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## SET-VALUED STOCHASTIC DIFFERENTIAL EQUATION IN M-TYPE 2 BANACH SPACE

ITARU MITOMA\*, YOSHIAKI OKAZAKI, AND JINPING ZHANG

ABSTRACT. Let  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$  be a complete probability space with filtration  $\{\mathcal{F}_t\}$ ,  $(\mathcal{X}, \mathcal{H}, \mu)$  an abstract Wiener space of M-type 2, and  $\{B_t : t \geq 0\}$  an  $\mathcal{X}$ -valued Brownian motion such that the distribution of the random function  $t^{-1/2}B_t : \Omega \rightarrow \mathcal{X}$  is  $\mu$  for any  $t > 0$ . We consider the strong solutions to a set-valued stochastic differential equation with a set-valued drift and a single valued diffusion driven by  $dB_t$ . Under some suitable conditions, the existence and uniqueness of strong solutions are obtained.

### 1. Introduction

Theory of classical stochastic differential equations is widely used and generalized in several ways. There are several references that consider stochastic integrals and stochastic differential equations in abstract Banach spaces (see for example [2, 4, 3, 7, 8, 9, 16, 17, 23, 22] etc.).

Among these references, Da Prato, G. and Zabczyk, J. ([7]) presented a comprehensive theory of abstract stochastic integrals and stochastic differential equations in Hilbert spaces, in which the integrator is a cylindrical Brownian motion. The paper [22] is devoted to the stochastic integral in certain Banach spaces named UMD, where the integrator is a cylindrical Brownian motion and the integrand is operator valued. In [23], the authors defined stochastic integration of operator-valued functions with respect to the Banach space-valued Brownian motion in a weak sense connected with the dual operator. In a separable M-type 2 Banach space, Brzezniak [2] studied the stochastic integral with respect to a finite dimensional Brownian motion and the corresponding stochastic partial differential equations. Stochastic integration of operator-valued functions with respect to the Banach space-valued Wiener process is described briefly in [4, 3].

Theory of stochastic differential inclusions is another way to generalize classical stochastic differential equations, which has widespread applications to mathematical economics, stochastic control theory etc. In this area, we would refer to nice surveys [14, 15].

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Another way to extend the classical theory is to consider set-valued differential equations directly rather than inclusions ([20, 24, 26]).

In this paper, we continue to proceed the set-valued stochastic differential equation in a separable Banach space  $\mathcal{X}$  along this line. At first, mainly based on the work [4, 3], we study precisely stochastic integrals of  $B(\mathcal{X}, \mathcal{X})$ -valued random functions with respect to an  $\mathcal{X}$ -valued Brownian motion, where  $B(\mathcal{X}, \mathcal{X})$  is the set of all bounded linear operators from  $\mathcal{X}$  to  $\mathcal{X}$ . Let  $L^2((\mathcal{X}, \mu), \mathcal{X})$  denote a separable Banach space of all Borel measurable functions  $f : (\mathcal{X}, \mathcal{B}(\mathcal{X}), \mu) \rightarrow (\mathcal{X}, \mathcal{B}(\mathcal{X}))$  such that the norm of  $f$  is defined by

$$\|f\| := \left( \int_{\mathcal{X}} \|f(x)\|^2 \mu(dx) \right)^{1/2}.$$

The completion of  $B(\mathcal{X}, \mathcal{X})$  in norm  $\|\cdot\|$  is a separable Banach space. And then, based on the work of [26], we study the existence and uniqueness of strong solutions to a set-valued stochastic differential equation such that

$$X_t = cl\left(X_0 + \int_0^t a(s, X_s) ds + \int_0^t b(s, X_s) dB_s\right), \quad t \in [0, T], \quad (1.1)$$

where  $cl$  stands for the closure in  $\mathcal{X}$ , both  $X_s$  and  $a(s, X_s)$  are set-valued,  $b(s, X_s)$  is  $B(\mathcal{X}, \mathcal{X})$ -valued, and  $\{B_t\}$  is an  $\mathcal{X}$ -valued Brownian motion. The sum of a set  $X$  and an single point  $y$  is defined as  $X + y = \{x + y; x \in X\}$ .

There exist quite a few works treating stochastic differential or integral inclusions, and most of them deal with these subjects in finite dimensional settings. However the set-valued stochastic differential equation of the type (1.1) is a rather new subject if we compare it with those stochastic inclusions, and further the space, where (1.1) is considered, belongs to a certain class of Banach spaces including Hilbert spaces.

Under these circumstances, we obtain

**Theorem 1.** *Suppose  $a(\cdot, \cdot)$  and  $b(\cdot, \cdot)$  are jointly measurable,  $H$ -bounded and satisfy Lipschitz conditions in the following sense :*

$$H(\{0\}, a(t, X)) + \|b(t, X)\| \leq C(1 + H(\{0\}, X)), \quad X \subset \mathcal{X}, t \in [0, T]$$

for some constant  $C$ , and

$$H(a(t, X), a(t, Y)) + \|b(t, X) - b(t, Y)\| \leq DH(X, Y), \quad X, Y \subset \mathcal{X}, t \in [0, T]$$

for some constant  $D$ , where  $H(A, B)$  is the Hausdorff distance between sets  $A$  and  $B$ . Then for any given  $L^2$ -integrably bounded set-valued random variable  $X_0$ , the equation (1.1) has a unique  $H$ -continuous solution.

The paper is organized as follows. Section 2 is preliminaries of set-valued random variables and set-valued stochastic processes. Section 3 is devoted to the stochastic integration of  $B(\mathcal{X}, \mathcal{X})$ -valued functions by an  $\mathcal{X}$ -valued Brownian motion. Section 4 is on set-valued integrals of set-valued stochastic processes with respect to the Lebesgue measure in time interval  $[s, t]$ . In Section 5, first we present set-valued stochastic differential equations and prove the existence and uniqueness of strong solutions.

## 2. Preliminaries

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space, and  $\{\mathcal{F}_t\}_{t \geq 0}$  a filtration satisfying the usual conditions such that  $\mathcal{F}_0$  includes all  $P$ -null sets in  $\mathcal{F}$ . The filtration is non-decreasing and right continuous. Let  $\mathcal{B}(E)$  be the Borel field of a topological space  $E$ ,  $E^*$  the dual space of  $E$ .  $\langle x, x^* \rangle$  (or  $\langle x^*, x \rangle$ ) denotes the canonical bilinear form for  $x \in E$  and  $x^* \in E^*$ . Let  $(\mathcal{X}, \|\cdot\|)$  be a separable Banach space  $\mathcal{X}$  equipped with the norm  $\|\cdot\|$  and  $\mathbf{K}(\mathcal{X})$  (resp.  $\mathbf{K}_b(\mathcal{X})$ ,  $\mathbf{K}_c(\mathcal{X})$ ) the family of all nonempty closed (resp. closed bounded, closed convex) subsets of  $\mathcal{X}$ . Let  $p$  be  $1 \leq p < +\infty$  and  $L^p(\Omega, \mathcal{F}, P; \mathcal{X})$ , denoted briefly by  $L^p(\Omega; \mathcal{X})$ , the Banach space of equivalence classes of  $\mathcal{X}$ -valued  $\mathcal{F}$ -measurable functions  $f : \Omega \rightarrow \mathcal{X}$  such that the norm

$$\|f\|_p = \left\{ \int_{\Omega} \|f(\omega)\|^p dP \right\}^{1/p}$$

is finite.  $f$  is called  $L^p$ -integrable if  $f \in L^p(\Omega; \mathcal{X})$ .

The expectation operator of a random function  $X$  will be denoted by  $\mathbb{E}[X]$ . A set-valued function  $F : \Omega \rightarrow \mathbf{K}(\mathcal{X})$  is said to be *measurable* if for any open set  $O \subset \mathcal{X}$ , the inverse  $F^{-1}(O) := \{\omega \in \Omega; F(\omega) \cap O \neq \emptyset\}$  is in  $\mathcal{F}$ . Such a function  $F$  is called a *set-valued random variable*. Let  $\mathcal{M}(\Omega, \mathcal{F}, P; \mathbf{K}(\mathcal{X}))$  be the family of all set-valued random variables, briefly denoted by  $\mathcal{M}(\Omega; \mathbf{K}(\mathcal{X}))$ .

A mapping  $g$  from a measurable space  $(E_1, \mathcal{A}_1)$  into another measurable space  $(E_2, \mathcal{A}_2)$  is called  $\mathcal{A}_1/\mathcal{A}_2$ -measurable if  $g^{-1}(B) = \{x \in E_1; g(x) \in B\}$  is in  $\mathcal{A}_1$  for all  $B \in \mathcal{A}_2$ .

For any open subset  $O \subset \mathcal{X}$ , set

$$Z_O := \{E \in \mathbf{K}(\mathcal{X}); E \cap O \neq \emptyset\},$$

$$\mathcal{C} := \{Z_O; O \subset \mathcal{X}, O \text{ is open}\},$$

and let  $\sigma(\mathcal{C})$  be the  $\sigma$ -algebra generated by  $\mathcal{C}$ . We have ,(see [26]),

**Proposition 2.1.** *A set-valued function  $F : \Omega \rightarrow \mathbf{K}(\mathcal{X})$  is measurable if and only if  $F$  is  $\mathcal{F}/\sigma(\mathcal{C})$ -measurable.*

For  $A, B \in \mathbf{K}(\mathcal{X})$ ,  $H(A, B) \geq 0$  is defined by

$$H(A, B) := \max\left\{ \sup_{x \in A} \inf_{y \in B} \|x - y\|, \sup_{y \in B} \inf_{x \in A} \|x - y\| \right\}.$$

If  $A, B \in \mathbf{K}_b(\mathcal{X})$ , then  $H(A, B)$  is called the *Hausdorff distance* of  $A$  and  $B$ . It is well known that  $\mathbf{K}_b(\mathcal{X})$  equipped with the  $H$ -metric, denoted by  $(\mathbf{K}_b(\mathcal{X}), H)$ , is a complete metric space.

The following results are also well known. (see for example [11], [18]).

