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A FIRST-PASSAGE PROBLEM FOR EXPONENTIAL INTEGRATED DIFFUSION PROCESSES

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Abstract. Let $\mathrm{d}Y(t) = Z(t)\,\mathrm{d}t$, where $Z(t)$ is a one-dimensional diffusion process, and $X(t) = X(0)\,e^{Y(t)-Y(0)}$. We denote by $T(x,z)$ the first time the two-dimensional process $(X(t), Z(t))$ leaves a rectangle located in the first quadrant. The problem of computing the moment-generating function M and the mean m of $T(x,z)$ is considered. Explicit results are obtained in important particular cases for the Laplace transform of M and of m. A generalization is also presented.

1. Introduction

Let $\{(Y(t), Z(t)), t \geq 0\}$ be the degenerate two-dimensional diffusion process defined by

$$\begin{aligned} \mathrm{d}Y(t) &= Z(t)\,\mathrm{d}t, \\ \mathrm{d}Z(t) &= f[Z(t)]\,\mathrm{d}t + \{v[Z(t)]\}^{1/2}\,\mathrm{d}B(t), \end{aligned}$$

where $\{B(t), t \geq 0\}$ is a standard Brownian motion and the functions f and v are such that $\{Z(t), t \geq 0\}$ is a diffusion process. The process $\{Y(t), t \geq 0\}$ is known as an integrated diffusion process.

First-passage problems for integrated diffusion processes are very difficult to solve explicitly. The case when $\{Z(t), t \geq 0\}$ is a Wiener process has been considered, in particular, by McKean (1963), Goldman (1971), Gor'kov (1975), Lefebvre and Léonard (1989), Lachal (1991 and 1993) and Lefebvre (2006). Lefebvre (1989) and Hesse (1991) treated the case when $\{Z(t), t \geq 0\}$ is an Ornstein-Uhlenbeck process. Papers on such problems for integrated geometric Brownian motions include those by Lefebvre (2004), Makasu (2009), Metzler (2013), Mansour *et al.* (2016) and Levy (2018). See also Lefebvre (2002) and Benedetto *et al.* (2013).

In this paper, we consider the exponential of integrated diffusion processes, which we denote by $X(t)$:

$$X(t) := X(0)\,e^{Y(t)-Y(0)} \quad \text{for } t \geq 0.$$

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The three-dimensional diffusion process $\{(X(t), Y(t), Z(t)), t \geq 0\}$ can thus be defined as follows:

$$\begin{aligned} \mathrm{d}X(t) &= X(t)\,Z(t)\,\mathrm{d}t, \\ \mathrm{d}Y(t) &= Z(t)\,\mathrm{d}t, \\ \mathrm{d}Z(t) &= f[Z(t)]\,\mathrm{d}t + \{v[Z(t)]\}^{1/2}\,\mathrm{d}B(t). \end{aligned}$$

Let R be the rectangle defined by

$$R = \{(x, z) \in \mathbb{R}^2 : 0 \leq a < x < b, 0 \leq c < z < d\} \tag{1.1}$$

and let $T(x, z)$ be the first-exit time from R for the two-dimensional process $\{(X(t), Z(t)), t \geq 0\}$:

$$T(x, z) := \inf\{t > 0 : (X(t), Z(t)) \notin R \mid (X(0), Z(0)) = (x, z) \in R\}. \tag{1.2}$$

Notice that because $X(t)$ and $Z(t)$ are both positive inside R, the stochastic process $(X(t), Z(t))$ cannot exit the rectangle through the boundary $x = a$. Indeed, $X(t)$ is strictly increasing inside R.

Next, assume that $\alpha > 0$ and denote the moment-generating function of $T(x, z)$ by $M(x, z; \alpha)$:

$$M(x, z; \alpha) := E\left[e^{-\alpha T(x,z)}\right].$$

Lemma 1.1. *The function M satisfies the Kolmogorov backward equation*

$$\frac{1}{2} v(z)\, M_{zz} + f(z)\, M_z + x\, z\, M_x = \alpha\, M \quad \textit{for } (x, z) \in R, \tag{1.3}$$

where $M_{zz} := \partial^2 M / \partial z^2$, etc., and is subject to the boundary condition

$$M(x, z; \alpha) = 1 \quad \textit{for } (x, z) \notin R. \tag{1.4}$$

Proof. The conditional transition density function

$$p(x, x_0; t, t_0) := f_{X(t)|X(t_0)}(x \mid x_0)$$

satisfies the Kolmogorov backward equation (see, for instance, Cox and Miller (1965, p. 247) or Karatzas and Shreve (1991, p. 282))

$$\frac{1}{2} v(z) p_{zz} + f(z) p_z + x\, z\, p_x = p_t \quad \text{for } (x, z) \in R.$$

