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SPECTRAL THEOREM APPROACH TO THE CHARACTERISTIC FUNCTION OF QUANTUM OBSERVABLES II

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Abstract. Using Stone's formula for finding the spectral resolution of an, either bounded or unbounded, self-adjoint operator on a Hilbert space, we compute the *vacuum characteristic function (quantum Fourier transform)* of the *anti-commutator* $XP+PX$ and of the *Hamiltonian* $\frac{1}{2}(X^2+P^2)$ quantum random variables . We also show how Stone's formula is applied to the computation of the vacuum characteristic function of finite dimensional quantum random variables. The method is proposed as an analytical alternative to the algebraic (or Heisenberg) approach relying on the associated Lie algebra commutation relations.

1. Introduction

On page 373 of [11] it is stated tha"the actual determination of the resolution of the identity for a given operator is not an easy matter, in general". Some Functional Analysis texts that cover spectral integration, for example [14], give, at most, the spectral resolution of the multiplication and differentiation operators on the Hilbert space of square integrable functions. Few Functional Analysis texts, such as [5], give Stone's formula for computing the spectral resolution of a self-adjoint Hilbert space operator and even fewer, for example [9], give examples of how to use it.

In this paper, which is a sequel to [4], we illustrate in detail the use of Stone's formula for the spectral resolution, by applying it to the *anti-commutator operator* $XP+PX$ and the *quantum harmonic oscillator Hamiltonian operator* $\frac{1}{2}(X^2+P^2)$ of Quantum Mechanics. We see that it naturally leads to the appearance of several types of differential equations and special functions. After computing the spectral resolution, we compute the Vacuum Characteristic Function (Quantum Fourier Transform) of these operators and show that the results agree with those obtained, using Lie algebraic techniques, in [1].

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As in [4], we consider the (self-adjoint) *position, momentum* and *identity* operators, respectively, defined in $L^2(\mathbb{R},\mathbb{C})$ with inner product

$$\langle f,g\rangle=\frac{1}{\sqrt{\hbar}}\int_{\mathbb{R}}\overline{f(x)}g(x)\,dx\ ,$$

by

$$X\,f(x)=x\,f(x)\ ;\ P\,f(x)=-i\,\hbar\,f'(x)\ ;\ \mathbf{1}\,f(x)=f(x)\ ,$$

and satisfying the commutation relations

$$[P,X]=-i\hbar\mathbf{1}\ ,$$

on

$$\Omega=\operatorname{dom}(X)\cap\operatorname{dom}(P)\ ,$$

where,

$$\operatorname{dom}(X)=\{f\in L^2(\mathbb{R},\mathbb{C}))\ :\ \int_{\mathbb{R}}x^2\,|f(x)|^2\,dx<+\infty\}\ ,$$

and

$$\operatorname{dom}(P)=\{f\in L^2(\mathbb{R},\mathbb{C})\ :\ f\text{ is abs. cont. and }\int_{\mathbb{R}}|f'(x)|^2\ dx<+\infty\}\ ,$$

are respectively the, dense in $L^2(\mathbb{R},\mathbb{C})$, domains of X and P . Since it contains $C_0^\infty(\mathbb{R})$, Ω is nonempty and dense in $L^2(\mathbb{R},\mathbb{C})$. The *Schwartz class* $\mathcal{S}$ is a common **invariant** domain [7] of X and P which is also dense in $L^2(\mathbb{R},\mathbb{C})$ and contains $C_0^\infty(\mathbb{R})$, and is therefore suitable for defining XP and PX. In this paper we normalize to $\hbar=1$. Functions in the domain of P are continuous and vanish at infinity. As pointed out in [4],

$$\Phi=\Phi(x)=\pi^{-1/4}\,e^{-\frac{x^2}{2\hbar}}=\pi^{-1/4}\,e^{-\frac{x^2}{2}}\ ,$$

is a unit vector in Ω. For $a\in\mathbb{C}$ we denote $R(a;T)=(a-T)^{-1}$ the *resolvent* of T. The *spectral resolution* $\{E_\lambda\,|\,\lambda\in\mathbb{R}\}$ of a bounded or unbounded self-adjoint operator T in a complex separable Hilbert Space $\mathcal{H}$ can be computed using Stone's formula (see [5], Theorems X.6.1 and XII.2.10):

$$E\left((a,b)\right)=\lim_{\delta\to0^+}\lim_{\epsilon\to0^+}\frac{1}{2\pi i}\int_{a+\delta}^{b-\delta}\left(R(t-\epsilon i;T)-R(t+\epsilon i;T)\right)\,dt\ ,$$

where (a,b) is the open interval $a<\lambda<b$, and the limit is in the strong operator topology. For $a\to-\infty$ and $b=\lambda$ we have

$$\begin{aligned}E_\lambda=E\left((-\infty,\lambda]\right)&=\lim_{\rho\to0^+}E\left((-\infty,\lambda+\rho)\right)\\&=\lim_{\rho\to0^+}\lim_{\delta\to0^+}\lim_{\epsilon\to0^+}\frac{1}{2\pi i}\int_{-\infty}^{\lambda+\rho-\delta}\left(R(t-\epsilon i;T)-R(t+\epsilon i;T)\right)\,dt\ .\end{aligned}$$

For $f,g\in\mathcal{H}$, see [9],

$$\langle f,E_\lambda g\rangle=\lim_{\epsilon\to0^+}\frac{1}{2\pi i}\int_{-\infty}^{\lambda}\langle f,\left(R(t-\epsilon i;T)-R(t+\epsilon i;T)\right)g\rangle\,dt\ . \qquad (1.1)$$

Once the *vacuum resolution of the identity*

$$\langle \Phi, E_\lambda \Phi \rangle = \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{\lambda} \langle \Phi, \left(R(t - \epsilon i; T) - R(t + \epsilon i; T) \right) \Phi \rangle \, dt \ ,$$

corresponding to the operator T is known, equivalently, once its *vacuum differential*

$$d\langle \Phi, E_\lambda \Phi \rangle \ ,$$

is known, we can immediately compute its *vacuum characteristic function*

$$\langle \Phi, e^{itT} \Phi \rangle = \int_{\mathbb{R}} e^{it\lambda} \, d\langle \Phi, E_\lambda \Phi \rangle \ .$$

For a complex number z we denote $\mathrm{Re}\, z$, $\mathrm{Im}\, z$, its real and imaginary part, respectively. The following Lemma is very useful in computing the vacuum spectral resolution.