**Proposition 2.2.** (i) *For  $A, B, C, D \in \mathbf{K}(\mathcal{X})$ , we have*

$$H(A + B, C + D) \leq H(A, C) + H(B, D)$$

and also

$$H(\text{cl}(A + B), \text{cl}(C + D)) \leq H(A, C) + H(B, D).$$

(ii) *For  $A, B \in \mathbf{K}(\mathcal{X})$ ,  $\mu \in \mathbb{R}$ , we have*

$$H(\mu A, \mu B) = |\mu|H(A, B).$$

For  $F \in \mathcal{M}(\Omega, \mathbf{K}(\mathcal{X}))$ , the family of all  $L^p$ -integrable selections is defined by

$$S_F^p(\mathcal{F}) := \{f \in L^p(\Omega, \mathcal{F}, P; \mathcal{X}); f(\omega) \in F(\omega) \text{ a.s.}\}.$$

In the following,  $S_F^p(\mathcal{F})$  is denoted briefly by  $S_F^p$ . If  $S_F^p$  is nonempty,  $F$  is said to be  $L^p$ -integrable.  $F$  is called  $L^p$ -integrably bounded if there exists a function  $h \in L^p(\Omega, \mathcal{F}, P; \mathbb{R})$  such that  $\|x\| \leq h(\omega)$  for any  $x$  and  $\omega$  with  $x \in F(\omega)$ . It is equivalent to that  $\|F\|_{\mathbf{K}} \in L^p(\Omega; \mathbb{R})$ , where  $\|F(\omega)\|_{\mathbf{K}} := \sup \lim_{a \in F(\omega)} \|a\|$ . The family of all measurable  $\mathbf{K}(\mathcal{X})$ -valued (resp.  $\mathbf{K}_c(\mathcal{X})$ -valued)  $L^p$ -integrably bounded functions is denoted by  $L^p(\Omega, \mathcal{F}, P; \mathbf{K}(\mathcal{X}))$  (resp.  $L^p(\Omega, \mathcal{F}, P; \mathbf{K}_c(\mathcal{X}))$ ). Write them for brevity as  $L^p(\Omega; \mathbf{K}(\mathcal{X}))$  (resp.  $L^p(\Omega; \mathbf{K}_c(\mathcal{X}))$ ).

Let  $\Gamma$  be a set of measurable functions  $f : \Omega \rightarrow \mathcal{X}$ .  $\Gamma$  is called *decomposable* with respect to the  $\sigma$ -algebra  $\mathcal{F}$  if, for any finite  $\mathcal{F}$ -measurable partition  $A_1, \dots, A_n$  and for any  $f_1, \dots, f_n \in \Gamma$ ,  $\chi_{A_1} f_1 + \dots + \chi_{A_n} f_n$  is in  $\Gamma$ , where  $\chi_A$  is the indicator function of set  $A$ , i.e.,

$$\chi_A(\omega) := \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{if } \omega \notin A. \end{cases}$$

**Proposition 2.3.** (Hiai-Umegaki [11]) *Let  $\Gamma$  be a nonempty closed subset of  $L^p(\Omega, \mathcal{F}, P; \mathcal{X})$ . Then there exists an  $F \in \mathcal{M}(\Omega; \mathbf{K}(\mathcal{X}))$  such that  $\Gamma = S_F^p$  if and only if  $\Gamma$  is decomposable with respect to  $\mathcal{F}$ .*

**Lemma 2.4.** *Let  $F$  be in  $\mathcal{M}(\Omega; \mathbf{K}(\mathcal{X}))$ . Then  $F$  is  $L^p$ -integrably bounded if and only if  $S_F^p$  is nonempty and bounded in  $L^p(\Omega; \mathcal{X})$ .*

*Proof.* The case of  $p = 1$  is due to Hiai-Umegaki [11]. By a manner similar to that of  $p = 1$ , we can also prove the statement for  $1 < p < +\infty$ .  $\square$

Let  $\mathbb{R}_+$  be the set of all nonnegative real numbers and  $\mathcal{B}_+ = \mathcal{B}(\mathbb{R}_+)$ . An  $\mathcal{X}$ -valued stochastic process  $f = \{f_t; t \geq 0\}$  (or denoted by  $f = \{f(t); t \geq 0\}$ ) is defined as a function  $f : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{X}$  with  $\mathcal{F}$ -measurable section  $f_t$  for each  $t \geq 0$ . We say  $f$  is *measurable* if  $f$  is  $\mathcal{B}_+ \otimes \mathcal{F}$ -measurable. The process  $f = \{f_t; t \geq 0\}$  is called  $\mathcal{F}_t$ -*adapted* if  $f_t$  is  $\mathcal{F}_t$ -measurable for every  $t \geq 0$ .

In a fashion similar to the  $\mathcal{X}$ -valued stochastic process, a *set-valued stochastic process*  $F = \{F_t; t \geq 0\}$  is defined as a set-valued function  $F : \mathbb{R}_+ \times \Omega \rightarrow \mathbf{K}(\mathcal{X})$  with  $\mathcal{F}$ -measurable section  $F_t$  for each  $t \geq 0$ . It is called *measurable* if it is  $\mathcal{B}_+ \otimes \mathcal{F}$ -measurable, and  $\mathcal{F}_t$ -*adapted* if for any fixed  $t$ ,  $F_t$  is  $\mathcal{F}_t$ -measurable.

**Proposition 2.5.** *Let  $F = \{F_t : t \geq 0\}$  be an  $\mathcal{F}_t$ -adapted and measurable set-valued stochastic process. Then there exists an  $\mathcal{F}_t$ -adapted and measurable selection  $f = \{f_t : t \geq 0\}$  such that*

$$f_t(\omega) \in F_t(\omega) \text{ for all } (t, \omega) \in \mathbb{R}_+ \times \Omega.$$

*Proof.* Let  $\Sigma := \bigcap_{t \geq 0} \{Z \in \mathcal{B}_+ \otimes \mathcal{F} : Z_t \in \mathcal{F}_t\}$ , where  $Z_t = \{\omega; (t, \omega) \in Z\}$ . We know that  $\Sigma$  is a  $\sigma$ -algebra on  $\mathbb{R}_+ \times \Omega$ . A function  $f : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{X}$  (or a set-valued function  $F : \mathbb{R}_+ \times \Omega \rightarrow \mathbf{K}(\mathcal{X})$ ) is measurable and  $\mathcal{F}_t$ -adapted if and only if it is  $\Sigma$ -measurable. Therefore according to Kuratowski-Ryll-Nardzewski Measurable Selection Theorem (see e.g., [6]), for every  $\mathcal{F}_t$ -adapted and measurable  $\mathbf{K}(\mathcal{X})$ -valued stochastic process  $F = \{F_t : t \geq 0\}$ , there exists an

$\mathcal{F}_t$ -adapted and measurable  $\mathcal{X}$ -valued selection  $f = \{f_t : t \geq 0\}$  such that  $f_t(\omega) \in F_t(\omega)$  for all  $(t, \omega) \in \mathbb{R}_+ \times \Omega$ .  $\square$

### 3. Stochastic Integrals With Respect to a Banach Space Valued Brownian Motion

In this section, we study the stochastic integrals with respect to a Banach space valued Brownian motion.

In order to define stochastic integral in a Banach space with respect to a Banach space valued Brownian motion, we need some lemmata and known results.

**Lemma 3.1.** *Let  $(\Omega, \mathcal{F}, P)$  be a probability space,  $E$  a separable Banach space and  $X : \Omega \rightarrow E$  a random variable. If for every  $x^* \in E^*$ ,  $\langle x^*, X(\omega) \rangle = 0$  a.s., then  $X(\omega) = 0$  a.s.*

*Proof.* Let  $\nu$  be the distribution induced by  $X$ , that is  $\nu(A) := P(X^{-1}(A))$  for  $A \in \mathcal{B}(E)$ .

Since  $E$  is separable,  $\nu$  is a Radon measure. Let  $S$  be the support of  $\nu$ , i.e.,  $S$  is the minimal closed subset satisfying  $\nu(S) = 1$ .

$S$  can be obtained as follows: Let  $\mathcal{C}$  be the family of all closed subsets of  $E$  with  $\nu(C) = 1$ , i.e.

$$\mathcal{C} := \{C : \text{closed subset of } E \text{ with } \nu(C) = 1\}.$$

Set

$$S := \bigcap_{C \in \mathcal{C}} C,$$

then  $\nu(S) = 1$ .

We can show  $\nu(S^c) = \nu(\bigcup_{C \in \mathcal{C}} C^c) = 0$  since  $\nu(C^c) = 0$ , where  $C^c$  denotes the complement of  $C$  in  $E$  and  $C^c$  is open. In fact, for every compact set  $K \subset \bigcup_{C \in \mathcal{C}} C^c$ , we can find a finite open cover  $C_i^c, i = 1, 2, \dots, n, K \subset \bigcup_{i=1}^n C_i^c$ , and since  $\nu(C_i^c) = 0$ , we get  $\nu(K) = 0$ , so that by the fact that  $\nu$  is a Radon measure,  $\nu(\bigcup_{C \in \mathcal{C}} C^c) = 0$ . Set  $\Omega_0 := X^{-1}(S)$ . Then  $P(\Omega_0) = \nu(S) = 1$ .

Assume the statement ‘ $X(\omega) = 0$  a.s.’ does not hold, which implies  $S \neq \{0\}$  (Indeed, if  $S = \{0\}$ , then  $P(X^{-1}(\{0\})) = 1$  and  $X(\omega) = 0$  a.s.). So we can find  $x_0 \neq 0, x_0 \in S$ . Take  $x^* \in E^*$  such that  $\langle x^*, x_0 \rangle = 1$ . Then  $U := \{x \in E \mid \langle x^*, x \rangle \neq 0\}$  is open and  $x_0 \in U$ , which implies  $\nu(U) > 0$ . In fact, if  $\nu(U) = 0$ , then  $\nu(S \setminus U) = 1$ , which contradicts to the minimality of  $S$  since  $S \setminus U \subset S$  and  $S \setminus U \neq S$ . So it follows that

$$\nu(\{x \in E \mid \langle x^*, x \rangle \neq 0\}) = P(\{\omega \in \Omega \mid \langle x^*, X(\omega) \rangle \neq 0\}) > 0,$$

which contradicts the assumption and completes the proof.  $\square$

**Definition 3.2.** ([3]) Let  $(\mathcal{X}, \mathcal{H}, \mu)$  be an abstract Wiener space, (write briefly AWS), such that  $\mathcal{X}$  is a separable Banach space,  $\mathcal{H}$  is a separable Hilbert space,  $\mathcal{H}$  is continuously and densely embedded into  $\mathcal{X}$  and  $\mu$  is a Gaussian measure satisfying

$$\int_{\mathcal{X}} e^{\sqrt{-1}\langle x, x^* \rangle} \mu(dx) = e^{-\frac{1}{2}\|x^*\|_{\mathcal{H}}^2}$$

where  $\|\cdot\|_{\mathcal{H}}^2 = (\cdot, \cdot)$  and  $(\cdot, \cdot)$  is the inner product of  $\mathcal{H}$ . The Hilbert space  $\mathcal{H}$  is called the *reproducing kernel Hilbert space*, RKHS, of  $(\mathcal{X}, \mu)$ .

*Remark 3.3.* Let  $(\mathcal{X}, \|\cdot\|)$  be a separable Banach space with Gaussian measure  $\mu$ , by Theorem 4.4 in [17] or Theorem 2 in [21], there exists a Hilbert space  $\mathcal{H}$  densely embedded in  $\mathcal{X}$  such that  $(\mathcal{X}, \mathcal{H}, \mu)$  is an AWS. Therefore  $\mathcal{X}^* \subset \mathcal{H}^* = \mathcal{H} \subset \mathcal{X}$  and  $\mathcal{X}^*$  is densely embedded in  $\mathcal{H}$ .