Moreover, the probability density function $\rho(t; x, z)$ of the random variable $T(x, z)$ satisfies the above equation as well. Multiplying both sides of the equation by $e^{-\alpha t}$ and integrating from $t_0 = 0$ to ∞, we find that the function M is indeed a solution of Eq. (1.3). Finally, the boundary condition follows at once from the fact that $T(x, z) = 0$ if $(x, z) \notin R$. □

Corollary 1.2. *The mean first-exit time $m(x, z) := E[T(x, z)]$ satisfies the partial differential equation*

$$\frac{1}{2} v(z)\, m_{zz} + f(z)\, m_z + x\, z\, m_x = -1 \quad \textit{for } (x, z) \in R, \tag{1.5}$$

subject to the boundary condition

$$m(x, z) = 0 \quad \textit{for } (x, z) \notin R. \tag{1.6}$$

Proof. We have, under appropriate hypotheses,

$$M(x,z;\alpha) := E\left[e^{-\alpha T(x,z)}\right] = 1 - \alpha\, E[T(x,z)] + \frac{\alpha^2}{2}\, E[T^2(x,z)] - \dots$$

Substituting the above expression into Eq. (1.3), we deduce that $m(x,z)$ indeed satisfies Eq. (1.5). The boundary condition follows again from the fact that $T(x,z) = 0$ if $(x,z) \notin R$. □

For the sake of simplicity, let us assume that $a = 0$ and $b = 1$ in the definition of the rectangle R. Notice that there is no loss of generality, since we could simply define $X^*(t) = [X(t) - a]/(b-a)$. Moreover, let $w := -\ln(x)$. If we denote $M(-\ln(x), z; \alpha)$ by $N(w,z;\alpha)$ and $m(-\ln(x), z)$ by $n(w,z)$, then Eqs. (1.3) and (1.5) respectively become

$$\frac{1}{2}\, v(z)\, N_{zz} + f(z)\, N_z - z\, N_w = \alpha\, N$$

and

$$\frac{1}{2}\, v(z)\, n_{zz} + f(z)\, n_z - z\, n_w = -1.$$

These equations are valid in the infinite rectangle

$$R_w := \{(w,z) \in \mathbb{R}^2 : w > 0, 0 \le c < z < d\}$$

and are subject to the boundary conditions

$$N(w,z;\alpha) = 1 \quad \text{and} \quad n(w,z) = 0 \quad \text{for } (w,z) \notin R_w. \tag{1.7}$$

Proposition 1.3. *Assume that* $\beta > 0$ *and define*

$$L(z;\alpha,\beta) = \int_0^\infty e^{-\beta w}\, N(w,z;\alpha)\, \mathrm{d}w \quad \text{and} \quad l(z;\beta) = \int_0^\infty e^{-\beta w}\, n(w,z)\, \mathrm{d}w.$$

That is, $L(z;\alpha,\beta)$ *and* $l(z;\beta)$ *are the Laplace transforms of the functions* $N(w,z;\alpha)$ *and* $n(w,z)$*, respectively. We find that* $L(z;\alpha,\beta)$ *and* $l(z;\beta)$ *satisfy the following ordinary differential equations:*

$$\frac{1}{2}\, v(z)\, L_{zz} + f(z)\, L_z - z\,(-1 + \beta\, L) = \alpha\, L \tag{1.8}$$

and

$$\frac{1}{2}\, v(z)\, l_{zz} + f(z)\, l_z - \beta\, z\, l = -\frac{1}{\beta}. \tag{1.9}$$

The boundary conditions are

$$L(z;\alpha,\beta) = \frac{1}{\beta} \quad \text{and} \quad l(z;\beta) = 0 \quad \textit{for } z = c \textit{ or } d. \tag{1.10}$$

