum_{k \leq N} \vartheta_k \uparrow \infty$ and denote $I_N = [T_N, T_{N+1}]$. Let $\eta_k \geq 0$ be such that*

$$\sum_{N \geq 1} \eta_N \min \left(\frac{1}{\sqrt{T_{N+1} - T_N}}, \frac{1}{\sqrt{T_N}} \right) < \infty.$$

Then

$$\mathbf{P} \left\{ \inf_{t \in I_N} |W(t)| \geq \eta_N \text{ or } W(t) = 0 \text{ for some } t \in I_N, \quad N \text{ eventually} \right\} = 1.$$

The notation used means that almost surely, for all N large enough, either $\inf_{t \in I_N} |W(t)| \geq \eta_N$ or $W(t) = 0$ for some $t \in I_N$. The proof given at section 3 relies upon several intermediate results on infima of $|W|$, which are also of independent interest. These ones are stated and proved in Section 2.

Received 2010-1-21.

2000 *Mathematics Subject Classification.* Primary 60J65, 60G17 ; Secondary 60G15.

Key words and phrases. Brownian motion, zero set, sample paths properties, infima, level probability, Skorokhod embedding, Gaussian processes.

We conclude the paper by giving an application of some infima results from Section 2 to level probabilities of square integrable random variables. The proof uses the Skorokhod embedding.

Notation: Throughout the paper, the letter C will denote a absolute constant, which may change of value at each occurrence.

2. Local Infima of Brownian Motion

In this section, we collect some properties of the infimum of W over bounded intervals. Precise estimates of the probability

$$\mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| \geq c\right\}.$$

will be necessary. Notice preliminary, since $-W$ and W have same law that

$$\begin{aligned} \mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) + M| \geq c\right\} &= \mathbf{P}\left\{\inf_{a \leq t \leq b} |-W(t) + M| \geq c\right\} \\ &= \mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| \geq c\right\}, \end{aligned}$$

so that it is enough to consider the case $M \geq 0$. Put

$$\Psi(x) = \mathbf{P}\{W(1) > x\} = \int_x^\infty e^{-u^2/2} \frac{du}{\sqrt{2\pi}}, \quad x \in \mathbf{R}.$$

The lemma below is certainly well-known, although we could not find a reference. We included a proof for the sake of completeness.

Lemma 2.1. *Let $0 < a < b < \infty$. Then for any $c > 0$ and any real M*

$$\mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| \geq c\right\} = \int_{|v| > c} \left[1 - 2\Psi\left(\frac{|v| - c}{\sqrt{b-a}}\right)\right] \frac{e^{-\frac{(M+v)^2}{2a}}}{\sqrt{2\pi a}} dv.$$

Proof. By symmetry of the law of W it suffices to consider the case $M \geq 0$. By the intermediate values Theorem,

$$\mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| \geq c\right\} = \mathbf{P}\left\{\inf_{a \leq t \leq b} W(t) \geq M + c\right\} + \mathbf{P}\left\{\sup_{a \leq t \leq b} W(t) \leq M - c\right\}. \quad (2.1)$$

Let $x \geq 0$. Then $\mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| \geq c \mid W(a) = M \pm x\right\} = 0$, if $0 \leq x \leq c$; and if $x > c$,

$$\mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| \geq c \mid W(a) = M + x\right\} = \mathbf{P}\left\{\sup_{a \leq t \leq b} (W(a) - W(t)) \leq x - c\right\},$$

$$\mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| \geq c \mid W(a) = M - x\right\} = \mathbf{P}\left\{\sup_{a \leq t \leq b} (W(t) - W(a)) \leq x - c\right\}.$$

As for $y \geq 0$, ([2], Theorem 1.5.1)

$$\mathbf{P}\left\{\sup_{0 \leq t \leq T} W(t) > y\right\} = 2\mathbf{P}\{W(T) > y\},$$

we get if $|x| > c$,

$$\mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| \geq c \mid W(a) = M + x\right\} = 1 - 2\Psi\left(\frac{|x| - c}{\sqrt{b-a}}\right). \quad (2.2)$$

Therefore

$$\begin{aligned}
& \mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| \geq c\right\} \\
&= \int_{\mathbf{R}} \mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| \geq c \mid W(a) = u\right\} \frac{e^{-\frac{u^2}{2a}}}{\sqrt{2\pi a}} du \\
&= \int_{|u-M| > c} \mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| \geq c \mid W(a) = u\right\} \frac{e^{-\frac{u^2}{2a}}}{\sqrt{2\pi a}} du \\
&= \int_{|u-M| > c} \left[1 - 2\Psi\left(\frac{|u-M| - c}{\sqrt{b-a}}\right)\right] \frac{e^{-\frac{u^2}{2a}}}{\sqrt{2\pi a}} du \\
&= \int_{|v| > c} \left[1 - 2\Psi\left(\frac{|v| - c}{\sqrt{b-a}}\right)\right] \frac{e^{-\frac{(M+v)^2}{2a}}}{\sqrt{2\pi a}} dv, \tag{2.3}
\end{aligned}$$

as claimed. \square

Remark 2.2. It follows from Lemma 2.1 that

$$\begin{aligned}
\mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| > 0\right\} &= \lim_{c \downarrow 0} \mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| \geq c\right\} \\
&= \int_{\mathbf{R}} \left[1 - 2\Psi\left(\frac{|v|}{\sqrt{b-a}}\right)\right] \frac{e^{-\frac{(M+v)^2}{2a}}}{\sqrt{2\pi a}} dv. \tag{2.4}
\end{aligned}$$