Let  $(\mathcal{X}, \mathcal{H}, \mu)$  be an AWS. The set of all bounded linear operators from  $\mathcal{X}$  to  $\mathcal{X}$  is denoted by  $B(\mathcal{X}, \mathcal{X})$ . Let  $L^2((\mathcal{X}, \mu), \mathcal{X})$  denote the set of all Borel measurable functions  $f : (\mathcal{X}, \mathcal{B}(\mathcal{X}), \mu) \rightarrow (\mathcal{X}, \mathcal{B}(\mathcal{X}))$  such that the norm of  $f$  is defined by

$$\|f\| := \left( \int_{\mathcal{X}} \|f(x)\|^2 \mu(dx) \right)^{1/2} < +\infty. \quad (3.1)$$

Analogously, we define the set  $L^2((\mathcal{X}, \mu), \mathbb{R})$ .

*Remark 3.4.* Since for a separable Banach space  $\mathcal{X}$ , its cylindrical  $\sigma$ -algebra is equal to its Borel fields, for any  $f \in B(\mathcal{X}, \mathcal{X})$ ,  $f$  is continuous, therefore  $f$  is Borel measurable. By the Fernique Theorem (see e.g. [17]), there exists  $\alpha > 0$ , such that

$$\int_{\mathcal{X}} e^{\alpha \|f(x)\|^2} \mu(dx) < \infty,$$

thus  $B(\mathcal{X}, \mathcal{X})$  is a subspace of  $L^2((\mathcal{X}, \mu), \mathcal{X})$ . But  $B(\mathcal{X}, \mathcal{X})$  is not complete with respect to the norm  $\|\cdot\|$ .

Define  $R : \mathcal{X}^* \rightarrow L^2((\mathcal{X}, \mu), \mathbb{R})$  by  $R(x^*) = \langle x^*, \cdot \rangle_{\mathcal{X}^* \times \mathcal{X}}$ , the canonical map, then we have  $\mathcal{H} := \overline{\mathcal{X}^* L^2((\mathcal{X}, \mu), \mathbb{R})}$ . The transpose

$$i := R^* : \mathcal{H}^* = \mathcal{H} \rightarrow \mathcal{X}^{**}$$

satisfies  $i(\mathcal{H}) \subset \mathcal{X}$  by [1], Theorem 3.2.3.

Let  $\{x_n^*\} \subset \mathcal{X}^*$  such that  $\{R(x_n^*) : n \in \mathbb{N}\}$  is a C. O. N. S in  $\mathcal{H} = \overline{\mathcal{X}^* L^2((\mathcal{X}, \mu), \mathbb{R})}$ . Set  $e_n := R(x_n^*)$  and  $x_n := R^*(e_n) = i(e_n) \in \mathcal{X}$ . Then we have in  $\mathcal{X}$ ,

$$x = \sum_{n=1}^{\infty} \langle x_n^*, x \rangle x_n, \quad \mu \text{ a.s.}, \quad (3.2)$$

(see [1], Theorem 3.5.1).

**Lemma 3.5.** *Define a mapping*

$$\psi : (x, T) \in \mathcal{X} \times B(\mathcal{X}, \mathcal{X}) \rightarrow T(x) \in \mathcal{X},$$

where the  $\sigma$ -algebra on  $B(\mathcal{X}, \mathcal{X})$  is the Borel  $\sigma$ -algebra induced from the space  $L^2((\mathcal{X}, \mu), \mathcal{X})$ , which is denoted by  $\mathcal{B}(B(\mathcal{X}, \mathcal{X}))$ , Then  $\psi(x, T) = T(x)$  is bi-measurable, i.e. for any  $A \in \mathcal{B}(\mathcal{X})$ ,  $\psi^{-1}(A) \in \mathcal{B}(\mathcal{X}) \otimes \mathcal{B}(B(\mathcal{X}, \mathcal{X}))$ .

*Proof.* By (3.2), there exists a Borel set  $\mathcal{X}_0$  with  $\mu(\mathcal{X}_0) = 1$  such that

$$x = \sum_{n=1}^{\infty} \langle x_n^*, x \rangle x_n$$

for every  $x \in \mathcal{X}_0$ . So that for any  $T \in B(\mathcal{X}, \mathcal{X})$  and  $x \in \mathcal{X}_0$ , we have

$$T(x) = \sum_{n=1}^{\infty} \langle x_n^*, x \rangle T(x_n).$$

Remark that  $T(x_n) = \int_{\mathcal{X}} \langle x_n^*, u \rangle T(u) \mu(du)$  (the Bochner integral in  $\mathcal{X}$ ). In fact, for any  $u^* \in \mathcal{X}^*$ ,

$$\begin{aligned} \langle u^*, T(x_n) \rangle &= \langle u^*, T \circ i(e_n) \rangle \\ &= \langle u^* \circ T, i(e_n) \rangle = \langle R(u^* \circ T), e_n \rangle \\ &= \int_{\mathcal{X}} \langle u^*, T(u) \rangle \langle x_n^*, u \rangle \mu(du) \\ &= \langle u^*, \int_{\mathcal{X}} T(u) \langle x_n^*, u \rangle \mu(du) \rangle, \end{aligned}$$

where the Bochner integral  $\int_{\mathcal{X}} T(u) \langle x_n^*, u \rangle \mu(du) < \infty$  makes sense since

$$\begin{aligned} &\int_{\mathcal{X}} \|T(u) \langle x_n^*, u \rangle\| \mu(du) \\ &\leq \left( \int_{\mathcal{X}} \langle x_n^*, u \rangle^2 \mu(du) \right)^{1/2} \left( \int_{\mathcal{X}} \|T(u)\|^2 \mu(du) \right)^{1/2} < +\infty. \end{aligned}$$

Therefore  $T(x_n) = \int_{\mathcal{X}} \langle x_n^*, u \rangle T(u) \mu(du) \in \mathcal{X}$ .

For any fixed  $n$ ,

$$B(\mathcal{X}, \mathcal{X}) \ni T \longrightarrow T(x_n) \in \mathcal{X}$$

is continuous in the norm  $\|\cdot\|$ , since by the preceding result,

$$\|T(x_n)\| \leq \left( \int_{\mathcal{X}} \|T(u)\|^2 \mu(du) \right)^{1/2} = \|T\|.$$

Then for every  $n$ ,  $\langle x_n^*, x \rangle T(x_n)$  is bi-measurable.

Since for every  $T \in B(\mathcal{X}, \mathcal{X})$  and every  $x \in \mathcal{X}_0$

$$\psi(T, x) = \lim_{N \rightarrow \infty} \sum_{n=1}^N \langle x_n^*, x \rangle T(x_n),$$

it follows that  $\psi(T, x)$  is bi-measurable.  $\square$

Suppose  $(\Omega, \mathcal{F}, P)$  is a complete probability space and let  $(\mathcal{X}, \mathcal{H}, \mu)$  be an AWS.

An  $\mathcal{X}$ -valued continuous stochastic process  $\{B_t\}_{t \geq 0}$  is called an  $\mathcal{X}$ -valued Brownian motion if

- (i)  $B_0 = 0$  a.s.,
- (ii) the distribution of the random function  $t^{-1/2} B_t : \Omega \rightarrow \mathcal{X}$  is  $\mu$  for any  $t > 0$ ,
- (iii)  $(B_t - B_s)$  is independent of  $\mathcal{F}_s$  for any  $t > s \geq 0$ , where  $\mathcal{F}_t$  is the complete filtration including the  $\sigma$ -algebra generated by  $B_s, s \in [0, t]$ .

*Remark 3.6.* From the above definition, we know that  $\mathcal{X}$ -valued Brownian motion  $\{B_t\}_{t \geq 0}$  is an  $\mathcal{X}$ -valued martingale with mean-zero. For any  $x^* \in \mathcal{X}^*$ , the process  $\{\langle x^*, B_t \rangle\}_{t \geq 0}$  is a real valued Brownian motion.



For  $p \geq 1$ , we have

$$m_p := \mathbb{E} \left[ \left\| \frac{B_t - B_s}{(t-s)^{1/2}} \right\|^p \right] = \int_{\mathcal{X}} \|z\|^p \mu(dz) < +\infty. \quad (3.3)$$

**Definition 3.7.** ([3]) A Banach space  $(\mathcal{X}, \|\cdot\|)$  is called M-type 2 if and only if there exists a constant  $C_{\mathcal{X}} > 0$  such that, for any  $\mathcal{X}$ -valued martingale  $\{M_k\}$ , the inequality

$$\sup_k \mathbb{E}[\|M_k\|^2] \leq C_{\mathcal{X}} \sum_k \mathbb{E}[\|M_k - M_{k-1}\|^2] \quad (3.4)$$

holds.

Assume  $\mathcal{X}$  is a real separable M-type 2 Banach space and  $T \in (0, +\infty)$ . Let  $\mathcal{L}^2(\Omega \times [0, T]; B(\mathcal{X}, \mathcal{X}))$  be the space of measurable and  $\mathcal{F}_t$ -adapted operator valued stochastic processes

$$f : (\Omega \times [0, T], \mathcal{F} \otimes \mathcal{B}([0, T])) \rightarrow (B(\mathcal{X}, \mathcal{X}), \mathcal{B}(B(\mathcal{X}, \mathcal{X})))$$

which satisfy

$$\|f\|_{\mathcal{L}^2}^2 := \mathbb{E} \left[ \int_0^T \|f\|^2 dt \right] < +\infty. \quad (3.5)$$

We notice that  $\mathcal{L}^2(\Omega \times [0, T]; B(\mathcal{X}, \mathcal{X}))$  is not complete in  $\|\cdot\|_{\mathcal{L}^2}$ . For the simplicity,  $\mathcal{L}^2(\Omega \times [0, T]; B(\mathcal{X}, \mathcal{X}))$  is denoted by  $\mathcal{L}^2(B(\mathcal{X}, \mathcal{X}))$ .