Proof. We have, making use of the boundary condition (1.7) and the fact that $N(w,z;\alpha)$ is bounded,

$$\begin{aligned} \int_0^\infty e^{-\beta w}\, N_w(w,z;\alpha)\, \mathrm{d}w &= e^{-\beta w}\, N(w,z;\alpha)\Big|_0^\infty + \beta \int_0^\infty e^{-\beta w}\, N(w,z;\alpha)\, \mathrm{d}w \\ &= -1 + \beta\, L(z;\alpha,\beta), \end{aligned}$$

from which we deduce Eq. (1.8). Similarly, we can write that

$$\begin{aligned}\int_0^\infty e^{-\beta w}\, n_w(w,z)\,\mathrm{d}w &= e^{-\beta w}\, n(w,z)\Big|_0^\infty + \beta\int_0^\infty e^{-\beta w}\, n(w,z)\,\mathrm{d}w\\ &= 0+\beta\, l(z;\beta),\end{aligned}$$

which yields Eq. (1.9). Finally, the boundary conditions in (1.10) follow immediately from Eq. (1.7). □

In the next section, we will give the solutions to the above equations in the most important cases.

2. Particular Cases

We first consider the most important particular case, namely that when the diffusion process $\{Z(t), t \geq 0\}$ is a Wiener process.

Proposition 2.1. *If $v(z) \equiv \sigma^2$ (with $\sigma > 0$) and $f(z) \equiv \mu \in \mathbb{R}$ in Eq. (1.8), so that $\{Z(t), t \geq 0\}$ is a Wiener process, using the mathematical software* Maple*, we find that the general solution of this equation can be written as follows:*

$$\begin{aligned}L(z;\alpha,\beta) &= e^{-\mu z/\sigma^2}\,\{c_1\,\mathrm{Ai}[\varphi(z)] + c_2\,\mathrm{Bi}[\varphi(z)]\} + \frac{2^{2/3}\,\pi}{\sigma^{4/3}\,\beta^{1/3}}\,e^{-\mu z/\sigma^2}\\ &\quad\times\Big\{\mathrm{Ai}[\varphi(z)]\int \mathrm{Bi}[\varphi(z)]\,z\,e^{\mu z/\sigma^2}\,\mathrm{d}z\\ &\quad -\mathrm{Bi}[\varphi(z)]\int \mathrm{Ai}[\varphi(z)]\,z\,e^{\mu z/\sigma^2}\,\mathrm{d}z\Big\}, \qquad (2.1)\end{aligned}$$

where Ai *and* Bi *are Airy functions and*

$$\varphi(z) := \frac{2^{1/3}\,\left[\frac{1}{2}\,\mu^2 + \sigma^2\,(\alpha+\beta\,z)\right]}{\sigma^{8/3}\,\beta^{2/3}}.$$

Furthermore, the arbitrary constants c_1 and c_2 are uniquely determined from the boundary conditions $L(z;\alpha,\beta) = 1/\beta$ for $z = c$ or d.

Remark 2.2. (i) The notation $\int h(x)\,\mathrm{d}x$ denotes an indefinite integral of the function h, without a constant of integration.
(ii) In the case of a standard Brownian motion, the solution given in (2.1) reduces to

$$\begin{aligned}L(z;\alpha,\beta) &= \{c_1\,\mathrm{Ai}[\varphi_1(z)] + c_2\,\mathrm{Bi}[\varphi_1(z)]\} + \frac{2^{2/3}\,\pi}{\beta^{1/3}}\\ &\quad\times\Big\{\mathrm{Ai}[\varphi_1(z)]\int \mathrm{Bi}[\varphi_1(z)]\,z\,\mathrm{d}z - \mathrm{Bi}[\varphi_1(z)]\int \mathrm{Ai}[\varphi_1(z)]\,z\,\mathrm{d}z\Big\},\end{aligned}$$

where

$$\varphi_1(z) := \frac{2^{1/3}\,(\alpha+\beta\,z)}{\beta^{2/3}}.$$

(iii) The particular solution that satisfies the boundary conditions is not difficult to compute. The expression obtained is however obviously quite long. Therefore, we

did not write it out in the proposition. It will be the same for the other problems that will be solved below.
(iv) Inverting the Laplace transform to obtain the moment-generating function $M(x, z; \alpha)$ does not seem to be possible, even in the simplest case. Again, it will be the same in the next problems.