Lemma 1.1.

$$\langle \Phi, E_\lambda \Phi \rangle = \lim_{\epsilon \to 0^+} \frac{1}{\pi} \int_{-\infty}^{\lambda} \mathrm{Im} \, \langle \Phi, R(t - \epsilon i; T) \Phi \rangle \, dt \ .$$

Proof. Using the resolvent identity, see [5],

$$R(\lambda; T)^* = R(\bar{\lambda}; T^*) \ ,$$

we have

$$\begin{aligned}
\langle \Phi, E_\lambda \Phi \rangle &= \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{\lambda} \langle \Phi, \left(R(t - \epsilon i; T) - R(t + \epsilon i; T) \right) \Phi \rangle \, dt \\
&= \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{\lambda} \left(\langle \Phi, R(t - \epsilon i; T) \Phi \rangle - \langle R(t - \epsilon i; T) \Phi, \Phi \rangle \right) \, dt \\
&= \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{\lambda} \left(\langle \Phi, R(t - \epsilon i; T) \Phi \rangle - \overline{\langle \Phi, R(t - \epsilon i; T) \Phi \rangle} \right) \, dt \\
&= \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{\lambda} 2i \, \mathrm{Im} \, \langle \Phi, R(t - \epsilon i; T) \Phi \rangle \, dt \\
&= \lim_{\epsilon \to 0^+} \frac{1}{\pi} \int_{-\infty}^{\lambda} \mathrm{Im} \, \langle \Phi, R(t - \epsilon i; T) \Phi \rangle \, dt \ .
\end{aligned}$$

□

As in [4], we define the Fourier transform of f by

$$\hat{f}(t) = (Uf)(t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{i\lambda t} f(\lambda) d\lambda \ ,$$

and the inverse Fourier transform of $\hat{f}$ by

$$f(\lambda) = \left(U^{-1} \hat{f} \right)(\lambda) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-i\lambda t} \hat{f}(t) dt \ .$$

Finally, for $a \in \mathbb{R}$ we denote by δ_a and H_a , respectively, the *Dirac delta* and *Heaviside unit step* functions defined, for a *test function* f, by

$$\int_{\mathbb{R}} f(x)\delta_a(x)\,dx = \int_{\mathbb{R}} f(x)\delta(x-a)\,dx = f(a)\ ,$$

and

$$H_a(x) = H(x-a) = \begin{cases} 1\ , & x \ge a \\ 0\ , & x < a \end{cases} .$$

In the sense of distributions, δ_a is the derivative of H_a.

2. The Anti-commutator Operator $XP + PX$

Theorem 2.1. *The operator $XP+PX$ is symmetric on the Schwartz class $\mathcal{S}$ and admits a self-adjoint extension.*

Proof. Let $T = XP + PX$ and let $f, g \in \mathcal{S}$. Since $Xf, Pf \in \mathcal{S}$, T is well defined on $\mathrm{dom}(T) = \mathcal{S}$. Since X, P are self-adjoint,

$$\begin{aligned}\langle Tf, g\rangle =& \langle XPf, g\rangle + \langle PXf, g\rangle = \langle Pf, Xg\rangle + \langle Xf, Pg\rangle \\ =& \langle f, PXg\rangle + \langle f, XPg\rangle = \langle f, (PX+XP)g\rangle = \langle f, Tg\rangle\ ,\end{aligned}$$

so T is symmetric. To show that T admits a self-adjoint extension, we consider the *conjugation operator*

$$Kf(x) = \overline{f(-x)}\ .$$

If f is in the domain of T then so also is Kf and, since for a function g in the domain of T,

$$Tg(x) = -ig(x) - 2ix\frac{d}{dx}g(x)\ ,$$

we find

$$\begin{aligned}[T, K]f(x) &= TKf(x) - KTf(x) \\ &= \left(-i\overline{f(-x)} + 2ix\overline{f'(-x)}\right) - \left(-i\overline{f(-x)} + 2ix\overline{f'(-x)}\right) = 0\ ,\end{aligned}$$

i.e., in the context of Section 8 of [12], T is K-real. Thus, by Theorem 8.9 of [12], the symmetric operator T has a self-adjoint extension. $\square$

Theorem 2.2. *For* $\mathrm{Im} a > -1$ *and* $s \neq 0$*, the resolvent of* $XP + PX$ *is*

$$R(a; XP+PX)g(s) = \begin{cases} \frac{i}{2} s^{-\frac{a+i}{2i}} \int_s^{\infty} w^{\frac{a-i}{2i}} g(w)\,dw\ , & s > 0 \\ \frac{i}{2}(-s)^{-\frac{a+i}{2i}} \int_{-\infty}^{s} (-w)^{\frac{a-i}{2i}} g(w)\,dw\ , & s < 0 \end{cases} ,$$

where $g \in \mathcal{S}$*. For* $s = 0$*,*

$$R(a; XP+PX)g(0) = \frac{g(0)}{a+i}\ .$$

Proof. For $a \in \mathbb{C}$ with $\mathrm{Im} a > -1$ and for $s \in \mathbb{R}$,

$$\begin{aligned} R(a; XP + PX)g(s) = G(s) &\iff g(s) = (a - XP - PX)G(s) \\ &\iff g(s) = aG(s) + isG'(s) + i(G(s)) + sG'(s)) \\ &\iff 2isG'(s) + (a+i)G(s) = g(s) \ . \end{aligned}$$