Thus

$$\mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| \geq 0\right\} = 1 \neq \int_{\mathbf{R}} \left[1 - 2\Psi\left(\frac{|v|}{\sqrt{b-a}}\right)\right] \frac{e^{-\frac{(M+v)^2}{2a}}}{\sqrt{2\pi a}} dv,$$

yielding a discontinuity at 0. Take for instance $a = 1$, $b = 1 + \mu^2$; the integral above is

$$\int_{\mathbf{R}} \left[1 - 2\Psi\left(\frac{|v|}{\mu}\right)\right] \frac{e^{-\frac{(M+v)^2}{2}}}{\sqrt{2\pi}} dv \rightarrow 0 \quad \mu \rightarrow \infty.$$

And

$$\mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| = 0\right\} = 2 \int_{\mathbf{R}} \Psi\left(\frac{|v|}{\sqrt{b-a}}\right) \frac{e^{-\frac{(M+v)^2}{2a}}}{\sqrt{2\pi a}} dv. \tag{2.5}$$

We will also show

Lemma 2.3. *There exists an absolute constant C , such that for every real M , and $0 < a < b < \infty$,*

$$\begin{aligned}
\mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| = 0\right\} &= 2 \int_{\mathbf{R}} \Psi\left(\frac{|v|}{\sqrt{b-a}}\right) \frac{e^{-\frac{(M+v)^2}{2a}}}{\sqrt{2\pi a}} dv \\
&\leq C \min\left(1, \sqrt{\frac{b-a}{a}} e^{-\frac{M^2}{8 \max(a, b-a)}}\right).
\end{aligned}$$

In particular

$$\mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| = 0\right\} \leq C \min\left(1, \sqrt{\frac{b-a}{a}}\right).$$

Proof. By (2.6)

$$\begin{aligned}
& \mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| = 0\right\} \\
&= 2 \int_{\mathbf{R}} \Psi\left(\frac{|v|}{\sqrt{b-a}}\right) \frac{e^{-\frac{(M+v)^2}{2a}}}{\sqrt{2\pi a}} dv \\
&= 2 \left\{ \int_{|v| \leq \frac{M}{2}} + \int_{|v| > \frac{M}{2}} \right\} \Psi\left(\frac{|v|}{\sqrt{b-a}}\right) \frac{e^{-\frac{(M+v)^2}{2a}}}{\sqrt{2\pi a}} dv \\
&\leq \sqrt{\frac{2}{\pi a}} \left\{ e^{-\frac{M^2}{8a}} \int_{\mathbf{R}} \Psi\left(\frac{|v|}{\sqrt{b-a}}\right) dv + \int_{|v| > \frac{M}{2}} \Psi\left(\frac{|v|}{\sqrt{b-a}}\right) dv \right\} \\
&= \sqrt{\frac{2}{\pi a}} \left\{ e^{-\frac{M^2}{8a}} \sqrt{b-a} \int_{\mathbf{R}} \Psi(|w|) dw + \int_{|v| > \frac{M}{2}} \Psi\left(\frac{|v|}{\sqrt{b-a}}\right) dv \right\}. \quad (2.6)
\end{aligned}$$

Recall that the Mills' ratio $R(x) = e^{x^2/2} \int_x^\infty e^{-t^2/2} dt = (\sqrt{2\pi})e^{x^2/2}\Psi(x)$ verifies for all $x \geq 0$ $R(x) \leq \sqrt{\pi/2}$. See [8], p.177. Thus

$$\begin{aligned}
\int_{|v| > \frac{M}{2}} \Psi\left(\frac{|v|}{\sqrt{b-a}}\right) dv &= \sqrt{b-a} \int_{|w| > \frac{M}{2\sqrt{b-a}}} \Psi(|w|) dw \\
&\leq C\sqrt{b-a} \int_{|w| > \frac{M}{2\sqrt{b-a}}} e^{-w^2/2} dw \\
&\leq C\sqrt{b-a} e^{-\frac{M^2}{8(b-a)}}. \quad (2.7)
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| = 0\right\} &\leq C\sqrt{\frac{b-a}{a}} \left\{ e^{-\frac{M^2}{8a}} + e^{-\frac{M^2}{8(b-a)}} \right\} \\
&\leq C\sqrt{\frac{b-a}{a}} e^{-\frac{M^2}{8\max(a, b-a)}}. \quad (2.8)
\end{aligned}$$

□

It is possible to give an exact expression of the probability $\mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| = 0\right\}$, although for $M \neq 0$ this one is relatively more complicated. This is indicated in the Lemma below.

Lemma 2.4. *Let $0 < a \leq b < \infty$. We have*

$$\begin{aligned}
\mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| = 0\right\} &= -\sqrt{\frac{2}{\pi}} \frac{M}{\sqrt{a}} \int_{\sqrt{\frac{b-a}{b}}}^1 u e^{-\frac{(Mu)^2}{2a}} \left[\int_{|x| \leq M\sqrt{\frac{1-u^2}{a}}} \frac{e^{-\frac{x^2}{2}} dx}{\sqrt{2\pi}} \right] du \\
&\quad + \left[1 - \left(\frac{2}{\pi}\right) e^{-\frac{M^2}{2a}} \arctan \sqrt{\frac{a}{b-a}} \right].
\end{aligned}$$

In particular,

$$\mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t)| = 0\right\} = 1 - \frac{2}{\pi} \arctan \sqrt{a/(b-a)}.$$

Moreover, for every positive real c ,

$$\begin{aligned} \mathbf{P}\{0 < \inf_{a \leq t \leq b} |W(t)| < c\} &= 2 \int_0^{c/\sqrt{a}} (1 - 2\Psi(\frac{u\sqrt{a}}{\sqrt{b-a}})) \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du \\ &\quad + 4 \int_{c/\sqrt{a}}^\infty (\Psi(\frac{u\sqrt{a}-c}{\sqrt{b-a}}) - \Psi(\frac{u\sqrt{a}}{\sqrt{b-a}})) \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du. \end{aligned}$$

Remark 2.5. One can recover as a special case that (see [4] p.248)

$$\mathbf{P}\{ \inf_{a \leq t \leq b} |W(t)| = 0 \} = 1 - \frac{2}{\pi} \arctan \sqrt{\frac{a}{b-a}},$$

or, equivalently $\mathbf{P}\{W(t) \text{ has no zero in } (a, b)\} = (2/\pi) \arcsin \sqrt{a/b}$. See also [6] Theorems 44 and 45 for conditional refinements (namely conditionally to $W(\alpha) = W(\beta) = 0$, with $\alpha < a < b < \beta$).