Proposition 2.3. *Under the same assumptions as in Proposition 2.1, the general solution of Eq. (1.9) is given by*

$$\begin{aligned} l(z;\beta) &= e^{-\mu z/\sigma^2}\left\{c_1 \operatorname{Ai}[\psi(z)] + c_2 \operatorname{Bi}[\psi(z)]\right\} + \frac{2^{2/3}\,\pi}{\sigma^{4/3}\,\beta^{4/3}}\, e^{-\mu z/\sigma^2} \\ &\quad \times \Bigg\{ \operatorname{Ai}[\psi(z)] \int \operatorname{Bi}[\psi(z)]\, z\, e^{\mu z/\sigma^2}\, \mathrm{d}z \\ &\quad - \operatorname{Bi}[\psi(z)] \int \operatorname{Ai}[\psi(z)]\, z\, e^{\mu z/\sigma^2}\, \mathrm{d}z \Bigg\}, \end{aligned}$$

where

$$\psi(z) := \frac{2^{1/3}\left(\frac{1}{2}\mu^2 + \sigma^2\,\beta\, z\right)}{\sigma^{8/3}\,\beta^{2/3}}.$$

The arbitrary constants c_1 and c_2 can be determined from the boundary conditions $l(c;\beta) = l(d;\beta) = 0$.

Remark 2.4. (i) The explicit expression given in Proposition 2.3, as well as the ones in the next propositions in this section, were obtained by making use of *Maple*.
(ii) In addition to the above first-passage-time problems, we can also consider a first-passage-place problem. Let

$$p_0(x, z) := P[Z[T(x, z)]] = c. \tag{2.2}$$

Defining $q(w, z) = p_0(-\ln(x), z)$ and

$$\lambda(z;\beta) = \int_0^\infty e^{-\beta w}\, q(w, z)\, \mathrm{d}w, \tag{2.3}$$

we find that the function $\lambda(z;\beta)$ satisfies, in the general case, the ordinary differential equation

$$\frac{1}{2}\, v(z)\, \lambda_{zz} + f(z)\, \lambda_z - \beta\, z\, \lambda = 0, \tag{2.4}$$

subject to the boundary conditions

$$\lambda(c;\beta) = \frac{1}{\beta} \quad \text{and} \quad \lambda(d;\beta) = 0. \tag{2.5}$$

The general solution of Eq. (2.4) when $\{Z(t), t \geq 0\}$ is a Wiener process is

$$\lambda(z;\beta) = e^{-\mu z/\sigma^2}\left\{c_1 \operatorname{Ai}[\psi(z)] + c_2 \operatorname{Bi}[\psi(z)]\right\}.$$

Similarly, we can compute explicitly the Laplace transform of the functions $p_1(x, z) := P[X[T(x, z)]] = b$ and $p_2(x, z) := P[Z[T(x, z)]] = d$.

Assume next that $\{Z(t), t \geq 0\}$ is a geometric Brownian motion. As is well known, this diffusion process plays a very important role in mathematical finance.

Proposition 2.5. *If $v(z) = \sigma^2 z^2$ (with $\sigma > 0$) and $f(z) = \mu z$ (with $\mu \in \mathbb{R}$) in Eq. (1.8), so that $\{Z(t), t \geq 0\}$ is a geometric Brownian motion, the general solution of this equation can be expressed as follows:*

$$\begin{aligned} L(z;\alpha,\beta) &= z^{-\frac{\mu}{\sigma^2}+\frac{1}{2}} \left[c_1 I_\nu(\kappa z^{1/2}) + c_2 K_\nu(\kappa z^{1/2})\right] - \frac{2^{1/2}}{\sigma \beta^{1/2}} z^{-\frac{\mu}{\sigma^2}+\frac{1}{2}} \\ &\quad \times \left\{ I_\nu(\kappa z^{1/2}) \int \frac{K_\nu(\kappa z^{1/2})}{\Delta(z)} z^{\frac{\mu}{\sigma^2}-1} \,\mathrm{d}z \right. \\ &\quad \left. - K_\nu(\kappa z^{1/2}) \int \frac{I_\nu(\kappa z^{1/2})}{\Delta(z)} z^{\frac{\mu}{\sigma^2}-1} \,\mathrm{d}z \right\}, \end{aligned} \tag{2.6}$$