**Lemma 3.8.** Let  $\{B_t\}_{t \in [0, T]}$  be an  $\mathcal{X}$ -valued Brownian motion,  $s, t \in [0, T]$  and  $s < t$ . For any  $f \in \mathcal{L}^2(B(\mathcal{X}, \mathcal{X}))$ , we have

$$\mathbb{E}[\|f_s(B_t - B_s)\|^2] = (t-s) \mathbb{E} \left[ \int_{\mathcal{X}} \|f_s(y)\|^2 \mu(dy) \right].$$

*Proof.* Define the mapping

$$\psi : (x, \xi) \ni \mathcal{X} \times B(\mathcal{X}, \mathcal{X}) \longrightarrow \xi(x) \in \mathcal{X},$$

then by Lemma 3.5,  $\psi : \mathcal{X} \times B(\mathcal{X}, \mathcal{X}) \longrightarrow \mathcal{X}$  is bi-measurable, i.e. for any  $A \in \mathcal{B}(\mathcal{X})$ ,

$$\psi^{-1}(A) \in \mathcal{B}(\mathcal{X}) \otimes \mathcal{B}(B(\mathcal{X}, \mathcal{X})).$$

For any  $A \in \mathcal{B}(\mathcal{X})$  and  $B \in \mathcal{B}(B(\mathcal{X}, \mathcal{X}))$ , set

$$\mu_{B_t - B_s}(A) := P(B_t - B_s \in A), \quad \nu_{f_s}(B) := P(f_s \in B)$$

and

$$\gamma(A) := P(f_s(B_t - B_s) \in A) = P((f_s, B_t - B_s) \in \psi^{-1}(A)).$$

Since  $B_t - B_s$  and  $f_s$  are independent,

$$P((f_s, B_t - B_s) \in \psi^{-1}(A)) = \mu_{B_t - B_s} \otimes \nu_{f_s}(\psi^{-1}(A))$$

where  $\mu_{B_t - B_s} \otimes \nu_{f_s}$  is the product measure on  $\mathcal{X} \times B(\mathcal{X}, \mathcal{X})$ .

Therefore, we have

$$\begin{aligned}
\mathbb{E}[\|f_s(B_t - B_s)\|^2] &= \int_{\Omega} \|f_s(B_t - B_s)\|^2 dP \\
&= \int_{\mathcal{X}} \|y\|^2 \gamma(dy) \\
&= \int_{\mathcal{X}} \|y\|^2 \mu_{B_t - B_s} \otimes \nu_{f_s}(\psi^{-1}(dy)) \\
&= \int_{\mathcal{X} \times B(\mathcal{X}, \mathcal{X})} \|Tx\|^2 \mu_{B_t - B_s}(dx) \nu_{f_s}(dT) \\
&= \int_{B(\mathcal{X}, \mathcal{X})} \left( \int_{\mathcal{X}} \|Tx\|^2 \mu_{B_t - B_s}(dx) \right) \nu_{f_s}(dT) \\
&= (t-s) \int_{B(\mathcal{X}, \mathcal{X})} \left( \int_{\mathcal{X}} \|Tx\|^2 \mu(dx) \right) \nu_{f_s}(dT) \\
&= (t-s) \int_{B(\mathcal{X}, \mathcal{X})} \|T\|^2 \nu_{f_s}(dT) \\
&= (t-s) \int_{\Omega} \|T\|^2 P(f_s \in dT) \\
&= (t-s) \mathbb{E} \left[ \int_{\mathcal{X}} \|f_s(y)\|^2 \mu(dy) \right].
\end{aligned}$$

which completes the proof.  $\square$

**Lemma 3.9.** For  $f \in \mathcal{L}^2(B(\mathcal{X}, \mathcal{X}))$ ,  $0 \leq s < t \leq T$ , each  $x^* \in \mathcal{X}^*$ , we have

$$\langle \mathbb{E}[f_s(B_t - B_s)], x^* \rangle = 0, \quad (3.6)$$

and

$$\langle \mathbb{E}[f_s(B_t - B_s) | \mathcal{F}_s], x^* \rangle = 0 \text{ a.s.} \quad (3.7)$$

*Proof.* Since  $f$  is in  $\mathcal{L}^2(B(\mathcal{X}, \mathcal{X}))$ , for any  $s, t \in [0, T]$  and  $s < t$ , by Lemma 3.8, we have

$$\mathbb{E}[\|f_s(B_t - B_s)\|^2] = (t-s) \mathbb{E} \left[ \int_{\mathcal{X}} \|f_s(x)\|^2 \mu(dx) \right] < +\infty.$$

Thus one can take expectation and conditional expectation of  $f_s(B_t - B_s)$  under  $\sigma$ -algebra  $\mathcal{F}_s$ .

Take any  $x^* \in \mathcal{X}^*$ ,

$$\langle f_s(B_t - B_s), x^* \rangle = \langle B_t - B_s, f_s^* x^* \rangle,$$

where  $f_s^*$  is the adjoint operator of  $f_s$  and  $f_s^* x^*$  is in  $\mathcal{X}^*$ .

As we proved Lemma 3.8 by appealing the independentness of  $B_t - B_s$  and  $f_s$ , we get

$$\begin{aligned}
& \mathbb{E} \left[ \langle f_s(B_t - B_s), x^* \rangle^2 \right] \\
&= \int_{\mathcal{X}} \mathbb{E} \left[ \langle f_s(x), x^* \rangle^2 \right] \mu \left( \frac{dx}{\sqrt{t-s}} \right) \\
&= (t-s) \int_{\mathcal{X}} \mathbb{E} \left[ \langle f_s(x), x^* \rangle^2 \right] \mu(dx) \\
&= (t-s) \mathbb{E} \left[ \int_{\mathcal{X}} \langle f_s(x), x^* \rangle^2 \mu(dx) \right] \\
&= (t-s) \mathbb{E} [\|f_s^* x^*\|_{\mathcal{H}}^2].
\end{aligned}$$

Setting  $Y^k := \sum_{j=1}^k (f_s^* x^*, e_j) e_j$  and noticing that  $\{e_j : j = 1, 2, \dots\}$  forms a complete orthonormal system of  $\mathcal{H}$ , we have

$$\|f_s^* x^* - Y^k\|_{\mathcal{H}}^2 = \sum_{j=k+1}^{\infty} (f_s^* x^*, e_j)^2.$$

Therefore  $\|f_s^* x^* - Y^k\|_{\mathcal{H}}^2$  is monotonically decreasing, and  $E[\|f_s^* x^*\|_{\mathcal{H}}^2]$  is finite. By the Lebesgue monotone convergence theorem, we have

$$\lim_{k \rightarrow \infty} \mathbb{E} [\|f_s^* x^* - Y^k\|_{\mathcal{H}}^2] = 0.$$

Then we have

$$\begin{aligned}
& \lim_{k \rightarrow \infty} \mathbb{E} \left[ \langle B_t - B_s, f_s^* x^* - Y^k \rangle^2 \right] \\
& \leq \lim_{k \rightarrow \infty} (t-s) \mathbb{E} [\|f_s^* x^* - Y^k\|_{\mathcal{H}}^2] = 0,
\end{aligned}$$

which implies

$$\begin{aligned}
& \mathbb{E} \left[ \langle B_t - B_s, \sum_{j=1}^{\infty} (f_s^* x^*, e_j) e_j \rangle \right] \\
&= \lim_{k \rightarrow \infty} \mathbb{E} \left[ \langle B_t - B_s, \sum_{j=1}^k (f_s^* x^*, e_j) e_j \rangle \right].
\end{aligned}$$

Thus

$$\begin{aligned}
& \langle \mathbb{E}[f_s(B_t - B_s)], x^* \rangle \\
&= \mathbb{E} \left[ \langle f_s(B_t - B_s), x^* \rangle \right] \\
&= \mathbb{E} \left[ \langle B_t - B_s, f_s^* x^* \rangle \right] \\
&= \lim_{k \rightarrow \infty} \mathbb{E} \left[ \langle B_t - B_s, Y^k \rangle \right] \\
&= \lim_{k \rightarrow \infty} \sum_{j=1}^k \mathbb{E} \left[ \langle B_t - B_s, e_j \rangle (f_s^* x^*, e_j) \right] \text{ a.s.} \\
&= \lim_{k \rightarrow \infty} \sum_{j=1}^k \mathbb{E} [(f_s^* x^*, e_j)] \mathbb{E} [\langle B_t - B_s, e_j \rangle] = 0
\end{aligned}$$

since for each  $e_j$ ,  $\mathbb{E}[\langle B_t - B_s, e_j \rangle] = 0$ , which yields (3.6).

By Jensen's inequality for conditional expectation, we have

$$\left(\mathbb{E}[\langle B_t - B_s, f_s^* x^* \rangle | \mathcal{F}_s]\right)^2 \leq \mathbb{E}[\langle B_t - B_s, f_s^* x^* \rangle^2 | \mathcal{F}_s],$$

so that

$$\begin{aligned} & \lim_{k \rightarrow \infty} \mathbb{E}[(\mathbb{E}[\langle B_t - B_s, f_s^* x^* - Y^K \rangle | \mathcal{F}_s])^2] \\ & \leq \lim_{k \rightarrow \infty} \mathbb{E}[\mathbb{E}[\langle B_t - B_s, f_s^* x^* - Y^K \rangle^2 | \mathcal{F}_s]] \\ & \leq \lim_{k \rightarrow \infty} (t - s) \mathbb{E}[\|f_s^* x^* - Y^K\|_{\mathcal{H}}^2] = 0. \end{aligned}$$

Then, if necessary we take a subsequence, we have

$$\begin{aligned} & \mathbb{E}[\langle B_t - B_s, \sum_{j=1}^{\infty} (f_s^* x^*, e_j) e_j \rangle | \mathcal{F}_s] \\ & = \lim_{k \rightarrow \infty} \mathbb{E}[\langle B_t - B_s, \sum_{j=1}^k (f_s^* x^*, e_j) e_j \rangle | \mathcal{F}_s]. \end{aligned}$$

Thus by [10],

$$\begin{aligned} & \langle \mathbb{E}[f_s(B_t - B_s) | \mathcal{F}_s], x^* \rangle \\ & = \mathbb{E}[\langle f_s(B_t - B_s), x^* \rangle | \mathcal{F}_s] \\ & = \mathbb{E}[\langle B_t - B_s, f_s^* x^* \rangle | \mathcal{F}_s] \\ & = \lim_{k \rightarrow \infty} \mathbb{E}[\langle B_t - B_s, Y^k \rangle | \mathcal{F}_s] \\ & = \lim_{k \rightarrow \infty} \sum_{j=1}^k \mathbb{E}[\langle B_t - B_s, e_j \rangle (f_s^* x^*, e_j) | \mathcal{F}_s] \text{ a.s.} \\ & = \lim_{k \rightarrow \infty} \sum_{j=1}^k (f_s^* x^*, e_j) \mathbb{E}[\langle B_t - B_s, e_j \rangle | \mathcal{F}_s] = 0 \text{ a.s.}, \end{aligned}$$

which yields (3.7). □

**Corollary 3.10.** For  $f \in \mathcal{L}^2(B(\mathcal{X}, \mathcal{X}))$ ,  $0 \leq s < t \leq T$ , we have

$$\mathbb{E}[f_s(B_t - B_s)] = 0.$$

*Proof.* It can be obtained by (3.6). □

**Theorem 3.11.** For  $f \in \mathcal{L}^2(B(\mathcal{X}, \mathcal{X}))$ ,  $0 \leq s < t \leq T$ , we have

$$\mathbb{E}[f_s(B_t - B_s) | \mathcal{F}_s] = 0 \text{ a.s.}$$

*Proof.* By Lemma 3.1 and Lemma 3.9, immediately we get the result. □

Let  $\mathcal{L}_{step}^2(B(\mathcal{X}, \mathcal{X}))$  be the subspace of those  $f \in \mathcal{L}^2(B(\mathcal{X}, \mathcal{X}))$  for which there exists a partition  $0 = t_0 < t_1 < \dots < t_n = T$  such that  $f_t = f_{t_k}$  for  $t \in [t_k, t_{k+1})$ ,  $0 \leq k \leq n-1$ ,  $n \in \mathbb{N}$ .