We point out that G is differentiable since, being in the range of $R(a; XP + PX)$, it is in the domain of $XP + PX$. For $s = 0$ we find

$$G(0) = \frac{g(0)}{a+i} \ .$$

For $s \neq 0$, we have

$$G'(s) + \frac{a+i}{2is} G(s) = \frac{g(s)}{2is} \ ,$$

which is a first-order linear ordinary differential equation with complex coefficients. Multiplying by the integrating factor $|s|^{\frac{a+i}{2i}}$, replacing s by t and integrating from s to ∞ (if $s > 0$) and from $-\infty$ to s (if $s < 0$), using the fact that, since G is in the Schwartz class,

$$\lim_{t \to \pm\infty} |t^{\frac{a+i}{2i}} G(t)| = \lim_{t \to \pm\infty} |t^{\frac{\mathrm{Im} a + 1}{2}} G(t)| = 0 \ ,$$

we find

$$G(s) = R(a; XP + PX)g(s) = \begin{cases} \frac{i}{2} s^{-\frac{a+i}{2i}} \int_s^\infty w^{\frac{a-i}{2i}} g(w)\, dw \ , & s > 0 \\ \frac{i}{2} (-s)^{-\frac{a+i}{2i}} \int_{-\infty}^s (-w)^{\frac{a-i}{2i}} g(w)\, dw \ , & s < 0 \ . \end{cases}$$

□

Corollary 2.3. *For* $\mathrm{Im} a > -1$*, if* g *is even then so also is* $R(a; XP + PX)g$*.*

Proof. Let $s > 0$. By Theorem 2.2,

$$\begin{aligned} R(a; XP + PX)g(-s) &= \frac{i}{2} s^{-\frac{a+i}{2i}} \int_{-\infty}^{-s} (-w)^{\frac{a-i}{2i}} g(w)\, dw \\ &= \frac{i}{2} s^{-\frac{a+i}{2i}} \int_{-\infty}^{-s} (-w)^{\frac{a-i}{2i}} g(-w)\, (-1) d(-w) \ , \end{aligned}$$

which, letting $u = -w$ in the integral, yields

$$R(a; XP + PX)g(-s) = \frac{i}{2} s^{-\frac{a+i}{2i}} \int_s^\infty u^{\frac{a-i}{2i}} g(u)\, du = R(a; XP + PX)g(s) \ .$$

For $s < 0$,

$$R(a; XP + PX)g(-s) = \frac{i}{2} (-s)^{-\frac{a+i}{2i}} \int_{-s}^\infty w^{\frac{a-i}{2i}} g(w)\, dw \ ,$$

which, letting $u=-w$, yields

$$\begin{aligned}R(a;XP+PX)g(-s)&=\frac{i}{2}(-s)^{-\frac{a+i}{2i}}\int_s^{-\infty}(-u)^{\frac{a-i}{2i}}g(-u)\,(-1)du\\&=\frac{i}{2}(-s)^{-\frac{a+i}{2i}}\int_{-\infty}^{s}(-u)^{\frac{a-i}{2i}}g(u)\,du\\&=R(a;XP+PX)g(s)\ .\end{aligned}$$

□

Theorem 2.4. *The vacuum spectral resolution of $XP+PX$ is*

$$\langle\Phi,E_\lambda\Phi\rangle=\frac{1-i}{8\pi}\int_{-\infty}^{\lambda}\left(e^{-\frac{\pi t}{4}}\,B\left(-1;\frac{1-it}{4},\frac{1}{2}\right)+e^{\frac{\pi t}{4}}\,B\left(-1;\frac{1+it}{4},\frac{1}{2}\right)\right)\,dt\ ,$$

where

$$B(z;a,b)=\int_0^z t^{a-1}(1-t)^{b-1}\,dt=z^a\sum_{n=0}^{\infty}\frac{(1-b)_n}{n!\,(a+n)}z^n\ ,$$

is the incomplete Beta function and, for $x\in\mathbb{R}$, $(x)_n=x(x+1)(x+2)\cdots(x+n-1)$. Moreover, for $t\in\mathbb{R}$,

$$\langle\Phi,e^{it(XP+PX)}\Phi\rangle=(\operatorname{sech}2t)^{1/2}$$

Proof. We first present a proof without Stone's formula:

$$[P,X]=-i\mathbf{1}\implies PX=XP-i\mathbf{1}\ ,$$

so

$$\begin{aligned}\langle\Phi,e^{it(XP+PX)}\Phi\rangle&=\langle\Phi,e^{it(2XP-i\mathbf{1})}\Phi\rangle=e^t\langle\Phi,e^{2itXP}\Phi\rangle\\&=e^t\sum_{n=0}^{\infty}\frac{(2it)^n}{n!}\langle\Phi,(XP)^n\Phi\rangle\ .\end{aligned}$$

Using Lemma 8.4 of [4], trivially extended to $n=0$ and $\sum_{n=0}^{\infty}$ with the use of $S(0,0)=1$, we have

$$\begin{aligned}\langle\Phi,e^{it(XP+PX)}\Phi\rangle=&e^t\sum_{n=0}^{\infty}\frac{(2it)^n}{n!}\sum_{k=0}^{n}(-1)^{n-k}i^{n-k}S(n,k)\langle\Phi,X^kP^k\Phi\rangle\\=&e^t\sum_{n=0}^{\infty}\frac{(2it)^n}{n!}\sum_{k=0}^{\infty}(-1)^{n-k}i^{n-k}S(n,k)\langle X^k\Phi,P^k\Phi\rangle\ ,\end{aligned}$$

where we have used the fact that X is self-adjoint and the *Stirling numbers of the second kind* $S(n,k)$ satisfy $S(n,k)=0$ for $k>n$. As in the proof of Theorem 8.5 of [4],

$$\langle X^k\Phi,P^k\Phi\rangle=\frac{1}{\pi\sqrt{2}}\int_{\mathbb{R}}\int_{\mathbb{R}}\lambda^k\mu^k e^{-\frac{\lambda^2+\mu^2}{2}+i\lambda\mu}\,d\lambda\,d\mu\ .$$