Proof. Let $M_1 = M/\sqrt{a}$, and put $F(s) = \int_{\mathbf{R}} 2\Psi(|w|s) e^{-\frac{(M_1+w)^2}{2}} \frac{dw}{\sqrt{2\pi}}$. We have

$$\begin{aligned} \mathbf{P}\{ \inf_{a \leq t \leq b} |W(t) - M| = 0 \} &= \int_{\mathbf{R}} 2\Psi(\frac{|v|}{\sqrt{b-a}}) \frac{e^{-\frac{(M+v)^2}{2a}}}{\sqrt{2\pi a}} dv \\ &= \int_{\mathbf{R}} 2\Psi(|w| \sqrt{\frac{a}{b-a}}) \frac{e^{-\frac{(M_1+w)^2}{2}}}{\sqrt{2\pi}} dw \\ &= F\left(\sqrt{\frac{a}{b-a}}\right). \end{aligned} \tag{2.9}$$

As $\frac{\partial}{\partial s} \Psi(|w|s) = |w| \Psi'(|w|s) = -\frac{|w|}{\sqrt{2\pi}} e^{-|w|s)^2/2}$, we have

$$\begin{aligned} \frac{\partial}{\partial s} F(s) &= \int_{\mathbf{R}} 2 \frac{\partial}{\partial s} \Psi(|w|s) \frac{e^{-\frac{(M_1+w)^2}{2}}}{\sqrt{2\pi}} dw \\ &= -\sqrt{\frac{2}{\pi}} \int_{\mathbf{R}} |w| e^{-\frac{1}{2}[(ws)^2 + (M_1+w)^2]} \frac{dw}{\sqrt{2\pi}}. \end{aligned}$$

But $[(ws)^2 + (M_1+w)^2] = [w\sqrt{s^2+1} + \frac{M_1}{\sqrt{s^2+1}}]^2 + \frac{M_1^2 s^2}{s^2+1}$, hence

$$\begin{aligned} \frac{\partial}{\partial s} F(s) &= -\sqrt{\frac{2}{\pi}} e^{-\frac{M_1^2 s^2}{2(s^2+1)}} \int_{\mathbf{R}} |w| e^{-\frac{1}{2}[w\sqrt{s^2+1} + \frac{M_1}{\sqrt{s^2+1}}]^2} \frac{dw}{\sqrt{2\pi}} \\ &= -\sqrt{\frac{2}{\pi}} \frac{e^{-\frac{M_1^2 s^2}{2(s^2+1)}}}{s^2+1} \int_{\mathbf{R}} |z| e^{-\frac{1}{2}[z + \frac{M_1}{\sqrt{s^2+1}}]^2} \frac{dz}{\sqrt{2\pi}} \\ &= -\sqrt{\frac{2}{\pi}} \frac{e^{-\frac{M_1^2 s^2}{2(s^2+1)}}}{s^2+1} \mathbf{E} \left| g - \frac{M_1}{\sqrt{s^2+1}} \right|, \end{aligned} \tag{2.10}$$

where g denotes a Gaussian standard random variable. But for any real a

$$\mathbf{E} |g + a| = |a| \int_{-|a|}^{|a|} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} + \sqrt{\frac{2}{\pi}} e^{-a^2/2}. \tag{2.11}$$

Hence

$$\begin{aligned}
\frac{\partial}{\partial s} F(s) &= -\sqrt{\frac{2}{\pi}} \frac{e^{-\frac{M_1^2 s^2}{2(s^2+1)}}}{s^2+1} \left\{ \frac{M_1}{\sqrt{s^2+1}} \int_{|x| \leq \frac{M_1}{\sqrt{s^2+1}}} \frac{e^{-x^2/2} dx}{\sqrt{2\pi}} + \sqrt{\frac{2}{\pi}} e^{-\frac{M_1^2}{2(s^2+1)}} \right\} \\
&= -\sqrt{\frac{2}{\pi}} \frac{M_1 e^{-\frac{M_1^2 s^2}{2(s^2+1)}}}{(s^2+1)^{3/2}} \int_{|x| \leq \frac{M_1}{\sqrt{s^2+1}}} \frac{e^{-x^2/2} dx}{\sqrt{2\pi}} - \frac{2}{\pi} \frac{e^{-M_1^2/2}}{s^2+1}. \tag{2.12}
\end{aligned}$$

If we assume for this equation $M = 0$, (recall that $M_1 = M/\sqrt{a}$) we get the much simplified form

$$\frac{\partial}{\partial s} F(s) = -\frac{2}{\pi} \frac{1}{s^2+1}.$$

Furthermore $F(0) = \int_{\mathbf{R}} e^{-\frac{(M_1+w)^2}{2}} \frac{dw}{\sqrt{2\pi}} = 1$. Therefore

$$\begin{aligned}
F(s) - 1 &= -\sqrt{\frac{2}{\pi}} M_1 \int_0^s e^{-\frac{M_1^2 t^2}{2(t^2+1)}} \left[\int_{|x| \leq \frac{M_1}{\sqrt{t^2+1}}} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} \right] \frac{dt}{(t^2+1)^{3/2}} \\
&\quad - \frac{2}{\pi} e^{-M_1^2/2} \int_0^s \frac{dt}{t^2+1} \\
&= -e^{-M_1^2/2} \left\{ \sqrt{\frac{2}{\pi}} M_1 \int_0^s \left[\int_{|x| \leq \frac{M_1}{\sqrt{t^2+1}}} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} \right] \frac{e^{\frac{M_1^2}{2(t^2+1)}} dt}{(t^2+1)^{3/2}} \right. \\
&\quad \left. - \frac{2}{\pi} \arctan s \right\}. \tag{2.13}
\end{aligned}$$