For  $f \in \mathcal{L}_{step}^2(B(\mathcal{X}, \mathcal{X}))$ , define the  $\mathcal{X}$ -valued  $\mathcal{F}$ -measurable random function

$$I_T(f) := \sum_{k=0}^{n-1} f_{t_k}(B_{t_{k+1}} - B_{t_k}).$$

For  $t \in [t_k, t_{k+1}), 0 \leq k \leq n-1$ ,

$$I_t(f) := \sum_{i=0}^{k-1} f_{t_i}(B_{t_{i+1}} - B_{t_i}) + f_{t_k}(B_t - B_{t_k}).$$

We have the following lemmata which are crucial for defining the Itô integral successfully.

**Lemma 3.12.** *For  $f \in \mathcal{L}_{step}^2(B(\mathcal{X}, \mathcal{X}))$ , we have  $I_T(f) \in L^2(\Omega, \mathcal{F}, P; \mathcal{X})$ ,  $\mathbb{E}[I_T(f)] = 0$  and*

$$\mathbb{E}[\|I_T(f)\|^2] \leq C_{\mathcal{X}} \mathbb{E}\left[\int_0^T \|f_s\|^2 ds\right], \quad (3.8)$$

where  $C_{\mathcal{X}}$  is the constant appeared in 3.7.

*Proof.* For any integer  $0 \leq k \leq n-1$ , let  $M_k = \sum_{i=0}^{k-1} f_{t_i}(B_{t_{i+1}} - B_{t_i})$ . Then  $M_k$  is  $\mathcal{F}_{t_k}$ -measurable. By Corollary 3.10,  $\mathbb{E}[M_k] = 0$ ,  $\mathbb{E}[I_T(f)] = \mathbb{E}[M_{n-1}] = 0$  and

$$\begin{aligned} \mathbb{E}[M_k | \mathcal{F}_{t_{k-1}}] &= \mathbb{E}[M_{k-1} + f_{t_{k-1}}(B_{t_k} - B_{t_{k-1}}) | \mathcal{F}_{t_{k-1}}] \\ &= M_{k-1} + \mathbb{E}[f_{t_{k-1}}(B_{t_k} - B_{t_{k-1}}) | \mathcal{F}_{t_{k-1}}] \\ &= M_{k-1} \text{ a.s. (By Theorem 3.11)} \end{aligned}$$

That is to say,  $\{M_k, \mathcal{F}_{t_k} : 0 \leq k \leq n-1\}$  is an  $\mathcal{X}$ -valued martingale. According to (3.3) and (3.4), we obtain

$$\begin{aligned} \mathbb{E}[\|I_T(f)\|^2] &= \mathbb{E}\left[\left\|\sum_{i=0}^{n-1} f_{t_i}(B_{t_{i+1}} - B_{t_i})\right\|^2\right] \\ &\leq \sup_{0 \leq k \leq n-1} \mathbb{E}\left[\left\|\sum_{j=0}^{k-1} f_{t_j}(B_{t_{j+1}} - B_{t_j})\right\|^2\right] \\ &= \sup_{0 \leq k \leq n-1} \mathbb{E}[\|M_k\|^2] \\ &\leq C_{\mathcal{X}} \sum_k \mathbb{E}[\|M_k - M_{k-1}\|^2] \\ &= C_{\mathcal{X}} \sum_k \mathbb{E}[\|f_{t_{k-1}}(B_{t_k} - B_{t_{k-1}})\|^2] \\ &= C_{\mathcal{X}} \sum_k (t_k - t_{k-1}) \mathbb{E}\left[\int_{\mathcal{X}} \|f_{t_k}(x)\|^2 \mu(dx)\right] \text{ (by Lemma 3.8)} \\ &= C_{\mathcal{X}} \mathbb{E}\left[\sum_k (t_k - t_{k-1}) \|f_{t_k}\|^2\right] \\ &= C_{\mathcal{X}} \mathbb{E}\left[\int_0^T \|f_t\|^2 dt\right] < +\infty. \end{aligned}$$

Then  $I_T(f) \in L^2(\Omega, \mathcal{F}, P; \mathcal{X})$  and the proof is complete.  $\square$

**Lemma 3.13.** For any  $f \in \mathcal{L}^2(B(\mathcal{X}, \mathcal{X}))$ , there exists a sequence  $f_n \in \mathcal{L}_{step}^2(B(\mathcal{X}, \mathcal{X}))$  such that  $f_n \rightarrow f$  in  $\|\cdot\|_{\mathcal{L}^2}$ .

*Proof.* Suppose  $f \in \mathcal{L}^2(B(\mathcal{X}, \mathcal{X}))$ , then

$$\mathbb{E}\left[\int_0^T \|f_t\|^2 dt\right] < +\infty,$$

which implies

$$\int_0^T \|f_t\|^2 dt < +\infty \text{ a.s.}$$

Setting  $f_t(\omega) = 0 \in B(\mathcal{X}, \mathcal{X})$  for all  $t \notin [0, T]$ , we get

$$\int_{-\infty}^{+\infty} \|f_t\|^2 dt < +\infty \text{ a.s.}$$

Since the completion  $\overline{B(\mathcal{X}, \mathcal{X})}^{L^2((\mathcal{X}, \mu), \mathcal{X})}$  of  $B(\mathcal{X}, \mathcal{X})$  in  $L^2((\mathcal{X}, \mu), \mathcal{X})$  is a separable Banach space with respect to the norm  $\|\cdot\|$ , by the Lemma 4.2 in [25],  $f(\omega)$  is approximated by bounded continuous function, so that we have

$$\lim_{s \rightarrow 0} \int_{-\infty}^{+\infty} \|f_{t+s} - f_t\|^2 dt = 0.$$

By a manner similar to that of K.Itô (page 176-178, [13]), we can construct the desired sequence of step operators which approximates  $f$  in  $\|\cdot\|_{\mathcal{L}^2}$ .  $\square$

**Definition 3.14.** For any  $f \in \mathcal{L}^2(B(\mathcal{X}, \mathcal{X}))$ , by Lemma 3.13, there exists a sequence  $\{f^n : n = 1, 2, \dots\} \subset \mathcal{L}_{step}^2(B(\mathcal{X}, \mathcal{X}))$ , such that  $f^n \rightarrow f$  with respect to  $\|\cdot\|_{\mathcal{L}^2}$ . By (3.8), the corresponding integral sequence  $\{I_t(f^n) : n = 1, 2, \dots\}$  converges in  $L^2(\Omega, \mathcal{X})$ . The limit is denoted by  $\int_0^t f_s dB_s$ , which is called *stochastic integral* of  $f$  with respect to  $\mathcal{X}$ -valued Brownian motion. For any interval  $[s, t] \subset [0, T]$ , the integral  $\int_s^t f_u dB_u$  is defined similarly.

About the stochastic integral, we have the following results.

**Theorem 3.15.** Let  $\mathcal{X}$  be an AWS of M-type 2.  $\{B_t\}_{t \in [0, T]}$  is the  $\mathcal{X}$ -valued Brownian motion. For  $f \in \mathcal{L}^2(B(\mathcal{X}, \mathcal{X}))$ , we have

- (i)  $E[I_t(f)] = 0$ ,  $I_t(f) \in L^2(\Omega, \mathcal{F}, P; \mathcal{X})$  and  $\{I_t(f) : t \in [0, T]\}$  is a measurable  $\mathcal{F}_t$ -martingale,  
(ii)

$$\mathbb{E}[\|I_t(f)\|^2] \leq C_{\mathcal{X}} \mathbb{E}\left[\int_0^t \|f_t\|^2 dt\right]. \quad (3.9)$$

- (iii) there exists a  $t$ -continuous (in the norm of  $\mathcal{X}$ ) version of

$$\int_0^t f_s(\omega) dB_s(\omega) \text{ for } t \in [0, T],$$

that is, there exists a  $t$ -continuous  $\mathcal{X}$ -valued stochastic process  $J_t$  on  $(\Omega, \mathcal{F}, P)$  such that

$$P\left(J_t = \int_0^t f_s dB_s\right) = 1 \text{ for all } t, 0 \leq t \leq T.$$

*Proof.* By Lemma 3.12 and Definition 3.14, it is not difficult to get the result, see [12], [13].  $\square$

From now on, we always assume that  $\int_0^t f_s(\omega)dB_s(\omega)$  means a  $t$ -continuous version of the integral.

#### 4. Set-valued Integrals With Respect to Lebesgue Measure

This section is on set-valued integrals of set-valued stochastic processes with respect to the Lebesgue measure on time interval  $[s, t]$ , which are studied in detail in [26]. Now we review it briefly.

Let  $p$  be  $1 \leq p < \infty$ ,  $\mathcal{X}$  a separable Banach space,  $T$  a positive real number,  $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$  a complete probability space with filtration  $\{\mathcal{F}_t; t \in [0, T]\}$  and  $\lambda$  the Lebesgue measure on  $[0, T]$ . In the following, the Lebesgue integral  $\int_{[s,t]} f d\lambda$  will be denoted by  $\int_s^t f_\tau d\tau$  for  $[s, t] \subset [0, T]$ . Let  $L^p([0, T] \times \Omega, \mathcal{B}([0, T]) \otimes \mathcal{F}, \lambda \times P; \mathcal{X})$ , denoted briefly by  $L^p([0, T] \times \Omega; \mathcal{X})$ , be the Banach space of equivalence classes of  $\mathcal{X}$ -valued,  $\mathcal{B}([0, T]) \otimes \mathcal{F}$ -measurable functions  $f : [0, T] \times \Omega \rightarrow \mathcal{X}$  such that

$$\int_{[0,T] \times \Omega} \|f(t, \omega)\|^p d\lambda dP < +\infty. \quad (4.1)$$

Let  $\mathcal{L}^p(\mathcal{X})$  be the family of all  $\mathcal{B}([0, T]) \otimes \mathcal{F}$ -measurable,  $\mathcal{F}_t$ -adapted,  $\mathcal{X}$ -valued stochastic processes  $f = \{f_t, \mathcal{F}_t; t \in [0, T]\}$  such that

$$\mathbb{E} \left[ \int_0^T \|f_s\|^p ds \right] := \int_{[0,T] \times \Omega} \|f(t, \omega)\|^p d\lambda dP < +\infty,$$

and  $\mathcal{L}^p(\mathbf{K}(\mathcal{X}))$  the family of all  $\mathcal{B}([0, T]) \otimes \mathcal{F}$ -measurable,  $\mathcal{F}_t$ -adapted, set-valued stochastic processes  $F = \{F_t, \mathcal{F}_t; t \in [0, T]\}$  such that  $\{\|F_t\|_{\mathbf{K}}\}_{t \in [0, T]}$  is in  $\mathcal{L}^p(\mathbb{R})$ .