where I_ν and K_ν are Bessel functions,

$$\nu := \frac{\sqrt{\sigma^4 + (-4\mu + 8\alpha)\sigma^2 + 4\mu^2}}{\sigma^2},$$

$$\kappa := \frac{2\sqrt{2\beta}}{\sigma}$$

and

$$\Delta(z) := I_\nu(\kappa z^{1/2}) K_{\nu+1}(\kappa z^{1/2}) + I_{\nu+1}(\kappa z^{1/2}) K_\nu(\kappa z^{1/2}).$$

Again, the arbitrary constants c_1 and c_2 can be found by making use of the boundary conditions $L(c;\alpha,\beta) = L(d;\alpha,\beta) = 1/\beta$.

Remark 2.6. (i) With $\sigma = 1$ and $\mu = 0$, *Maple* is able to evaluate the integrals in Eq. (2.6). Let $\gamma := \sqrt{1+\alpha}$ and $\kappa_1 := 2\sqrt{2\beta}$. Then, we have

$$\begin{aligned} L(z;\alpha,\beta) &= \sqrt{z}\left[c_1 I_\gamma(\kappa_1 z^{1/2}) + c_2 K_\gamma(\kappa_1 z^{1/2})\right] + \frac{z}{2\alpha\gamma \sin(\gamma\pi)\Gamma(\gamma)} \\ &\times \left\{ I_\gamma(\kappa_1 z^{1/2}) (2\beta z)^{-\gamma/2} \gamma(\gamma+1)\Gamma^2(\gamma) \sin(\gamma\pi) H_1(z) \right. \\ &\left. +(\gamma-1)(2\beta z)^{\gamma/2} H_2(z) \left[2\sin(\gamma\pi) K_\gamma(\kappa_1 z^{1/2}) + \pi I_\gamma(\kappa_1 z^{1/2})\right] \right\}, \end{aligned}$$

in which

$$H_1(z) := \text{hypergeom}\left(\left[\frac{1-\gamma}{2}\right], \left[1-\gamma, \frac{3-\gamma}{2}\right], 2\beta z\right)$$

and

$$H_2(z) := \text{hypergeom}\left(\left[\frac{1+\gamma}{2}\right], \left[1+\gamma, \frac{3+\gamma}{2}\right], 2\beta z\right),$$

where "hypergeom" is a generalized hypergeometric function.

(ii) Because the origin is a natural boundary for a geometric Brownian motion, the constant c must be positive in the definition of the rectangle R.

(iii) We can obtain an explicit solution to Eq. (1.9) under the same assumptions as in Proposition 2.5. The expression is however quite involved. Similarly, we can compute the function $\lambda(z;\beta)$ defined in Eq. (2.3). We find that the general solution of Eq. (2.4) is

$$\lambda(z;\beta) = z^{-\nu_1/2}\left[c_1 I_{\nu_1}(\kappa z^{1/2}) + c_2 K_{\nu_1}(\kappa z^{1/2})\right],$$

where

$$\nu_1 := \frac{2\mu}{\sigma^2} - 1.$$

When $\mu = 0$, the unique solution that satisfies the boundary conditions in Eq. (2.5) is

$$\lambda(z;\beta) = \frac{\sqrt{z}}{\sqrt{c}\beta} \frac{I_1(\kappa\sqrt{d})\, K_1(\kappa\sqrt{z}) - K_1(\kappa\sqrt{d})\, I_1(\kappa\sqrt{z})}{I_1(\kappa\sqrt{d})\, K_1(\kappa\sqrt{c}) - K_1(\kappa\sqrt{d})\, I_1(\kappa\sqrt{c})}.$$

Finally, another important diffusion process is the Ornstein-Uhlenbeck process. We can prove the following proposition.