Consequently,

$$\begin{aligned}
\mathbf{P}\left\{ \inf_{a \leq t \leq b} |W(t) - M| = 0 \right\} &= F\left(\sqrt{\frac{a}{b-a}}\right) \\
&= -\sqrt{\frac{2}{\pi}} \frac{M}{\sqrt{a}} \int_{\sqrt{\frac{b-a}{a}}}^1 u e^{-\frac{(Mu)^2}{2a}} \left[\int_{|x| \leq M\sqrt{\frac{1-u^2}{a}}} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \right] du \\
&\quad + \left[1 - \frac{2}{\pi} e^{-\frac{M^2}{2a}} \arctan \sqrt{\frac{a}{b-a}} \right]. \tag{2.14}
\end{aligned}$$

If $M = 0$, this is simplified into

$$\mathbf{P}\left\{ \inf_{a \leq t \leq b} |W(t)| = 0 \right\} = 1 - \frac{2}{\pi} \arctan \sqrt{\frac{a}{b-a}}. \tag{2.15}$$

As concerning the second formula, it suffices to write

$$\begin{aligned}
\mathbf{P}\{0 < \inf_{a \leq t \leq b} |W(t)| \leq c\} &= \mathbf{P}\left\{ \inf_{a \leq t \leq b} |W(t)| \leq c \right\} - \mathbf{P}\left\{ \inf_{a \leq t \leq b} |W(t)| = 0 \right\} \\
&= \left(1 - 2 \int_{\frac{c}{\sqrt{a}}}^{\infty} \left(1 - 2\Psi\left(\frac{u\sqrt{a}-c}{\sqrt{b-a}}\right) \right) \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du \right) \\
&\quad - \left(1 - 2 \int_0^{\infty} \left(1 - 2\Psi\left(\frac{u\sqrt{a}}{\sqrt{b-a}}\right) \right) \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du \right)
\end{aligned}$$

$$\begin{aligned}
&= -2 \int_{\frac{c}{\sqrt{a}}}^{\infty} \left(1 - 2\Psi\left(\frac{u\sqrt{a}-c}{\sqrt{b-a}}\right)\right) \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du \\
&\quad + 2 \int_0^{\frac{c}{\sqrt{a}}} \left(1 - 2\Psi\left(\frac{u\sqrt{a}}{\sqrt{b-a}}\right)\right) \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du \\
&= 2 \int_0^{\frac{c}{\sqrt{a}}} \left(1 - 2\Psi\left(\frac{u\sqrt{a}}{\sqrt{b-a}}\right)\right) \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du \\
&\quad + 4 \int_{\frac{c}{\sqrt{a}}}^{\infty} \left(\Psi\left(\frac{u\sqrt{a}-c}{\sqrt{b-a}}\right) - \Psi\left(\frac{u\sqrt{a}}{\sqrt{b-a}}\right)\right) \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du.
\end{aligned}$$

□

Notice that $\frac{\pi}{2} = \int_0^{\infty} \frac{dt}{1+t^2}$, and so

$$\begin{aligned}
1 - \frac{2}{\pi} \arctan s &= \frac{2}{\pi} \left[\int_0^{\infty} \frac{dt}{1+t^2} - \int_0^s \frac{dt}{1+t^2} \right] \\
&= \frac{2}{\pi} \int_s^{\infty} \frac{dt}{1+t^2} \leq C \min(1, s^{-1}).
\end{aligned}$$

We deduce the bound obtained in Lemma 2.3

$$\mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t)| = 0\right\} \leq C \min\left(1, \sqrt{\frac{b-a}{a}}\right).$$

In what follows we shall be interested in finding estimates of the delicate random variable $\beta_{[a,b]}^{-\alpha} \cdot \chi_{\{\beta_{[a,b]} > 0\}}$, $0 \leq \alpha < 1$, where we set

$$\beta_{[a,b]} := \inf_{a \leq t \leq b} |W(t)|.$$

Proposition 2.6. *Let $b > a > 0$, and $\eta > 0$. Then,*

$$\mathbf{P}\{0 < \beta_{[a,b]} \leq \eta\} \leq \frac{10\eta}{\sqrt{2\pi}} \min\left(\frac{1}{\sqrt{b-a}}, \frac{1}{\sqrt{a}}\right).$$

Furthermore, for any real α , $0 \leq \alpha < 1$

$$\mathbf{E}\left\{\frac{1}{\beta_{[a,b]}^{\alpha}} \cdot \chi_{\{\beta_{[a,b]} > 0\}}\right\} \leq \frac{29}{1-\alpha} \min\left(\frac{1}{\sqrt{b-a}}, \frac{1}{\sqrt{a}}\right) + 1.$$

Remark 2.7. The result gives a control which is uniform in $b-a$. We have $\beta_{[a,b]} \rightarrow 0$ as $b-a \rightarrow \infty$, but in the same time the constraint $\beta_{[a,b]} > 0$ becomes also stronger, making the probability of the set $\{\beta_{[a,b]} > 0\}$ small. When $b-a \rightarrow 0$, $\beta_{[a,b]} \rightarrow |W(a)|$, and this is reflected by the term $1/\sqrt{a}$ in our estimate.