For a  $\mathcal{B}([0, T]) \otimes \mathcal{F}$ -measurable set-valued stochastic process  $\{F_t, \mathcal{F}_t; t \in [0, T]\}$ , a  $\mathcal{B}([0, T]) \otimes \mathcal{F}$ -measurable selection  $f = \{f_t, \mathcal{F}_t; t \in [0, T]\}$  is called  $\mathcal{L}^p$ -selection if  $f = \{f_t, \mathcal{F}_t; t \in [0, T]\}$  is in  $\mathcal{L}^p(\mathcal{X})$ . The family of all  $\mathcal{L}^p$ -selections is denoted by  $S^p(F(\cdot))$ .

In fact, letting  $F$  be in  $\mathcal{L}^p(\mathbf{K}(\mathcal{X}))$  and setting

$$\Sigma_T := \bigcap_{t \in [0, T]} \{Z \in \mathcal{B}([0, T]) \otimes \mathcal{F} : Z_t \in \mathcal{F}_t\},$$

we have, by a manner similar to the proof of Proposition 2.5, that  $S^p(F(\cdot))$  is nonempty and

$$S^p(F(\cdot)) = \left\{ f \in L^p([0, T] \times \Omega, \Sigma_T, \lambda \times P; \mathcal{X}); f_t(\omega) \in F_t(\omega) \text{ for a.e. } (t, \omega) \right\}, \quad (4.2)$$

where  $L^p([0, T] \times \Omega, \Sigma_T, \lambda \times P; \mathcal{X})$  is the Banach space of equivalence classes of  $\mathcal{X}$ -valued,  $\Sigma_T$ -measurable functions  $f : [0, T] \times \Omega \rightarrow \mathcal{X}$  satisfying (4.1).

For a set-valued stochastic process  $\{F_t, \mathcal{F}_t; t \in [0, T]\} \in \mathcal{L}^p(\mathbf{K}(\mathcal{X}))$ , and for  $0 \leq s \leq t \leq T$ , define

$$\Lambda_{s,t} := \left\{ \int_s^t f_u du; (f_u)_{u \in [0, T]} \in S^p(F(\cdot)) \right\}, \quad (4.3)$$

where  $\int_s^t f_u(\omega)du$  is the Bochner integral with respect to the Lebesgue measure  $\lambda$  on the interval  $[s, t]$ . By [5, Theorem 9.41],  $f$  is Bochner integrable on the interval  $[s, t]$  if and only if its norm function  $\|f\|$  is Lebesgue integrable, that is,  $\int_s^t \|f_u\|du < +\infty$ .

For each  $f \in S^p(F(\cdot))$ , we have  $\int_0^T \|f_u\|^p du < +\infty$  a.s., which means there is a  $P$ -null set  $N_f$ , such that for all  $\omega \in \Omega \setminus N_f$  and for  $0 \leq s < t \leq T$ ,  $\int_s^t \|f(u)\|^p du < +\infty$ . For  $\omega \in N_f$ , we define  $\int_s^t f_u du = 0$ . Then for each  $f \in S^p(F(\cdot))$ ,  $\int_s^t f_u du$  is well defined for all  $\omega \in \Omega$ . Moreover, the process  $\{\int_0^t f_u du : t \in [0, T]\}$  is continuous, measurable and  $\mathcal{F}_t$ -adapted. Hence  $\Lambda_{s,t}$  is a subset of  $L^p(\Omega, \mathcal{F}_t, P; \mathcal{X}) \subset L^p(\Omega; \mathcal{X})$ .

We define the *decomposable closed hull* of  $\Lambda_{s,t}$  with respect to  $\mathcal{F}_t$  by

$$\begin{aligned} \overline{\text{de}}\Lambda_{s,t} &:= \left\{ g \in L^p(\Omega, \mathcal{F}_t, P; \mathcal{X}); \text{ for any } \varepsilon > 0, \text{ there exist a finite} \right. \\ &\quad \mathcal{F}_t\text{-measurable partition } \{A_1, \dots, A_n\} \text{ of } \Omega \text{ and} \\ &\quad f^1, \dots, f^n \in S^p(F(\cdot)) \text{ such that} \\ &\quad \left. \left\| g - \sum_{i=1}^n \chi_{A_i} \int_s^t f_u^i du \right\|_{L^p(\Omega, \mathcal{F}_t, P; \mathcal{X})} < \varepsilon \right\}. \end{aligned}$$

By Proposition 2.3,  $\overline{\text{de}}\Lambda_{s,t}$  determines an  $\mathcal{F}_t$ -measurable set-valued function

$$I_{s,t}(F) : \Omega \rightarrow \mathbf{K}(\mathcal{X})$$

such that the family of all  $L^p$ -integrable selections of  $I_{s,t}(F)$  is

$$S_{I_{s,t}(F)}^p(\mathcal{F}_t) = \overline{\text{de}}\Lambda_{s,t}.$$

Particularly,  $I_{0,t}(F)$  will be denoted by  $I_t(F)$  for brevity. Therefore  $\{I_t(F); t \in [0, T]\}$  is an  $\mathcal{F}_t$ -adapted set-valued stochastic process. The joint measurability of  $\{I_t(F); t \in [0, T]\}$  will be discussed in Lemma 4.5.

**Definition 4.1.** For a set-valued stochastic process  $\{F_t, \mathcal{F}_t; t \in [0, T]\}$  belongs to  $\mathcal{L}^p(\mathbf{K}(\mathcal{X}))$ , the set-valued random variable  $I_{s,t}(F)$  defined as above is called the *set-valued integral* of  $\{F_t, \mathcal{F}_t; t \in [0, T]\}$  with respect to the Lebesgue measure on the interval  $[s, t]$ . We denote it by  $\int_s^t F_u du := I_{s,t}(F)$ .

**Theorem 4.2.** For a set-valued stochastic process  $\{F_t, \mathcal{F}_t; t \in [0, T]\}$  belongs to  $\mathcal{L}^p(\mathbf{K}(\mathcal{X}))$ , the set-valued integral  $\int_0^T F_s(\omega)ds$  is convex a.s.

*Proof.* We have proved this theorem in [26]. Here we give only a sketch of the proof. Obviously,  $S_{I_T(F)}^p(\mathcal{F}_T)$  is nonempty. According to [6, Corollary 1.6], it suffices to prove that

$$S_{I_T(F)}^p(\mathcal{F}_T) = \overline{\text{de}}\left\{ \int_0^T f_s ds; f \in S^p(F(\cdot)) \right\}$$

is convex. It is noticed that if  $cl\left\{ \int_0^T f_s ds; f \in S^p(F(\cdot)) \right\}$  is convex, then the set of all selections  $S_{I_T(F)}^p(\mathcal{F}_T)$  is convex, where  $cl$  denotes the closure in  $L^p(\Omega; \mathcal{X})$ .



Define an  $(L^p(\Omega; \mathcal{X}), L^p(\Omega; \mathcal{X}))$ -valued measure  $\mu$  on  $\mathcal{B}([0, T])$  by

$$\mu(A) = \left( \int_A g_s ds, \int_A h_s ds \right), \quad A \in \mathcal{B}([0, T]).$$

The space  $([0, T], \mathcal{B}([0, T]), \lambda)$  is non-atomic. Hence by the result in [24, p.162], the closure of the range of  $\mu$  is convex in  $(L^p(\Omega; \mathcal{X}), L^p(\Omega; \mathcal{X}))$ . It is not difficult to get the set  $cl\left\{\int_0^T f_s ds; f \in S^p(F(\cdot))\right\}$  is convex, which yields that the set  $S_{I_T(F)}^p(\mathcal{F}_T)$  is convex. Then the integral  $I_T(F)(\omega)$  is convex a.s.  $\square$

If  $\mathcal{F}$  is separable with respect to the probability measure  $P$ , then the space  $L^p\left([0, T] \times \Omega, \mathcal{B}([0, T]) \otimes \mathcal{F}, \lambda \times P; \mathcal{X}\right)$  is separable by the same reason as in the proof of the separability of  $L^p(\Omega; \mathcal{X})$  in [25]. Therefore  $S^p(F(\cdot))$  is separable since it is a closed subset of  $L^p([0, T] \times \Omega; \mathcal{X})$ . Hence we can find a sequence  $\{f^n = (f_t^n)_{t \in [0, T]}; n \in \mathbb{N}\} \subset S^p(F(\cdot))$  such that

$$S^p(F(\cdot)) = cl\{f^n; n \in \mathbb{N}\},$$

where  $cl$  stands for the closure in  $L^p([0, T] \times \Omega; \mathcal{X})$ . Moreover, for  $0 \leq s \leq t \leq T$ , the equality

$$S_{I_{s,t}(F)}^p(\mathcal{F}_t) = \overline{de}\left\{\int_s^t f_u^n du; n \in \mathbb{N}\right\}$$

holds.

**Theorem 4.3.** *Assume  $\mathcal{F}$  is separable with respect to the probability measure  $P$ . Then for a set-valued stochastic process  $\{F_t, \mathcal{F}_t; t \in [0, T]\} \in \mathcal{L}^p(\mathbf{K}(\mathcal{X}))$ , there exists a sequence  $\{f^n; n \in \mathbb{N}\} \subset S^p(F(\cdot))$  such that*

$$F_t(\omega) = cl\{f_t^n(\omega); n \in \mathbb{N}\} \text{ for a.e. } (t, \omega),$$

and, for  $0 \leq s \leq t \leq T$ ,

$$I_{s,t}(F)(\omega) = cl\left\{\int_s^t f_u^n(\omega) du; n \in \mathbb{N}\right\} \text{ a.s.},$$

where  $cl$  denotes the closure in  $\mathcal{X}$ .

*Proof.* Here also we will give only a sketch of the proof. For  $0 \leq s \leq t \leq T$ ,  $I_{s,t}(F)$  is in  $\mathcal{M}(\Omega; \mathbf{K}(\mathcal{X}))$  and  $S_{I_{s,t}(F)}^p(\mathcal{F}_t)$  is nonempty. Then by [6, Theorem 1.0], there exists a sequence  $\{g_{s,t}^i; i \in \mathbb{N}\} \subset S_{I_{s,t}(F)}^p(\mathcal{F}_t)$  such that

$$I_{s,t}(F)(\omega) = cl\{g_{s,t}^i(\omega); i \in \mathbb{N}\} \text{ for all } \omega \in \Omega.$$

Note that in the above equation, the sequence depends on  $s$  and  $t$ .