Proposition 2.7. *If we choose $v(z) \equiv 1$ and $f(z) = -z$ in Eq. (1.8), so that $\{Z(t), t \geq 0\}$ is an Ornstein-Uhlenbeck process, then we find that*

$$\begin{aligned} L(z;\alpha,\beta) &= e^{-\beta z}\left[c_1 M\left(\delta, 1/2, (z+\beta)^2\right) + c_2 U\left(\delta, 1/2, (z+\beta)^2\right)\right] \\ &+ \frac{2}{\delta} e^{-\beta z}\left\{ M\left(\delta, 1/2, (z+\beta)^2\right) \int \frac{U\left(\delta, 1/2, (z+\beta)^2\right) z(z+\beta) e^{\beta z}}{\Omega(z)} \mathrm{d}z \right. \\ &\left. - U\left(\delta, 1/2, (z+\beta)^2\right) \int \frac{M\left(\delta, 1/2, (z+\beta)^2\right) z(z+\beta) e^{\beta z}}{\Omega(z)} \mathrm{d}z \right\}, \end{aligned}$$

where $M(\cdot,\cdot,\cdot)$ and $U(\cdot,\cdot,\cdot)$ are Kummer (or confluent hypergeometric) functions,

$$\delta := \frac{\alpha}{2} - \frac{\beta^2}{4}$$

and

$$\begin{aligned} \Omega(z) &:= (2\delta+1) M\left(\delta, 1/2, (z+\beta)^2\right) U\left(\delta+1, 1/2, (z+\beta)^2\right) \\ &\quad - 2U\left(\delta, 1/2, (z+\beta)^2\right) M\left(\delta+1, 1/2, (z+\beta)^2\right). \end{aligned}$$

Remark 2.8. (i) We can actually give the solution of Eq. (1.8) in the general case $v(z) \equiv \sigma^2$ and $f(z) = -\mu z$, with $\sigma, \mu > 0$.

(ii) With $v(z) \equiv 1$ and $f(z) = -z$, the general solution of Eq. (1.9) is given by

$$\begin{aligned} l(z;\beta) &= e^{-\beta z}\left[c_1 M\left(\delta_1, 1/2, (z+\beta)^2\right) + c_2 U\left(\delta_1, 1/2, (z+\beta)^2\right)\right] \\ &+ \frac{16}{\beta^3} e^{-\beta z}\left\{ M\left(\delta_1, 1/2, (z+\beta)^2\right) \int \frac{U\left(\delta_1, 1/2, (z+\beta)^2\right) (z+\beta) e^{\beta z}}{\Omega_1(z)} \mathrm{d}z \right. \\ &\left. - U\left(\delta_1, 1/2, (z+\beta)^2\right) \int \frac{M\left(\delta_1, 1/2, (z+\beta)^2\right) (z+\beta) e^{\beta z}}{\Omega_1(z)} \mathrm{d}z \right\}, \end{aligned}$$

where

$$\delta_1 := -\frac{\beta^2}{4}$$

and

$$\begin{aligned} \Omega_1(z) &:= (\beta^2 - 2) M\left(\delta_1, 1/2, (z+\beta)^2\right) U\left(\delta_1+1, 1/2, (z+\beta)^2\right) \\ &\quad + 4U\left(\delta_1, 1/2, (z+\beta)^2\right) M\left(\delta_1+1, 1/2, (z+\beta)^2\right). \end{aligned}$$

(iii) If we choose, for the sake of simplicity, $c = 0$ and $d = 1$ in Eq. (2.5), then we find that

$$\lambda(z;\beta) = \frac{1}{\beta}\, e^{-\beta z}\, \frac{M^*(1)\, U^*(z) - U^*(1)\, M^*(z)}{M^*(1)\, U^*(0) - U^*(1)\, M^*(0)},$$

where

$$M^*(z) := M\left(\delta_1, 1/2, (z+\beta)^2\right) \quad \text{and} \quad U^*(z) := U\left(\delta_1, 1/2, (z+\beta)^2\right).$$

In the next section, a generalization of the results obtained in Section 1 will be presented.