Thus, switching the order of summation,

$$\begin{aligned}\langle \Phi, e^{it(XP+PX)}\Phi\rangle =&\frac{e^t}{\pi\sqrt{2}}\int_{\mathbb{R}}\int_{\mathbb{R}}\sum_{k=0}^{\infty} i^k\left(\sum_{n=0}^{\infty}\frac{(2t)^n}{n!}S(n,k)\right)\lambda^k\mu^k e^{-\frac{\lambda^2+\mu^2}{2}+i\lambda\mu}\,d\lambda\,d\mu\\ =&\frac{e^t}{\pi\sqrt{2}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\lambda\mu(e^{2t}-1)}e^{-\frac{\lambda^2+\mu^2}{2}+i\lambda\mu}\,d\lambda\,d\mu\\ =&\frac{e^t}{\pi\sqrt{2}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-\frac{\lambda^2+\mu^2}{2}+i\lambda\mu e^{2t}}\,d\lambda\,d\mu\ ,\end{aligned}$$

where we have used the identities

$$\sum_{n=0}^{\infty}\frac{(2t)^n}{n!}S(n,k)=\frac{(e^{2t}-1)^k}{k!}\ ,$$

and

$$\sum_{k=0}^{\infty}\frac{\left(i(e^{2t}-1)\lambda\mu\right)^k}{k!}=e^{i\lambda\mu(e^{2t}-1)}\ .$$

Thus, using the integration formula

$$\int_{\mathbb{R}}\int_{\mathbb{R}} e^{\alpha x^2+\beta y^2+i\gamma xy}\,dxdy=\frac{2\pi}{\sqrt{\gamma^2+4\alpha\beta}}\ ,\mathrm{Re}\,\beta<0\ ,\ \mathrm{Re}\left(4\alpha+\frac{\gamma^2}{\beta}\right)<0\ ,$$

we obtain

$$\langle \Phi, e^{it(XP+PX)}\Phi\rangle=\frac{e^t}{\pi\sqrt{2}}\frac{2\pi}{\sqrt{e^{4t}+1}}=\sqrt{\frac{2}{e^{2t}+e^{-2t}}}=\sqrt{\mathrm{sech}\,2\mathrm{t}}\ .$$

The vacuum spectral resolution can be found using the inverse Fourier transform (see Theorem 9.5 of [4]).

We can derive the formula for the vacuum spectral resolution by a direct application of Stone's formula as follows: By Lemma 1.1 and Theorem 2.2, since, by Corollary 2.3,

$$\mathrm{Im}\ (\Phi(s)R(t-\epsilon i;T)\Phi(s))$$

is an even function of s we have,

$$\begin{aligned}\langle\Phi,E_\lambda\Phi\rangle=&\frac{1}{\pi}\lim_{\epsilon\to0^+}\int_{-\infty}^{\lambda}\mathrm{Im}\,\langle\Phi,R(t-\epsilon i;T)\Phi\rangle\,dt\\ =&\frac{1}{\pi}\lim_{\epsilon\to0^+}\int_{-\infty}^{\lambda}\int_{-\infty}^{\infty}\mathrm{Im}\ (\Phi(s)R(t-\epsilon i;T)\Phi(s))\ ds\,dt\\ =&\frac{2}{\pi}\lim_{\epsilon\to0^+}\int_{-\infty}^{\lambda}\int_{0}^{\infty}\mathrm{Im}\ (\Phi(s)R(t-\epsilon i;T)\Phi(s))\ ds\,dt\\ =&\frac{1}{\pi^{3/2}}\lim_{\epsilon\to0^+}\int_{-\infty}^{\lambda}\int_{0}^{\infty}\mathrm{Im}\left(ie^{-\frac{s^2}{2}}s^{\frac{it+\epsilon-1}{2}}\int_s^{\infty}w^{\frac{-it-\epsilon-1}{2}}e^{-\frac{w^2}{2}}\,dw\right)ds\,dt\\ =&\frac{1}{\pi^{3/2}}\lim_{\epsilon\to0^+}\int_{-\infty}^{\lambda}\int_{0}^{\infty}\mathrm{Re}\left(e^{-\frac{s^2}{2}}s^{\frac{it+\epsilon-1}{2}}\int_s^{\infty}w^{\frac{-it-\epsilon-1}{2}}e^{-\frac{w^2}{2}}\,dw\right)ds\,dt\\ =&\frac{1}{\pi^{3/2}}\lim_{\epsilon\to0^+}\int_{-\infty}^{\lambda}\int_{0}^{\infty}\int_s^{\infty}e^{-\frac{s^2+w^2}{2}}\mathrm{Re}\left(s^{\frac{it+\epsilon-1}{2}}w^{\frac{-it-\epsilon-1}{2}}\right)dw\,ds\,dt\ .\end{aligned}$$

Since, for $s > 0$,

$$s^{\frac{it+\epsilon-1}{2}} = s^{\frac{\epsilon-1}{2}}\left(\cos\left(\frac{t}{2}\ln s\right)+i\sin\left(\frac{t}{2}\ln s\right)\right) \ ,$$
$$w^{\frac{-it-\epsilon-1}{2}} = w^{-\frac{\epsilon+1}{2}}\left(\cos\left(\frac{t}{2}\ln w\right)-i\sin\left(\frac{t}{2}\ln w\right)\right) \ ,$$

we see that

$$\mathrm{Re}\left(s^{\frac{it+\epsilon-1}{2}}w^{\frac{-it-\epsilon-1}{2}}\right) = s^{\frac{\epsilon-1}{2}}w^{-\frac{\epsilon+1}{2}}\cos\left(\frac{t}{2}\ln\frac{s}{w}\right) \ .$$

Mathematica computes

$$\int_0^\infty\int_s^\infty e^{-\frac{s^2+w^2}{2}}s^{\frac{\epsilon-1}{2}}w^{-\frac{\epsilon+1}{2}}\cos\left(\frac{t}{2}\ln\frac{s}{w}\right)dw\,ds = \left(\frac{\pi}{2}\right)^{1/2} \cdot\left(\frac{1}{1-it+\epsilon}\,{}_2F_1\left(\frac{1}{2},\frac{1-it+\epsilon}{4};\frac{5-it+\epsilon}{4};-1\right)+\frac{1}{1+it+\epsilon}\,{}_2F_1\left(\frac{1}{2},\frac{1+it+\epsilon}{4};\frac{5+it+\epsilon}{4};-1\right)\right) \ ,$$

so

$$\lim_{\epsilon\to0^+}\int_0^\infty\int_s^\infty e^{-\frac{s^2+w^2}{2}}s^{\frac{\epsilon-1}{2}}w^{-\frac{\epsilon+1}{2}}\cos\left(\frac{t}{2}\ln\frac{s}{w}\right)dw\,ds = \left(\frac{\pi}{2}\right)^{1/2} \cdot\left(\frac{1}{1-it}\,{}_2F_1\left(\frac{1}{2},\frac{1-it}{4};\frac{5-it}{4};-1\right)+\frac{1}{1+it}\,{}_2F_1\left(\frac{1}{2},\frac{1+it}{4};\frac{5+it}{4};-1\right)\right) \ ,$$