Proof. Write $\beta = \beta_{[a,b]}$. By using the second formula in Lemma 2.4

$$\begin{aligned}
\mathbf{P}\{0 < \beta \leq \eta\} &= 2 \int_0^{\eta/\sqrt{a}} \left(1 - 2\Psi\left(\frac{u\sqrt{a}}{\sqrt{b-a}}\right)\right) \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du \\
&\quad + 4 \int_{\eta/\sqrt{a}}^{\infty} \left(\Psi\left(\frac{u\sqrt{a}-\eta}{\sqrt{b-a}}\right) - \Psi\left(\frac{u\sqrt{a}}{\sqrt{b-a}}\right)\right) \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du. \quad (2.16)
\end{aligned}$$

As $1 - 2\Psi(x) = \int_{-x}^x e^{-u^2/2} \frac{du}{\sqrt{2\pi}} \leq \min((2/\pi)^{1/2}x, 1)$, $x \geq 0$,

$$\begin{aligned}
& \int_0^{\eta/\sqrt{a}} \left(1 - 2\Psi\left(\frac{u\sqrt{a}}{\sqrt{b-a}}\right)\right) \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du \\
& \leq \int_0^{\eta/\sqrt{a}} \min\left(\left(\frac{2}{\pi}\right)^{1/2} \frac{u\sqrt{a}}{\sqrt{b-a}}, 1\right) \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du \\
& \leq \min\left(\left(\frac{2}{\pi}\right)^{1/2} \frac{\sqrt{a}}{\sqrt{b-a}} \int_0^{\eta/\sqrt{a}} u e^{-\frac{u^2}{2}} \sqrt{2\pi} du, \frac{\eta}{\sqrt{2\pi a}}\right) \\
& \leq \min\left(\frac{\eta \max_{u \geq 0} u e^{-\frac{u^2}{2}}}{\pi \sqrt{b-a}}, \frac{\eta}{\sqrt{2\pi a}}\right) = \eta \min\left(\frac{1}{\pi \sqrt{b-a}}, \frac{1}{\sqrt{2\pi a}}\right). \quad (2.17)
\end{aligned}$$

Furthermore

$$\Psi\left(\frac{u\sqrt{a}-\eta}{\sqrt{b-a}}\right) - \Psi\left(\frac{u\sqrt{a}}{\sqrt{b-a}}\right) = \int_{\frac{u\sqrt{a}-\eta}{\sqrt{b-a}}}^{\frac{u\sqrt{a}}{\sqrt{b-a}}} e^{-\frac{u^2}{2}} \frac{du}{\sqrt{2\pi}} \leq \frac{\eta}{\sqrt{2\pi(b-a)}}, \quad (2.18)$$

which implies

$$\begin{aligned}
\int_{\eta/\sqrt{a}}^{\infty} \left(\Psi\left(\frac{u\sqrt{a}-\eta}{\sqrt{b-a}}\right) - \Psi\left(\frac{u\sqrt{a}}{\sqrt{b-a}}\right)\right) \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du & \leq \frac{\eta}{\sqrt{2\pi(b-a)}} \int_{\eta/\sqrt{a}}^{\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du \\
& \leq \frac{\eta}{\sqrt{2\pi(b-a)}}. \quad (2.19)
\end{aligned}$$

Besides, with the variable change $u = v\sqrt{\frac{b-a}{a}}$, letting $v_\eta = \frac{\eta}{\sqrt{b-a}}$,

$$\begin{aligned}
& \int_{\eta/\sqrt{a}}^{\infty} \left(\Psi\left(\frac{u\sqrt{a}-\eta}{\sqrt{b-a}}\right) - \Psi\left(\frac{u\sqrt{a}}{\sqrt{b-a}}\right)\right) \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du \\
& = \sqrt{\frac{b-a}{a}} \int_{v_\eta}^{\infty} (\Psi(v - v_\eta) - \Psi(v)) \frac{e^{-\frac{v^2(b-a)}{2a}}}{\sqrt{2\pi}} dv.
\end{aligned}$$

We have

$$\sqrt{\frac{b-a}{a}} \int_{v_\eta}^{2v_\eta} (\Psi(v - v_\eta) - \Psi(v)) \frac{e^{-\frac{v^2(b-a)}{2a}}}{\sqrt{2\pi}} dv \leq \sqrt{\frac{b-a}{a}} \frac{v_\eta}{\sqrt{2\pi}} = \frac{\eta}{\sqrt{2\pi a}}.$$

Now if $v \geq 2v_\eta$, then $v - v_\eta \geq v - v/2 = v/2$. Consequently

$$\begin{aligned}
& \sqrt{\frac{b-a}{a}} \int_{2v_\eta}^{\infty} (\Psi(v - v_\eta) - \Psi(v)) \frac{e^{-\frac{v^2(b-a)}{2a}}}{\sqrt{2\pi}} dv \\
& = \sqrt{\frac{b-a}{a}} \int_{2v_\eta}^{\infty} \left\{ \int_{v-v_\eta}^v \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx \right\} \frac{e^{-\frac{v^2(b-a)}{2a}}}{\sqrt{2\pi}} dv \\
& \leq \sqrt{\frac{b-a}{a}} v_\eta \int_{2v_\eta}^{\infty} \frac{e^{-v^2/8}}{\sqrt{2\pi}} \frac{e^{-\frac{v^2(b-a)}{2a}}}{\sqrt{2\pi}} dv \\
& \leq \sqrt{\frac{b-a}{a}} v_\eta \int_0^{\infty} \frac{e^{-v^2/8}}{2\pi} dv = \frac{\eta}{\sqrt{2\pi a}}.
\end{aligned}$$

It follows that

$$\begin{aligned} \int_{\eta/\sqrt{a}}^{\infty} \left(\Psi\left(\frac{u\sqrt{a}-\eta}{\sqrt{b-a}}\right) - \Psi\left(\frac{u\sqrt{a}}{\sqrt{b-a}}\right) \right) \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du &\leq \min\left(\frac{2\eta}{\sqrt{2\pi a}}, \frac{\eta}{\sqrt{2\pi(b-a)}}\right) \\ &\leq \frac{2\eta}{\sqrt{2\pi}} \min\left(\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b-a}}\right). \end{aligned} \quad (2.20)$$