3. Generalization

We now consider the degenerate diffusion process $\{(X(t), Y(t), Z(t)), t \geq 0\}$ defined by the system

$$\begin{aligned} \mathrm{d}X(t) &= g[X(t)]\, Z(t)\, \mathrm{d}t, \\ \mathrm{d}Y(t) &= Z(t)\, \mathrm{d}t, \\ \mathrm{d}Z(t) &= f[Z(t)]\, \mathrm{d}t + \{v[Z(t)]\}^{1/2}\, \mathrm{d}B(t). \end{aligned} \tag{3.1}$$

Let

$$G(x) := \int \frac{1}{g(x)}\, \mathrm{d}x$$

and assume that the inverse function G^{-1} exists. The process $\{X(t), t \geq 0\}$ can be expressed as follows:

$$X(t) = G^{-1}\{G[X(0)] + Y(t) - Y(0)\} = G^{-1}\left\{G[X(0)] + \int_0^t Z(s)\, \mathrm{d}s\right\}.$$

Notice that the case considered in Section 1 is the one when $g[X(t)] = X(t)$, so that $G(x) = \ln(x)$.

The problem of computing the Laplace transforms of the functions $M(x,z;\alpha) := E\left[e^{-\alpha T(x,z)}\right]$ and $m(x,z) := E[T(x,z)]$, where $T(x,z)$ is defined in (1.2), can sometimes be reduced to that of solving an ordinary differential equation, subject to the appropriate boundary conditions.

The function M now satisfies the Kolmogorov backward equation

$$\frac{1}{2}\, v(z)\, M_{zz} + f(z)\, M_z + g(x)\, z\, M_x = \alpha\, M \quad \text{for } (x,z) \in R,$$

subject to the boundary condition in Eq. (1.4), whereas $m(x,z)$ satisfies the partial differential equation

$$\frac{1}{2}\, v(z)\, m_{zz} + f(z)\, m_z + g(x)\, z\, m_x = -1 \quad \text{for } (x,z) \in R,$$

subject to the boundary condition in (1.6).

Proposition 3.1. *Let $u := G(x)$ and denote $M(G(x), z;\alpha)$ and $m(G(x), z)$ respectively by $C(u,z;\alpha)$ and $c(u,z)$. If G is a strictly increasing function, then the rectangle R defined in (1.1) becomes*

$$R_u := \{(u,z) \in \mathbb{R}^2 : G(a) < u < G(b), 0 \leq c < z < d\},$$

while

$$R_u := \{(u,z) \in \mathbb{R}^2 : G(b) < u < G(a), 0 \le c < z < d\}$$

if G is strictly decreasing.

We find at once that

$$\frac{1}{2} v(z)\, C_{zz} + f(z)\, C_z + z\, C_u = \alpha\, C$$

and

$$\frac{1}{2} v(z)\, c_{zz} + f(z)\, c_z + z\, c_u = -1.$$

These equations are valid inside R_u and are subject to the boundary conditions

$$C(u,z;\alpha) = 1 \quad \text{and} \quad c(u,z) = 0 \quad \textit{for } (u,z) \notin R_u.$$

We can now state the following proposition.

Proposition 3.2. *Assume that there exists an affine function $h(u) := a_1\, u + a_0$ such that*

$$h[G(a)] = 0 \text{ and } h[G(b)] = \infty \quad \text{or} \quad h[G(b)] = 0 \text{ and } h[G(a)] = \infty.$$

Let $\rho := h(u)$ and define

$$\mathcal{L}(z;\alpha,\beta) = \int_0^\infty e^{-\beta\rho}\, C(\rho,z;\alpha)\, \mathrm{d}\rho$$

and

$$\mathcal{M}(z;\beta) = \int_0^\infty e^{-\beta\rho}\, c(\rho,z)\, \mathrm{d}\rho.$$

The functions $\mathcal{L}(z;\alpha,\beta)$ and $\mathcal{M}(z;\beta)$ both satisfy a linear second-order ordinary differential equation:

$$\frac{1}{2} v(z)\, \mathcal{L}_{zz} + f(z)\, \mathcal{L}_z + a_1\, z\, (-1 + \beta\, \mathcal{L}) = \alpha\, \mathcal{L}$$

and

$$\frac{1}{2} v(z)\, \mathcal{M}_{zz} + f(z)\, \mathcal{M}_z + a_1\, \beta\, z\, \mathcal{M} = -\frac{1}{\beta}.$$

These equations are subject to the boundary conditions

$$\mathcal{L}(z;\alpha,\beta) = \frac{1}{\beta} \quad \text{and} \quad \mathcal{M}(z;\beta) = 0 \quad \textit{for } z = c \textit{ or } d. \tag{3.2}$$

Remark 3.3. (i) In Section 2, the function $h(u)$ was given by $h(u) = -u$.
(ii) We could also compute the Laplace transforms of the first-passage-place probability $P[Z[T(x,z)]] = c$ (see Eq. (2.2)) and of the other probabilities defined in Section 2.