where

$${}_2F_1(a,b;c;z) = \sum_{n=0}^\infty\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} \ ,$$

is *Gauss's hypergeometric function.* Using the identity

$${}_2F_1(a,b;a+1;z) = \frac{a}{z^a}B(z;a,1-b) \ ,$$

we obtain

$$\lim_{\epsilon\to0^+}\int_0^\infty\int_s^\infty e^{-\frac{s^2+w^2}{2}}s^{\frac{\epsilon-1}{2}}w^{-\frac{\epsilon+1}{2}}\cos\left(\frac{t}{2}\ln\frac{s}{w}\right)dw\,ds = \left(\frac{\pi}{2}\right)^{1/2}\frac{1}{4} \cdot\left((-1)^{-\frac{1-it}{4}}B\left(-1;\frac{1}{4}(1-it),\frac{1}{2}\right)+(-1)^{-\frac{1+it}{4}}\,B\left(-1;\frac{1+it}{4},\frac{1}{2}\right)\right) \ ,$$

which, since

$$(-1)^{-\frac{1-it}{4}} = \frac{\sqrt{2}}{2}(1-i)e^{-\frac{t\pi}{4}} \ , \ (-1)^{-\frac{1+it}{4}} = \frac{\sqrt{2}}{2}(1-i)e^{\frac{t\pi}{4}} \ ,$$

gives

$$\lim_{\epsilon\to0^+}\int_0^\infty\int_s^\infty e^{-\frac{s^2+w^2}{2}}s^{\frac{\epsilon-1}{2}}w^{-\frac{\epsilon+1}{2}}\cos\left(\frac{t}{2}\ln\frac{s}{w}\right)dw\,ds = \frac{\pi^{1/2}(1-i)}{8} \cdot\left(e^{-\frac{t\pi}{4}}\,B\left(-1;\frac{1}{4}(1-it),\frac{1}{2}\right)+e^{\frac{t\pi}{4}}\,B\left(-1;\frac{1+it}{4},\frac{1}{2}\right)\right) \ .$$

Thus

$$\langle \Phi, E_\lambda \Phi \rangle = \frac{1}{\pi^{3/2}} \int_{-\infty}^{\lambda} \frac{\pi^{1/2}(1-i)}{8}$$

$$\cdot \left(e^{-\frac{t\pi}{4}} \; B\left(-1; \frac{1}{4}\left(1-it\right), \frac{1}{2}\right) + e^{\frac{t\pi}{4}} \; B\left(-1; \frac{1+it}{4}, \frac{1}{2}\right)\right) \, dt$$

$$= \frac{1-i}{8\pi} \int_{-\infty}^{\lambda} \left(e^{-\frac{\pi t}{4}} \, B\left(-1; \frac{1-it}{4}, \frac{1}{2}\right) + e^{\frac{\pi t}{4}} \, B\left(-1; \frac{1+it}{4}, \frac{1}{2}\right)\right) \, dt \; .$$

Therefore, for the vacuum characteristic function we have

$$\langle \Phi, e^{it(XP+PX)} \Phi \rangle = \int_{\mathbb{R}} e^{it\lambda} \, d\langle \Phi, E_\lambda \Phi \rangle$$

$$= \frac{1-i}{8\pi} \int_{\mathbb{R}} e^{it\lambda} \left(e^{-\frac{\pi\lambda}{4}} \, B\left(-1; \frac{1-i\lambda}{4}, \frac{1}{2}\right) + e^{\frac{\pi\lambda}{4}} \, B\left(-1; \frac{1+i\lambda}{4}, \frac{1}{2}\right)\right) \, d\lambda \; ,$$

which, using the series representation of the incomplete Beta function, yields

$$\langle \Phi, e^{it(XP+PX)} \Phi \rangle = \frac{2}{\pi^{1/2}} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n (-1)^n (4n+1)}{n!}$$

$$\cdot \left((2\pi)^{-1/2} \int_{\mathbb{R}} \frac{e^{it\lambda}}{(4n+1)^2 + \lambda^2} \, d\lambda \right) \; .$$

From Fourier transform tables, wee see that

$$(2\pi)^{-1/2} \int_{\mathbb{R}} \frac{e^{it\lambda}}{(4n+1)^2 + \lambda^2} \, d\lambda = \frac{1}{4n+1} \left(\frac{\pi}{2}\right)^{1/2} e^{-(4n+1)|t|}.$$

Thus

$$\langle \Phi, e^{it(XP+PX)} \Phi \rangle = \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n (-1)^n e^{-(4n+1)|t|}}{n!}$$

$$= \frac{1}{\sqrt{2}} \frac{e^{-|t|}}{1 + e^{-4|t|}} = \sqrt{\frac{2}{e^{2|t|} + e^{-2|t|}}}$$

$$= \sqrt{\frac{2}{e^{2t} + e^{-2t}}} = \sqrt{\operatorname{sech} 2t} \; .$$

□

Remark 2.5. As pointed out in [4], using

$$X = \frac{a + a^\dagger}{\sqrt{2}} \; , \; P = \frac{a - a^\dagger}{\sqrt{2} i} \; ,$$

where

$$[a, a^\dagger] = \mathbf{1},$$

we find that

$$XP + PX = i\left((a^\dagger)^2 - a^2\right) \; ,$$

so the result of Theorem 2.4 is in agreement with Proposition 3.4 of [1] (see also Proposition 3.9 of [2], Proposition 4.1.1 of [6] and Proposition 4 of [3]), and Proposition 9.5 of [4], where it is shown that $XP + PX$ is a *hyperbolic secant process.*

We remark also that, in the statement of Proposition 9.5 of [4], a factor of 2 is missing.