By reporting,

$$\begin{aligned} \mathbf{P}\{0 < \beta \leq \eta\} &\leq 2\eta \min\left(\frac{1}{\pi\sqrt{b-a}}, \frac{1}{\sqrt{2\pi a}}\right) + \frac{8\eta}{\sqrt{2\pi}} \min\left(\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b-a}}\right) \\ &\leq \frac{10\eta}{\sqrt{2\pi}} \min\left(\frac{1}{\sqrt{b-a}}, \frac{1}{\sqrt{a}}\right). \end{aligned} \quad (2.21)$$

Now let X be a random variable such that $X \geq 0$ almost surely. As $\chi\{[0, 1]\} = \sum_{n=0}^{\infty} \chi\{[\frac{1}{2^{n+1}}, \frac{1}{2^n}]\}$, we have by the Beppo-Levi theorem

$$\mathbf{E} \frac{1}{X^\alpha} \cdot \chi\{X > 0\} = \sum_{n=0}^{\infty} \mathbf{E} \left(\frac{1}{X^\alpha} \cdot \chi\left\{\frac{1}{2^{n+1}} < X \leq \frac{1}{2^n}\right\} \right) + \mathbf{E} \frac{1}{X^\alpha} \cdot \chi\{X > 1\}, \quad (2.22)$$

where "=" means that the previous expressions are equal when finite or both infinite. Moreover,

$$\begin{aligned} &\sum_{n=0}^{\infty} \mathbf{E} \left(\frac{1}{X^\alpha} \cdot \chi\left\{\frac{1}{2^{n+1}} < X \leq \frac{1}{2^n}\right\} \right) + \mathbf{E} \frac{1}{X^\alpha} \cdot \chi\{X > 1\} \\ &\leq \sum_{n=0}^{\infty} 2^{\alpha(n+1)} \mathbf{P}\{0 < X \leq \frac{1}{2^n}\} + 1. \end{aligned} \quad (2.23)$$

Apply this with $X = \beta$. Then $\mathbf{E} \beta^{-\alpha} \cdot \chi\{\beta > 0\}$ will be finite once we prove that the series

$$\sum_{n=0}^{\infty} 2^{\alpha(n+1)} \mathbf{P}\{0 < \beta \leq \frac{1}{2^n}\}$$

is convergent. By using (2.21) with $\eta = 2^{-n}$, we get

$$\begin{aligned} \sum_{n=0}^{\infty} 2^{\alpha(n+1)} \mathbf{P}\{0 < \beta \leq \frac{1}{2^n}\} &\leq \frac{2^{\alpha+4}}{\sqrt{2\pi}} \min\left(\frac{1}{\sqrt{b-a}}, \frac{1}{\sqrt{a}}\right) \sum_{n=0}^{\infty} 2^{-n(1-\alpha)} \\ &\leq \frac{20}{2^{1-\alpha} - 1} \min\left(\frac{1}{\sqrt{b-a}}, \frac{1}{\sqrt{a}}\right) \\ &\leq \frac{29}{(1-\alpha)} \min\left(\frac{1}{\sqrt{b-a}}, \frac{1}{\sqrt{a}}\right). \end{aligned} \quad (2.24)$$

And we conclude that

$$\mathbf{E} \frac{1}{\beta^\alpha} \cdot \chi\{\beta > 0\} \leq \frac{29}{1-\alpha} \min\left(\frac{1}{\sqrt{b-a}}, \frac{1}{\sqrt{a}}\right) + 1,$$

as claimed. \square

Remark 2.8. (Hitting probabilities) Given some bounded interval I and reals $0 < a < b < \infty$, we define the restricted hitting probability for the brownian motion by

$$G(a, b, I) = \mathbf{P}\{\exists t; a \leq t \leq b : W(t) \in I\}.$$

When $a = 0$, this coincides with the usual hitting probability

$$G(b, I) = \mathbf{P}\{\exists t \leq b : W(t) \in I\}.$$

The results obtained in this section have a direct application to hitting probabilities, since for instance, by Lemma 2.1 (writing $I = [M - c, M + c]$),

$$\begin{aligned} G(a, b, I) &= 1 - \mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| \geq c\right\} \\ &= 1 - \int_{|v| > c} \left[1 - 2\Psi\left(\frac{|v| - c}{\sqrt{b - a}}\right)\right] \frac{e^{-\frac{(M+v)^2}{2a}}}{\sqrt{2\pi a}} dv. \end{aligned}$$

3. Proof of Theorem

Recall that $I_N = [T_N, T_{N+1}]$. It is now easy. As

$$\sum_{N \geq 1} \mathbf{P}\{0 < \beta_{I_N} \leq \eta_N\} \leq C \sum_{N \geq 1} \eta_N \min\left(\frac{1}{\sqrt{T_{N+1} - T_N}}, \frac{1}{\sqrt{T_N}}\right) < \infty,$$

we deduce from Borel-Cantelli lemma that

$$\mathbf{P}\left\{\inf_{t \in I_N} |W(t)| \geq \eta_N \text{ or } W(t) = 0 \text{ for some } t \in I_N, \quad N \text{ eventually}\right\} = 1.$$

4. An Application to Level Sets

The notion of level sets associated to a given random variable X , namely the sets of type $\{X = M\}$ where M is any real, is very important in probability theory. There are easy examples yielding notably their role in the study of diophantine equations. Let for instance $N_n(P)$ denote the number of solutions diophantine equation

$$x_1 + \dots + x_n = y_1 + \dots + y_n, \quad (4.1)$$

in which x_i, y_j are integers and $0 \leq x_i, y_j \leq P$. Let also $u(k)$ denote the number of solutions of the equation $x - y = k$, $0 \leq x \leq P$, $0 \leq y \leq P$. We have

$$u(k) = \begin{cases} P + 1 - |k|, & \text{if } |k| \leq P, \\ 0, & \text{if } |k| > P. \end{cases} \quad (4.2)$$