Since

$$S_{I_{s,t}(F)}^p(\mathcal{F}_t) = \overline{de}\left\{\int_s^t f_u du; f \in S^p(F(\cdot))\right\},$$

by the separability of  $S^p(F(\cdot))$ , there exists a dense sequence  $\{f^n; n \in \mathbb{N}\}$  in  $S^p(F(\cdot))$  such that

$$S_{I_{s,t}(F)}^p(\mathcal{F}_t) = \overline{de}\left\{\int_s^t f_u^n du; n \in \mathbb{N}\right\} \text{ for } 0 \leq s \leq t \leq T.$$

It is not difficult to prove that the sequence  $\{f^n; n \in N\}$  is just the desired sequence.  $\square$

Then we have the following theorem (see [26]).

**Theorem 4.4.** *Assume  $\mathcal{F}$  is separable with respect to  $P$ . Then for a set-valued stochastic process  $\{F_t, \mathcal{F}_t; t \in [0, T]\} \in \mathcal{L}^p(\mathbf{K}(\mathcal{X}))$ ,  $S_{I_{s,t}(F)}^p(\mathcal{F}_t)$  is nonempty and bounded in  $L^p(\Omega, \mathcal{F}_t, P; \mathcal{X})$  for  $0 \leq s \leq t \leq T$ .*

**Lemma 4.5.** *Assume  $\mathcal{F}$  is separable with respect to  $P$ . Let a set-valued stochastic process  $\{F_t, \mathcal{F}_t; t \in [0, T]\}$  be in  $\mathcal{L}^p(\mathbf{K}(\mathcal{X}))$ . Then there exists a  $\mathcal{B}([0, T]) \otimes \mathcal{F}$ -measurable version  $\{\widetilde{I}_{s,t}(F); t \in [s, T]\}$  of  $\{I_{s,t}(F); t \in [s, T]\}$  such that  $I_{s,t}(F)(\omega)$  equals to  $\widetilde{I}_{s,t}(F)(\omega)$  a.s. and  $\widetilde{I}_{s,t}(F)(\omega) \in \mathbf{K}_b(\mathcal{X})$  for all  $s \leq t \leq T$  and almost sure  $\omega$ , where  $s \in [0, T]$  is arbitrarily fixed.*

*Proof.* If  $\mathcal{F}$  is separable, it can be obtained by using Theorem 4.3.  $\square$

From now on, if  $\mathcal{F}$  is separable, we will always assume that the set-valued integral of  $\{F_t, \mathcal{F}_t; t \in [0, T]\} \in \mathcal{L}^p(\mathbf{K}(\mathcal{X}))$  means the  $\mathcal{B}([0, T]) \otimes \mathcal{F}$ -measurable version  $\{\widetilde{I}_{s,t}(F); t \in [s, T]\}$ . For convenience, we still denote  $\widetilde{I}_{s,t}(F)(\omega)$  by  $I_{s,t}(F)(\omega)$  or  $\int_s^t F_u(\omega) du$ .

By Lemma 4.5 and properties of Hausdorff distance, we obtain

**Theorem 4.6.** *Assume  $\mathcal{F}$  is separable with respect to  $P$ . For set-valued stochastic processes  $\{F_t, \mathcal{F}_t; t \in [0, T]\}$  and  $\{G_t, \mathcal{F}_t; t \in [0, T]\} \in \mathcal{L}^p(\mathbf{K}(\mathcal{X}))$ , set*

$$\phi(t, \omega) := H\left(\int_0^t F_s(\omega) ds, \int_0^t G_s(\omega) ds\right) : [0, T] \times \Omega \rightarrow \mathbb{R}.$$

Then  $\phi(\cdot, \cdot)$  is  $\mathcal{B}([0, T]) \otimes \mathcal{F}$ -measurable.

**Proposition 4.7.** *Assume  $\mathcal{F}$  is separable with respect to  $P$ . Then for a set-valued stochastic process  $\{F_t, \mathcal{F}_t; t \in [0, T]\} \in \mathcal{L}^p(\mathbf{K}(\mathcal{X}))$ , the equality*

$$I_t(F)(\omega) = cl\left\{I_s(F)(\omega) + I_{s,t}(F)(\omega)\right\}$$

holds for  $0 \leq s < t \leq T$  and almost sure  $\omega$ , where  $cl$  stands for the closure in  $\mathcal{X}$ .

*Proof.* By Theorem 4.3 and Lemma 4.5, it is not difficult to get the desired result.  $\square$

By properties of Hausdorff distance and Proposition 4.7, we have

**Lemma 4.8.** *Assume  $\mathcal{F}$  is separable with respect to  $P$ . Then for a set-valued stochastic process  $\{F_t, \mathcal{F}_t; t \in [0, T]\} \in \mathcal{L}^p(\mathbf{K}(\mathcal{X}))$ , the set-valued integral  $\{I_t(F); t \in [0, T]\}$  is  $H$ -continuous in  $t$  a.s.*

The following lemma is important to study the solution of differential equation in the next section. It was proved in [26] in detail. Here we give the outline of the proof.

**Lemma 4.9.** *Assume  $\mathcal{F}$  is separable with respect to  $P$ . For set-valued stochastic processes  $\{F_t\}_{t \in [0, T]}$ ,  $\{G_t\}_{t \in [0, T]} \in \mathcal{L}^p(\mathbf{K}(\mathcal{X}))$ , and for all  $t$ , we have*

$$H^p\left(\int_0^t F_s(\omega) ds, \int_0^t G_s(\omega) ds\right) \leq t^{p-1} \int_0^t H^p(F_s(\omega), G_s(\omega)) ds \text{ a.s.}$$

*Proof.* Let  $\{f^i : i \in \mathbb{N}\}$  be the dense subset of  $S^p(F(\cdot))$ . Then we have

$$F_t(\omega) = cl\{f_t^i(\omega) : i \in \mathbb{N}\} \text{ a.e. } (t, \omega)$$

and for each  $t \in [0, T]$  and  $\omega \in \Omega$

$$\int_0^t F_s(\omega) ds = cl\left\{\int_0^t f_s^i(\omega) ds : i \in \mathbb{N}\right\}.$$

For each  $i \geq 1$ , we can choose a sequence  $\{g^{ij} : j \in \mathbb{N}\} \subset S^p(G(\cdot))$ , such that

$$\|f^i - g^{ij}\|_{L^p([0, T] \times \Omega; \mathcal{X})} \downarrow d(f^i, S^p(G(\cdot))) (j \rightarrow +\infty).$$

By Theorem 2.2 in [11], we have

$$\begin{aligned} d(f^i, S^p(G(\cdot))) &= \inf_{g \in S^p(G(\cdot))} \|f^i - g\|_{L^p(\Omega \times [0, T], \mathcal{X})} \\ &= \left(\int_{\Omega} \int_0^T \inf_{y \in G_s(\omega)} \|f_s^i(\omega) - y\|^p ds dP\right)^{1/p} \\ &= \left(\int_{\Omega} \int_0^T d^p(f_s^i(\omega), G_s(\omega)) ds dP\right)^{1/p}. \end{aligned}$$

Moreover, for each  $f^i$ , we can find a subsequence  $\{g^{i j_k} : k \in \mathbb{N}\} \subset \{g^{ij} : j \in \mathbb{N}\}$ , such that

$$\|f_s^i(\omega) - g_s^{i j_k}(\omega)\|^p \rightarrow d^p(f_s^i(\omega), G_s(\omega)) \text{ (} k \rightarrow +\infty \text{) a.e. } (s, \omega). \quad (4.4)$$

By (4.4), the Lebesgue dominated convergence theorem and the Hölder inequality, we can obtain that

$$\begin{aligned} &\sup_{x \in \int_0^t F_s(\omega) ds} d^p\left(x, \int_0^t G_s(\omega) ds\right) \\ &\leq t^{p-1} \int_0^t \sup_i d^p(f_s^i(\omega), G_s(\omega)) ds. \end{aligned}$$

Similarly, let  $\{g^m : m \in \mathbb{N}\}$  be a dense sequence of  $S^p(G(\cdot))$ . Then

$$G_t(\omega) = cl\{g_t^m(\omega) : m \in \mathbb{N}\} \text{ a.e. } (t, \omega)$$

and for each  $t \in [0, T]$

$$\int_0^t G_s(\omega) ds = cl\left\{\int_0^t g_s^m(\omega) ds : m \in \mathbb{N}\right\}.$$

By using the same way as the above, we obtain that for all  $t$  and almost sure  $\omega$

$$\begin{aligned} & \sup_{y \in \int_0^t G_s(\omega) ds} d^p \left( y, \int_0^t F_s(\omega) ds \right) \\ & \leq t^{p-1} \int_0^t \sup_m d^p (g_s^m(\omega), F_s(\omega)) ds \end{aligned}$$

Therefore, the inequality

$$\begin{aligned} & H^p \left( \int_0^t F_s(\omega) ds, \int_0^t G_s(\omega) ds \right) \\ & \leq t^{p-1} \int_0^t H^p (F_s(\omega), G_s(\omega)) ds \end{aligned}$$

holds for all  $t$  and almost sure  $\omega$ . □

### 5. Set-valued Stochastic Differential Equations

In this section, we study strong solutions to a set-valued stochastic differential equation. Let  $(\Omega, \mathcal{F}, \mathcal{F}_t; P)$  be a filtered probability space,  $(\mathcal{X}, \|\cdot\|)$  an AWS of M-type 2 and  $\{B_t\}$  the  $\mathcal{X}$ -valued  $\mathcal{F}_t$ -adapted Brownian motion. Let the functions  $a(\cdot, \cdot) : [0, T] \times \mathbf{K}(\mathcal{X}) \rightarrow \mathbf{K}(\mathcal{X})$  be  $(\mathcal{B}([0, T]) \otimes \sigma(\mathcal{C})) / \sigma(\mathcal{C})$ -measurable and  $b(\cdot, \cdot) : [0, T] \times \mathbf{K}(\mathcal{X}) \rightarrow B(\mathcal{X}, \mathcal{X})$  be  $(\mathcal{B}([0, T]) \otimes \sigma(\mathcal{C})) / \mathcal{B}(B(\mathcal{X}, \mathcal{X}))$ -measurable.