3. The Quantum Harmonic Oscillator Hamiltonian Operator $\frac{1}{2}(X^2+P^2)$

Theorem 3.1. *The operator* $\frac{1}{2}(X^2+P^2)$ *is symmetric on the Schwartz class* $\mathcal{S}$ *and admits a self-adjoint extension.*

Proof. Let $T=\frac{1}{2}(X^2+P^2)$ and let $f,g\in\mathcal{S}$. As in Theorem 2.1, since $Xf, Pf\in\mathcal{S}$, T is well defined on $\mathrm{dom}(T)=\mathcal{S}$. Since X,P are self-adjoint,

$$\begin{aligned}\langle Tf,g\rangle =&\frac{1}{2}\langle X^2f,g\rangle+\frac{1}{2}\langle P^2f,g\rangle=\frac{1}{2}\langle Xf,Xg\rangle+\frac{1}{2}\langle Pf,Pg\rangle\\ =&\frac{1}{2}\langle f,X^2g\rangle+\frac{1}{2}\langle f,P^2g\rangle=\frac{1}{2}\langle f,\left(X^2+P^2\right)g\rangle=\langle f,Tg\rangle\ ,\end{aligned}$$

so T is symmetric. To show that T admits a self-adjoint extension, we notice that if f is in the domain of T then so also is its complex conjugate $\bar{f}$. Moreover, if $f(x)$ is a real function in the domain of T then $Tf(x)=\frac{1}{2}\left(x^2f(x)-f''(x)\right)$ is also real. Thus, by a theorem due to von Neumann (see [14], Chapter XI, Section 7, Theorem 1), T admits a self-adjoint extension. □

Theorem 3.2. *The resolvent of* $\frac{1}{2}\left(X^2+P^2\right)$ *is*

$$\begin{aligned}R\left(a;\frac{1}{2}\left(X^2+P^2\right)\right)g(s)&=c_1(a)\,M_1(s\sqrt{2};a)+c_2(a)\,M_2(s\sqrt{2};a)\\ +\int_{-\infty}^{s\sqrt{2}}g\left(\frac{w}{\sqrt{2}}\right)&\left(M_1(w;a)M_2(s\sqrt{2};a)-M_1(s\sqrt{2};a)M_2(w;a)\right)\,dw\ .\end{aligned}$$

where $g\in\mathcal{S}$, $c_1(a),c_2(a)\in\mathbb{C}$,

$$M_1(z;a)=e^{-\frac{z^2}{4}}\,{}_1F_1\left(-\frac{a}{2}+\frac{1}{4};\frac{1}{2};\frac{z^2}{2}\right)\ ,\ M_2(z;a)=ze^{-\frac{z^2}{4}}\,{}_1F_1\left(-\frac{a}{2}+\frac{3}{4};\frac{3}{2};\frac{z^2}{2}\right)\ ,$$

and

$${}_1F_1\left(a;c;z\right)=\sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}\frac{z^n}{n!}\ ,$$

is Kummer's confluent hypergeometric function.

Proof. We notice that for $s\in\mathbb{R}$, with all derivatives taken with respect to s,

$$\begin{aligned}R\left(a;\frac{1}{2}\left(X^2+P^2\right)\right)g(s)=G(s;a)&\iff g(s)=\left(a-\frac{1}{2}\left(X^2+P^2\right)\right)G(s;a)\\ &\iff g(s)=\left(a-\frac{1}{2}s^2\right)G(s;a)+\frac{1}{2}G''(s;a)\\ &\iff G''(s;a)-\left(s^2-2a\right)G(s;a)=2g(s)\ ,\end{aligned}$$

i.e. G satisfies a *non-homogeneous Weber differential equation* [8, 13, 15].

Letting $M(s;a) := G\left(\frac{s}{\sqrt{2}};a\right)$, we see that M satisfies a non-homogeneous Weber differential equation in *canonical form* [8, 13],

$$M''(s;a) - \left(\frac{1}{4}s^2 - a\right) M(s;a) = g\left(\frac{s}{\sqrt{2}}\right) \ .$$

The general solution of the associated homogeneous differential equation

$$M''(s;a) - \left(\frac{1}{4}s^2 - a\right) M(s;a) = 0 \ ,$$

is, see [15] and [8],

$$M(s;a) = c_1(a)\, M_1(s;a) + c_2(a)\, M_2(s;a) \ ,$$

where, $c_1(a), c_2(a) \in \mathbb{C}$ and

$$M_1(s;a) = e^{-\frac{s^2}{4}}\, {}_1F_1\left(-\frac{a}{2}+\frac{1}{4};\frac{1}{2};\frac{s^2}{2}\right) \ , \ M_2(s;a) = s e^{-\frac{s^2}{4}}\, {}_1F_1\left(-\frac{a}{2}+\frac{3}{4};\frac{3}{2};\frac{s^2}{2}\right) \ .$$

The Wronskian $W(M_1,M_2)(s)$ of M_1 and M_2 is identically equal to 1 (an easy way to show this is by showing that $\frac{dW}{ds}(M_1,M_2)(s;a) = 0$ and then computing $W(M_1,M_2)(0;a) = 1$). By the well-known *variation of parameters formula*, a solution of the non-homogeneous Weber differential equation is

$$M_p(s;a) = \int_{-\infty}^{s} g\left(\frac{w}{\sqrt{2}}\right)\left(M_1(w;a)M_2(s;a) - M_1(s;a)M_2(w;a)\right) dw \ .$$

Thus, the general solution of the non-homogeneous Weber differential equation, in canonical form, is

$$\begin{aligned} M(s;a) &= c_1(a)\, M_1(s;a) + c_2(a)\, M_2(s;a) \\ &\quad + \int_{-\infty}^{s} g\left(\frac{w}{\sqrt{2}}\right)\left(M_1(w;a)M_2(s;a) - M_1(s;a)M_2(w;a)\right) dw \ . \end{aligned}$$

Thus,

$$\begin{aligned} G(s;a) &= M(s\sqrt{2};a) = c_1(a)\, M_1(s\sqrt{2};a) + c_2(a)\, M_2(s\sqrt{2};a) \\ &\quad + \int_{-\infty}^{s\sqrt{2}} g\left(\frac{w}{\sqrt{2}}\right)\Big(M_1(w;a)M_2(s\sqrt{2};a) - M_1(s\sqrt{2};a)M_2(w;a)\Big)\, dw \ , \end{aligned}$$

where,

$$\begin{aligned} M_1(s\sqrt{2};a) &= e^{-\frac{s^2}{2}}\, {}_1F_1\left(-\frac{a}{2}+\frac{1}{4};\frac{1}{2};s^2\right) \ , \\ M_2(s\sqrt{2};a) &= s\sqrt{2}e^{-\frac{s^2}{2}}\, {}_1F_1\left(-\frac{a}{2}+\frac{3}{4};\frac{3}{2};s^2\right) \ . \end{aligned}$$