Let X be a (centered) random variable defined by

$$\mathbf{P}\{X = k\} = \frac{u(k)}{(P + 1)^2}, \quad k \in \mathbf{Z}.$$

Let X_1, \dots, X_n be independent copies of X and denote $S_n = X_1 + \dots + X_n$. Then

$$\mathbf{P}\{S_n = 0\} = \sum_{k_1 + \dots + k_n = 0, |k_i| \leq P} \frac{u(k_1) \dots u(k_n)}{(P + 1)^{2n}} = \frac{N_n(P)}{(P + 1)^{2n}}.$$

By Gamkrelidze's inequality, ([10], p.56) we have

$$N_n(P) \leq C \frac{(P + 1)^{2n}}{\sqrt{n}}. \quad (4.3)$$

In fact, it is possible to show that there exist absolute constants

$$C' \frac{P^{2n-1}}{\sqrt{n}} \leq N_n(P) \leq C'' \frac{P^{2n}}{\sqrt{n}},$$

where C' and C'' are absolute constants.

There are nearly optimal estimates of the level probabilities of i.i.d. sums. In the independent case however, the picture is much less clear, even for weighted sums of i.i.d. random variables. The purpose of this section is to show that some of the infima results proved before can be combined with the Skorokhod embedding to establish a new level probability inequality for general centered square integrable random variables.

If \mathcal{Z} is such a random variable, by the Skorokhod embedding there exists a stopping time T such that $W(T) \stackrel{\mathcal{D}}{=} \mathcal{Z}$ and $\mathbf{E}T = \mathbf{E}\mathcal{Z}^2$. In fact, any centered measure μ on the real line embeds into W : there exists a stopping time τ such that $W(\tau) \stackrel{\mathcal{D}}{=} \mu$, and further $\{W(\tau \wedge t) : t \geq 0\}$ is a uniformly bounded martingale. Moreover, by the Burkholder, Davis, Gundy and Millar inequalities (see Proposition 2.1 in [9] and [1], [3] p.697 and estimates (1.10) in [12]), for any $1 \leq r < \infty$, there exist universal constants c_r, C_r such that

$$c_r^r \mathbf{E} \tau^{r/2} \leq \mathbf{E} |W(\tau)|^r = \int_{\mathbf{R}} |x|^r \mu(dx) \leq C_r^r \mathbf{E} \tau^{r/2}, \quad (4.4)$$

and $C_r^r = 2(8/\pi^2)^{r-1} \Gamma(r+1)$. There exists an abundant literature on this powerful approach and a still extensive work. We may refer to [9] p.332 where the explicit construction of T , which is the Skorokhod stopping time, is given, but also to [11], [12], and to the less known result [5], when specializing to weighted sums of i.i.d. random variables (see [7] as well and [13], p.527).

Having recalled these important facts, we can now state

Theorem 4.1. *Let \mathcal{Z} be a centered random variable such that $\mathbf{E}\mathcal{Z}^{2s} < \infty$ for some $1 \leq s < \infty$. Then there exists an absolute constant C_0 such that for every reals M and $0 < \eta < 1$,*

$$\mathbf{P}\{\mathcal{Z} = M\} \leq \frac{1}{\eta^s} \frac{\mathbf{E}|T - \mathbf{E}T|^s}{(\mathbf{E}T)^s} + C_0 \sqrt{\frac{\eta}{1-\eta}} e^{-\frac{M^2}{8 \max(2\eta, 1-\eta) \mathbf{E}\mathcal{Z}^2}}.$$

Remark 4.2. In many specific cases, like weighted i.i.d. sums, the ratio $\frac{\mathbf{E}|T - \mathbf{E}T|^s}{(\mathbf{E}T)^s}$ can be efficiently estimated by means of (4.4) and Rosenthal's inequality (see for instance [13], p.352).

Proof. Let $\varepsilon > 0$. Then

$$\begin{aligned} \mathbf{P}\{0 \leq |\mathcal{Z} - x| \leq \varepsilon\} &\leq \mathbf{P}\left\{\left|\frac{T}{\mathbf{E}T} - 1\right| > \eta\right\} \\ &+ \mathbf{P}\left\{|W(T) - x| \leq \varepsilon, \left|\frac{T}{\mathbf{E}T} - 1\right| \leq \eta\right\}. \\ &\leq \mathbf{P}\left\{\left|\frac{T}{\mathbf{E}T} - 1\right| > \eta\right\} + \mathbf{P}\left\{\inf_{|t - \mathbf{E}T| \leq \eta \mathbf{E}T} |W(t) - x| \leq \varepsilon\right\}. \end{aligned}$$

Letting ε tend to 0 gives

$$\mathbf{P}\{\mathcal{Z} = x\} \leq \mathbf{P}\left\{\left|\frac{T}{\mathbf{E}T} - 1\right| > \eta\right\} + \mathbf{P}\left\{\inf_{|t - \mathbf{E}T| \leq \eta \mathbf{E}T} |W(t) - x| = 0\right\}. \quad (4.5)$$

Now apply Lemma 2.3 with the choices $b = (1 + \eta)\mathbf{E}T$, $a = (1 - \eta)\mathbf{E}T$. We get

$$\mathbf{P}\left\{\inf_{|t - \mathbf{E}T| \leq \eta \mathbf{E}T} |W(t) - M| = 0\right\} \leq C \sqrt{\frac{\eta}{1 - \eta}} e^{-\frac{M^2}{8 \max(2\eta, 1 - \eta) \mathbf{E}T}}.$$

And so by (4.5) and Tchebycheff's inequality,

$$\mathbf{P}\{\mathcal{Z} = M\} \leq \frac{1}{\eta^s} \frac{\mathbf{E}|T - \mathbf{E}T|^s}{(\mathbf{E}T)^s} + C \sqrt{\frac{\eta}{1 - \eta}} e^{-\frac{M^2}{8 \max(2\eta, 1 - \eta) \mathbf{E}T}}.$$

Since $\mathbf{E}T = \mathbf{E}Z^2$, this provides the claimed result. \square

We conclude by indicating how to derive a more specific estimate for the limited range of values $|M| \leq B\|\mathcal{Z}\|_2$, under suitable integrability assumptions.