**Lemma 5.1.** *Let  $\{X_t : t \in [0, T]\}$  be an  $\mathcal{F}_t$ -adapted, measurable set-valued stochastic process, then the following statements hold:*

- (i)  $a(t, X_t(\omega)) : [0, T] \times \Omega \rightarrow \mathbf{K}(\mathcal{X})$  is  $(\mathcal{B}([0, T]) \otimes \mathcal{F}) / \sigma(\mathcal{C})$ -measurable and for fixed  $t \in [0, T]$ ,  $a(t, X_t(\cdot))$  is  $\mathcal{F}_t / \sigma(\mathcal{C})$ -measurable, and
- (ii)  $b(t, X_t(\omega)) : [0, T] \times \Omega \rightarrow B(\mathcal{X}, \mathcal{X})$  is  $(\mathcal{B}([0, T]) \otimes \mathcal{F}) / \mathcal{B}(B(\mathcal{X}, \mathcal{X}))$ -measurable and for fixed  $t \in [0, T]$ ,  $b(t, X_t(\cdot))$  is  $\mathcal{F}_t / \mathcal{B}(B(\mathcal{X}, \mathcal{X}))$ -measurable.

*Proof.* It can be proved by using the property of composition of measurable functions. For the detail of the proof, one might refer to Lemma 4.1 in [26]. □

Assume the above functions  $a(\cdot, \cdot)$  and  $b(\cdot, \cdot)$  also satisfy the following conditions:

$$\begin{aligned} & H(\{0\}, a(t, X)) + \| \|b(t, X)\| \| \\ & \leq C(1 + H(\{0\}, X)), \quad X \in \mathbf{K}(\mathcal{X}), t \in [0, T] \end{aligned} \tag{5.1}$$

for some constant  $C$ , and

$$\begin{aligned} & H(a(t, X), a(t, Y)) + \| \|b(t, X) - b(t, Y)\| \| \\ & \leq DH(X, Y), \quad X, Y \in \mathbf{K}(\mathcal{X}), t \in [0, T] \end{aligned} \tag{5.2}$$

for some constant  $D$ .

Let  $X_0$  be an  $L^2$ -integrably bounded set-valued random variable. Assume  $a(\cdot, \cdot)$  is  $(\mathcal{B}([0, T]) \otimes \sigma(\mathcal{C})) / \sigma(\mathcal{C})$ -measurable,  $b(\cdot, \cdot)$  is  $(\mathcal{B}([0, T]) \otimes \sigma(\mathcal{C})) / \mathcal{B}(B(\mathcal{X}, \mathcal{X}))$ -measurable and both  $a(\cdot, \cdot)$  and  $b(\cdot, \cdot)$  satisfy the conditions (5.1) and (5.2). Then, by Lemma 5.1, it is reasonable to define the set-valued stochastic differential equation as follows:

**Definition 5.2.** An  $\mathcal{F}_t$ -adapted,  $H$ -continuous in  $t$  almost surely and measurable set-valued process  $\{X_t; t \in [0, T]\}$  that satisfies the equation

$$X_t = cl\left(X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dB_s\right) \text{ for } t \in [0, T] \text{ a.s.} \quad (5.3)$$

is called a *strong solution* of the equation (5.3).

In the following, we will study the existence and uniqueness of the solutions to (5.3).

**Theorem 5.3.** *Assume  $\mathcal{F}$  is separable with respect to  $P$ . Let  $T > 0$ , and let  $a(\cdot, \cdot) : [0, T] \times \mathbf{K}(\mathcal{X}) \rightarrow \mathbf{K}(\mathcal{X})$  and  $b(\cdot, \cdot) : [0, T] \times \mathbf{K}(\mathcal{X}) \rightarrow B(\mathcal{X}, \mathcal{X})$  be measurable functions satisfying conditions (5.1) and (5.2). Then for any given  $L^2$ -integrably bounded initial set-valued random variable  $X_0$ , there exists a strong solution to (5.3).*

*Proof.* As a manner similar to that of solving the ordinary stochastic differential equation, we can use the successive approximation method to construct a solution to equation (5.3).

Define  $Y_t^0 = X_0$ . Then we can define  $Y_t^k = Y_t^k(\omega)$  inductively as follows:

$$Y_t^{k+1} = cl\left(X_0 + \int_0^t a(s, Y_s^k)ds + \int_0^t b(s, Y_s^k)dB_s\right). \quad (5.4)$$

By Proposition 2.2, Lemma 4.9 and condition (5.2), we have

$$\begin{aligned} & \mathbb{E}\left[H^2(Y_t^{k+1}, Y_t^k)\right] \\ &= \mathbb{E}\left[H^2\left(cl\left(X_0 + \int_0^t a(s, Y_s^k)ds + \int_0^t b(s, Y_s^k)dB_s\right), \right. \right. \\ & \quad \left. \left. cl\left(X_0 + \int_0^t a(s, Y_s^{k-1})ds + \int_0^t b(s, Y_s^{k-1})dB_s\right)\right)\right] \\ &\leq \mathbb{E}\left[\left(H(X_0, X_0) + H\left(\int_0^t a(s, Y_s^k)ds, \int_0^t a(s, Y_s^{k-1})ds\right) \right. \right. \\ & \quad \left. \left. + H\left(\int_0^t b(s, Y_s^k)dB_s, \int_0^t b(s, Y_s^{k-1})dB_s\right)\right)^2\right] \\ &= \mathbb{E}\left[\left(H\left(\int_0^t a(s, Y_s^k)ds, \int_0^t a(s, Y_s^{k-1})ds\right) \right. \right. \\ & \quad \left. \left. + \left\|\int_0^t b(s, Y_s^k)dB_s - \int_0^t b(s, Y_s^{k-1})dB_s\right\|\right)^2\right] \\ &\leq 2\mathbb{E}\left[H^2\left(\int_0^t a(s, Y_s^k)ds, \int_0^t a(s, Y_s^{k-1})ds\right)\right] \\ & \quad + 2\mathbb{E}\left[\left\|\int_0^t b(s, Y_s^k)dB_s - \int_0^t b(s, Y_s^{k-1})dB_s\right\|^2\right] \end{aligned}$$

$$\begin{aligned}
 &\leq 2t\mathbb{E}\left[\int_0^t H^2(a(s, Y_s^k), a(s, Y_s^{k-1}))ds\right] \\
 &\quad + 2C_{\mathcal{X}}\mathbb{E}\left[\int_0^t \|(b(s, Y_s^k) - b(s, Y_s^{k-1}))\|^2 ds\right] \\
 &\leq 2tD^2\mathbb{E}\left[\int_0^t H^2(Y_s^k, Y_s^{k-1})ds\right] \\
 &\quad + 2C_{\mathcal{X}}D^2\mathbb{E}\left[\int_0^t H^2(Y_s^k, Y_s^{k-1})ds\right] \\
 &\leq 2D^2(T + C_{\mathcal{X}})\int_0^t \mathbb{E}\left[H^2(Y_s^k, Y_s^{k-1})\right]ds \text{ for } k \geq 1, t \in [0, T].
 \end{aligned}$$

Similarly, we have, by the  $L^2$ -integrably boundedness of  $X_0$ ,

$$\mathbb{E}\left[H^2(Y_t^1, Y_t^0)\right] \leq A_1 t,$$

where  $A_1 < +\infty$  and is independent of  $t$ . So by the induction on  $k$  we obtain

$$\mathbb{E}\left[H^2(Y_t^{k+1}, Y_t^k)\right] \leq \frac{A_2^{k+1}t^{k+1}}{(k+1)!}, \quad k \geq 0, t \in [0, T],$$

where  $A_2 := \max\{1, A_1, (2D^2(T + C_{\mathcal{X}}))\}$ .

For  $m > n > 0$ , by the above inequality, we have

$$\begin{aligned}
 &\left(\mathbb{E}\left[H^2(Y_t^m, Y_t^n)\right]\right)^{1/2} = \|H(Y_t^m, Y_t^n)\|_{L^2} \\
 &\leq \|H(Y_t^m, Y_t^{m-1}) + H(Y_t^{m-1}, Y_t^{m-2}) + \dots + H(Y_t^{n+1}, Y_t^n)\|_{L^2} \\
 &= \left\| \sum_{k=n}^{m-1} H(Y_t^{k+1}, Y_t^k) \right\|_{L^2} \\
 &\leq \sum_{k=n}^{m-1} \|H(Y_t^{k+1}, Y_t^k)\|_{L^2} \\
 &\leq \sum_{k=n}^{\infty} \|H(Y_t^{k+1}, Y_t^k)\|_{L^2} \\
 &\leq \sum_{k=n}^{\infty} \left(\frac{A_2^{k+1}t^{k+1}}{(k+1)!}\right)^{1/2} \rightarrow 0 \text{ as } n \rightarrow +\infty,
 \end{aligned}$$

which means  $\{Y_t^n; n \in \mathbb{N}\}$  is a Cauchy sequence in the complete metric space  $L^2(\Omega; (K_b(\mathcal{X}), H))$ , so that the sequence  $\{Y_t^n; n \in \mathbb{N}\}$  converges to a limit  $\tilde{Y}_t$  in the sense that

$$\lim_{n \rightarrow +\infty} \mathbb{E}[H^2(Y_t^n, \tilde{Y}_t)] = 0 \text{ for every } t \in [0, T].$$

Taking Proposition 2.2, Lemma 4.9 and Doob's inequality into account, we can prove the convergence is also uniform in  $t \in [0, T]$  by the manner exact similar to that in [26]. Therefore the sequence  $\{Y_t^k; k \in \mathbb{N}\}$  converges to  $Y_t$  uniformly in  $[0, T]$  a.s. By Lemma 4.8 and the definition of the stochastic integral, every  $Y_t^k$  is

$H$ -continuous in  $t$ , so that the limit  $Y_t$  is also  $H$ -continuous in  $t$  a.s. Further it is clear that for any fixed  $t$ ,  $Y_t = \tilde{Y}_t$  a.s., so that we can replace  $\tilde{Y}_t$  by  $Y_t$ .

It remains to show that  $Y_t$  satisfies equation (5.3). Indeed, for each integer  $n \geq 0$ ,

$$Y_t^{n+1} = cl\left(X_0 + \int_0^t a(s, Y_s^n) ds + \int_0^t b(s, Y_s^n) dB_s\right). \quad (5.5)$$

Similarly, we have

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ H^2 \left( \int_0^t a(s, Y_s^n) ds, \int_0^t a(s, Y_s) ds \right) \right] = 0$$

and

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ H^2 \left( \int_0^t b(s, Y_s^n) dB_s, \int_0^t b(s, Y_s) dB_s \right) \right] = 0,$$

which, together with (5.5), verify the existence of a solution to (5.3).  $\square$

Again taking Proposition 2.2, Lemma 4.9 and Gronwall's lemma, we have, by the manner exact similar to that in [26],

**Theorem 5.4.** *Under the same condition as that in Theorem 5.3, the solution to equation (5.3) is strongly unique in the sense that*

$$P \left( H(X_t, \hat{X}_t) = 0 \text{ for all } t \in [0, T] \right) = 1$$

if  $X_t$  and  $\hat{X}_t$  are solutions to (5.3) with the same initial value  $X_0$ .

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