□

Theorem 3.3. *The vacuum spectral resolution of* $\frac{1}{2}\left(X^2+P^2\right)$ *is*

$$\langle \Phi, E_\lambda \Phi \rangle = H_{1/2}(\lambda) \ .$$

Moreover, for $t \in \mathbb{R}$*,*

$$\langle \Phi, e^{it\frac{1}{2}\left(X^2+P^2\right)} \Phi \rangle = e^{\frac{it}{2}} \ ,$$

i.e., the probability distribution of $\frac{1}{2}\left(X^2+P^2\right)$ *is degenerate with pdf* $\delta_{\frac{1}{2}}$.

Proof. For $\Phi(s)=\pi^{-1/4}\,e^{-\frac{s^2}{2}}$,

$$\begin{aligned}\langle\Phi,E_\lambda\Phi\rangle &= \frac{1}{\pi}\lim_{\epsilon\to 0^+}\int_{-\infty}^{\lambda}\mathrm{Im}\,\langle\Phi,R\left(t-\epsilon i;\frac{1}{2}\left(X^2+P^2\right)\right)\Phi\rangle\,dt\\ &= \frac{1}{\pi}\lim_{\epsilon\to 0^+}\int_{-\infty}^{\lambda}\mathrm{Im}\,\left(\int_{-\infty}^{\infty}\Phi(s)R\left(t-\epsilon i;\frac{1}{2}\left(X^2+P^2\right)\right)\Phi(s)\,ds\right)\,dt\ .\end{aligned}$$

For $a=t-i\epsilon$, in the notation of Theorem 3.2, we have,

$$G(s;t-i\epsilon)=R\left(t-i\epsilon;\frac{1}{2}\left(X^2+P^2\right)\right)\Phi(s)\ ,$$

where

$$G''(s;t-i\epsilon)-\left(s^2-2(t-i\epsilon)\right)G(s;t-i\epsilon)=2\pi^{-1/4}\,e^{-\frac{s^2}{2}}\ .$$

Letting

$$\begin{aligned}G(s;t-i\epsilon)&=K(s;t-i\epsilon)+iL(s;t-i\epsilon)\ ,\\ K(s;t-i\epsilon)&=\mathrm{Re}\ G(s;t-i\epsilon)\ ,\\ L(s;t-i\epsilon)&=\mathrm{Im}\ G(s;t-i\epsilon)\ ,\end{aligned}$$

we find that the real-valued functions K,L satisfy the system of ODE's:

$$\begin{aligned}K''(s;t-i\epsilon)-\left(s^2-2t\right)K(s;t-i\epsilon)+2\epsilon L&=2\pi^{-1/4}\,e^{-\frac{s^2}{2}}\ ,\\ L''(s;t-i\epsilon)-\left(s^2-2t\right)L(s;t-i\epsilon)-2\epsilon K&=0\ .\end{aligned}$$

For $\epsilon\to 0$ the system uncouples into,

$$\begin{aligned}K''(s;t)-\left(s^2-2t\right)K(s;t)&=2\pi^{-1/4}\,e^{-\frac{s^2}{2}}\ ,\\ L''(s;t)-\left(s^2-2t\right)L(s;t)&=0\ .\end{aligned}$$

The equation satisfied by L is a *homogeneous Weber differential equation* [8, 15]. Thus, in the notation of Theorem 3.2,

$$L(s;t)=c_1(t)\,M_1(s\sqrt{2};t)+c_2(t)\,M_2(s\sqrt{2};t)\ ,$$

where, $c_1(t),c_2(t)\in\mathbb{R}$. Therefore,

$$\langle\Phi,E_\lambda\Phi\rangle=\frac{1}{\pi}\int_{-\infty}^{\lambda}\int_{-\infty}^{\infty}\Phi(s)L(s;t)\,ds\,dt\ .$$

Using,

$$\int_{\mathbb{R}}s^{2n}e^{-s^2}\,ds=\Gamma\left(n+\frac{1}{2}\right)\ ,\ \int_{\mathbb{R}}s^{2n+1}e^{-s^2}\,ds=0\,,\ n\in\{0,1,2,...\}\ ,$$

we obtain

$$\langle\Phi,E_\lambda\Phi\rangle=\frac{1}{\pi^{3/4}}\int_{-\infty}^{\lambda}c_1(t)\sum_{n=0}^{\infty}\frac{\left(\frac{1}{4}-\frac{t}{2}\right)_n}{n!}\,dt\ .$$

To determine $c_1(t)$ we will use the fact that

$$\langle\Phi,E_\infty\Phi\rangle=1\ .$$

The partial sums of the series

$$\sum_{n=0}^{\infty} \frac{(x)_n}{n!}$$

are

$$s_k = \sum_{n=0}^{k} \frac{(x)_n}{n!} = \frac{(1+k)\Gamma(1+x+k)}{\Gamma(x+1)\Gamma(2+k)} \ .$$

Thus,

$$\sum_{n=0}^{\infty} \frac{(x)_n}{n!} = \lim_{k\to\infty} s_k = \begin{cases} 0 \ , & x<0 \\ 1 \ , & x=0 \\ \infty \ , & x>0 \end{cases} \ .$$

Thus, in order for

$$\langle \Phi, E_\infty \Phi \rangle = 1 \ ,$$

we must interpret $c_1(t)$ in the distribution sense as,

$$c_1(t) = \pi^{3/4} \delta_{1/2}(t) \ ,$$

in which case,

$$\langle \Phi, E_\lambda \Phi \rangle = \int_{-\infty}^{\lambda} \delta_{1/2}(t) \sum_{n=0}^{\infty} \frac{\left(\frac{1}{4} - \frac{t}{2}\right)_n}{n!} \, dt = \left\{ \begin{array}{ll} 1 \ , & \lambda \geq \frac{1}{2} \\ 0 \ , & \lambda < \frac{1}{2} \end{array} \right\} = H_{1/2}(\lambda) \ .$$