To complete this section, we present an application of Proposition 3.2. We consider the three-dimensional process $\{(X(t), Y(t), Z(t)), t \ge 0\}$ defined by

$$\begin{aligned} \mathrm{d}X(t) &= X^2(t)\, Z(t)\, \mathrm{d}t, \\ \mathrm{d}Y(t) &= Z(t)\, \mathrm{d}t, \\ \mathrm{d}Z(t) &= f[Z(t)]\, \mathrm{d}t + \{v[Z(t)]\}^{1/2}\, \mathrm{d}B(t). \end{aligned}$$

That is, $g[X(t)] = X^2(t)$ in Eq. (3.1). It follows that

$$G(x) = -\frac{1}{x} \quad \text{and} \quad G^{-1}(x) = -\frac{1}{x} = G(x).$$

Hence, we can write that

$$\begin{aligned} X(t) &= -\frac{1}{G[X(0)] + Y(t) - Y(0)} = -\frac{1}{-\frac{1}{X(0)} + Y(t) - Y(0)} \\ &= \frac{X(0)}{1 - X(0)\,[Y(t) - Y(0)]}, \end{aligned}$$

in which we assume that $X(0) > 0$.

Next, suppose that $a = 0$ in Eq. (1.1). We have $G(a) = G(0) = -\infty$ and $G(b) = -1/b$. Therefore, we can choose the function $h(u) = -u - \frac{1}{b}$ to obtain $h[G(0)] = \infty$ and $h[G(b)] = 0$. Thus, the functions $\mathcal{L}(z;\alpha,\beta)$ and $\mathcal{M}(z;\beta)$ satisfy the following ordinary differential equations:

$$\frac{1}{2}v(z)\mathcal{L}_{zz} + f(z)\mathcal{L}_z - z(-1 + \beta\mathcal{L}) = \alpha\mathcal{L}$$

and

$$\frac{1}{2}v(z)\mathcal{M}_{zz} + f(z)\mathcal{M}_z - \beta z\mathcal{M} = -\frac{1}{\beta},$$

and are subject to the boundary conditions in Eq. (3.2). Notice that the above equations are respectively the same as Eq. (1.8) and Eq. (1.9), which entails that we can use the results presented in Section 3 to obtain explicit expressions for $\mathcal{L}$ and $\mathcal{M}$ in the cases considered in that section. Although $\mathcal{L}$ and $\mathcal{M}$ would be the same as the corresponding ones in Section 3, the functions $M(x,z;\alpha)$ and $m(x,z)$ (if we could compute them explicitly) would be different, because $G(x)$ (as well as $h(u)$) in this case is not the same as in Section 1.

4. Conclusion

In this paper, we computed the Laplace transforms of various functions of a first-exit time for degenerate three-dimensional diffusion processes. This type of problem entails finding the solution to a partial differential equation known as Kolmogorov backward equation, subject to the appropriate boundary conditions. In more than one dimension, the problems that can be solved explicitly for diffusion processes and especially for integrated diffusion processes are still few, and the solutions are generally very complicated.

Here, we were able to give exact expressions for the Laplace transforms of the quantities of interest in the most important cases for applications. Ideally, we would like to be able to invert the expressions obtained to get the functions we are looking for. However, because these expressions involve ratios of special functions, this task is obviously really hard. If we consider instead of R a region of the form

$$R_1 = \{(x,z) \in \mathbb{R}^2 : 0 \le a < x < b, z < d\}$$

or

$$R_2 = \{(x,z) \in \mathbb{R}^2 : 0 \le a < x < b, z > c\},$$

that is, problems for which there is a single boundary for the diffusion process $\{Z(t), t \geq 0\}$, then the expressions for the Laplace transforms would be simpler and perhaps we could at least invert some of them.

Finally, we could try to generalize the results obtained in this paper. In particular, we could try to reduce the first-passage problems to finding the solutions of second-order ordinary differential equations by making use of other integral transforms, for instance Fourier or Mellin transforms.

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