Corollary 4.3. *Let \mathcal{Z} be centered and such that $\mathbf{E}Z^{2s} < \infty$ for some $1 \leq s < \infty$. For any $\lambda \geq 1$, $0 < \theta < 1$ satisfying $e^\lambda \leq (C_0/2^{s+\frac{1}{2}}) \log \frac{1}{\theta}$, C_0 being the same constant as in Theorem 4.1, and*

$$\|T - \mathbf{E}T\|_s \leq \theta \mathbf{E}T, \quad (4.6)$$

we have for $|M| \leq 2\sqrt{\lambda}\|\mathcal{Z}\|_2$,

$$\mathbf{P}\{\mathcal{Z} = M\} \leq 2(2sC_0^{2s})^{\frac{1}{2s+1}} \theta^{\frac{s}{2s+1}} e^{-\frac{s}{2(2s+1)} \frac{M^2}{\mathbf{E}Z^2}}.$$

Remark 4.4. The application of Corollary 4.3 to the study of some families of diophantine equations will be made elsewhere. Notice that if $|M| > 2\sqrt{\lambda}\|\mathcal{Z}\|_2$, by giving to η in Theorem 4.1 any fixed value, we also have $\mathbf{P}\{\mathcal{Z} = M\} \leq C_1(\theta^s + e^{-\lambda})$, for some absolute constant C_1 .

Proof. Let $0 < \eta \leq 1/2$. By Theorem 4.1 and assumption (4.6),

$$\mathbf{P}\{\mathcal{Z} = M\} \leq \theta^s \eta^{-s} + C_0 \eta^{1/2} e^{-\frac{M^2}{4\mathbf{E}Z^2}}.$$

The function $\varphi(\eta) = A\eta^{-s} + B\eta^{1/2}$, $A > 0$, $B > 0$ reaches its minimum at the value $\eta_0 = (2sA/B)^{\frac{2}{2s+1}}$ and

$$\varphi(\eta_0) \leq \left(\frac{2s+1}{2s}\right)(2sA)^{1/(2s+1)} B^{2s/(2s+1)}.$$

In our case $A = \theta^s$, $B = C_0 e^{-M^2/4\mathbf{E}Z^2}$, and so $\eta_0 \leq 1/2$ if

$$\frac{2sA}{B} = \frac{2s\theta^s}{C_0} e^{\frac{M^2}{4\mathbf{E}Z^2}} \leq \left(\frac{1}{2}\right)^{s+\frac{1}{2}}.$$

The function $f(s) = s\theta^s$ being decreasing over the interval $[1/|\log \theta|, \infty)$ with maximal value $e^{-1}/|\log \theta|$, we have by assumption

$$\frac{2s\theta^s}{C_0} e^\lambda \leq \frac{2e^{-1}e^\lambda}{C_0|\log \theta|} \leq \frac{e^\lambda}{C_0|\log \theta|} \leq \left(\frac{1}{2}\right)^{s+\frac{1}{2}}.$$

We can thus take $\eta = \eta_0$, and obtain

$$\mathbf{P}\{Z = M\} \leq \left(\frac{2s+1}{2s}\right) (2sC_0^{2s})^{\frac{1}{2s+1}} \theta^{\frac{s}{2s+1}} e^{-\frac{sM^2}{2(2s+1)\mathbf{E}Z^2}},$$

which easily implies the claimed result. \square

Acknowledgment. I thank István Berkes for valuable remarks and a careful reading of the paper. I also thank the referees for constructive remarks.

References

1. Burkholder, D. and Gundy, R.: Extrapolation and interpolation of quasi-linear operators on martingales, *Acta Math.* **124** (1970) 249–304.
2. Csörgö, M. and Révész, P.: *Strong Approximations in Probability and Statistics*, Akadémiai Kiadó, Budapest, 1981.
3. Davis, B.: On the L_p norms of stochastic integrals and other martingales, *Duke Math. J.* **43** (1976) 697–704.
4. Durrett, R.: *Stochastic Calculus: A Practical Introduction*, Prob.& Stoch., 1996.
5. Fisher, E.: A Skorokhod representation and an invariance principle for sums of weighted i.i.d. random variables, *Rocky Mount. J. Math.* **22** (1992) 169–179.
6. Lévy, P.: *Processus Stochastiques et Mouvement Brownien*, Gauthier-Villars, Paris, 1948.
7. Lin, M. and Weber, M.: Weighted laws of the iterated logarithm for sums of iid random variables, *Contemporary Math.* **485** (2009) 115–129.
8. Mitrinović, D. S.: *Analytic Inequalities*, Springer Verlag **165**, 1970.
9. Obloj, J. The Skorokhod embedding problem and its offspring, *Probability Surveys* **1** (2004) 321–382.
10. Petrov, V. V.: *Sums of Independent Random Variables*, Ergebnisse der Math. und ihre Grenzgebiete **82**, Springer, 1975.
11. Sawyer, S.: A uniform rate of convergence for the maximum absolute value of partial sums in probability, *Comm. Pure Appl. Math.* **20** (1967) 647–658.
12. Sawyer, S.: The Skorokhod representation, *Rocky Mountain J. of Math.* **4** (1974) 579–596.
13. Weber, M.: *Dynamical Systems and Processes*, European Mathematical Society Publishing House, IRMA Lectures in Mathematics and Theoretical Physics **14**, 2009.

MICHEL WEBER: IRMA, UNIVERSITÉ LOUIS-PASTEUR ET C.N.R.S., 7 RUE RENÉ DESCARTES, 67084 STRASBOURG CEDEX, FRANCE.

E-mail address: michel.weber@math.unistra.fr

URL: <http://www-irma.u-strasbg.fr/~weber/>