Thus,

$$\begin{aligned} \langle \Phi, e^{it\frac{1}{2}\left(X^2+P^2\right)} \Phi \rangle &= \int_{\mathbb{R}} e^{it\lambda} \, d\langle \Phi, E_\lambda \Phi \rangle \\ &= \int_{\mathbb{R}} e^{it\lambda} \, dH_{1/2}(\lambda) = \int_{\mathbb{R}} e^{it\lambda} \, \delta_{1/2}(\lambda) \, d\lambda = e^{\frac{it}{2}} \ , \end{aligned}$$

meaning that the probability distribution of $\frac{1}{2}\left(X^2+P^2\right)$ is degenerate. □

Remark 3.4. In terms of the creation and annihilation operators $a^\dagger$ and a, respectively, where

$$[a, a^\dagger] = \mathbf{1} \ ,$$

using

$$X = \frac{a + a^\dagger}{\sqrt{2}} \ , \ P = \frac{a - a^\dagger}{\sqrt{2} i} \ ,$$

we find that

$$\frac{1}{2}\left(X^2 + P^2\right) = \frac{1}{2} + a^\dagger a \ .$$

The formula for the characteristic function given in Theorem 3.3 is precisely a special case of Remark 3.7 of Proposition 3.4 of [1], where $\frac{1}{2} + a^\dagger a$ is denoted by S_1^1.

4. Stone's Formula in the Finite Dimensional Case

To illustrate the use of Stone's formula in the finite dimensional case, we consider the *Heisenberg observable*

$$H = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} ,$$

of Section 6 of [4]. The eigenvalues of H are: $\lambda_1 = 2$, with multiplicity one, and $\lambda_2 = -1$ with multiplicity two. It was shown in [4] that, if $\Phi = (a, b, c)$ is a unit vector in $\mathbb{R}^3$ then, for $t \in \mathbb{R}$:

$$\langle \Phi, e^{itH}\Phi \rangle = \left(1 - \frac{(a+b+c)^2}{3}\right) e^{-it} + \frac{(a+b+c)^2}{3} e^{2it},$$

i.e., H follows a Bernoulli distribution with probability density function

$$p_{a,b,c}(\lambda) = \left(1 - \frac{(a+b+c)^2}{3}\right) \delta_{-1}(\lambda) + \frac{(a+b+c)^2}{3} \delta_2(\lambda).$$

We will show how the same result can be obtained with the use of Stone's formula.

For $a \in \mathbb{C} \setminus \{-1, 2\}$ we have

$$R\,(a; H) = \frac{1}{(a+1)(a-2)} \begin{pmatrix} a-1 & 1 & 1 \\ 1 & a-1 & 1 \\ 1 & 1 & a-1 \end{pmatrix} ,$$

and

$$R(t - i\epsilon; H) - R(t + i\epsilon; H) = \frac{2i\epsilon}{((t-2)^2 + \epsilon^2)((t+1)^2 + \epsilon^2)}$$
$$\cdot \begin{pmatrix} 3 + t(t-2) + \epsilon^2 & 2t-1 & 2t-1 \\ 2t-1 & 3 + t(t-2) + \epsilon^2 & 2t-1 \\ 2t-1 & 2t-1 & 3 + t(t-2) + \epsilon^2 \end{pmatrix} .$$

For a unit vector $\Phi = (a, b, c)$ in $\mathbb{R}^3$, we find,

$$\langle \Phi, (R(t - i\epsilon; H) - R(t + i\epsilon; H))\,\Phi \rangle = \frac{2i\epsilon}{((t-2)^2 + \epsilon^2)((t+1)^2 + \epsilon^2)}$$
$$\cdot \left((t-2)^2 + \epsilon^2 + (2t-1)(a+b+c)^2\right) ,$$

and

$$\frac{1}{2\pi i} \int \langle \Phi, (R(t - i\epsilon; H) - R(t + i\epsilon; H))\Phi \rangle \, dt$$
$$= \frac{1}{3\pi} \left(x^2 \arctan\left(\frac{t-2}{\epsilon}\right) - (x^2 - 3) \arctan\left(\frac{t+1}{\epsilon}\right)\right) ,$$

where,

$$x = a + b + c .$$

Since,

$$\lim_{y \to -\infty} \frac{1}{3\pi} \left(x^2 \arctan\left(\frac{y-2}{\epsilon}\right) - (x^2 - 3) \arctan\left(\frac{y+1}{\epsilon}\right)\right) = -\frac{1}{2} ,$$

we find that

$$\frac{1}{2\pi i}\int_{-\infty}^{\lambda}\langle\Phi,(R(t-i\epsilon;H)-R(t+i\epsilon;H))\Phi\rangle\,dt$$
$$=\frac{1}{3\pi}\left(x^2\arctan\left(\frac{\lambda-2}{\epsilon}\right)-(x^2-3)\arctan\left(\frac{\lambda+1}{\epsilon}\right)\right)+\frac{1}{2}\ .$$

Since,

$$\lim_{\epsilon\to0^+}\frac{1}{3\pi}\left(x^2\arctan\left(\frac{\lambda-2}{\epsilon}\right)-(x^2-3)\arctan\left(\frac{\lambda+1}{\epsilon}\right)\right)$$
$$=\begin{cases}\frac{1}{2}\ , & \lambda>2\\ -\frac{x^2}{3}+\frac{1}{2}\ , & -1<\lambda<2\\ -\frac{1}{2}\ , & \lambda<-1\end{cases}\ ,$$

we have,

$$\langle\Phi,E_\lambda\Phi\rangle=\begin{cases}1\ , & \lambda>2\\ -\frac{x^2}{3}+1\ , & -1<\lambda<2\\ 0\ , & \lambda<-1\end{cases}\ .$$

Using the right-continuity of the spectral resolution, we extend to $\lambda=2$ and $\lambda=-1$, to obtain the vacuum resolution of the identity,

$$\langle\Phi,E_\lambda\Phi\rangle=\left\{\begin{array}{ll}1\ , & \lambda\geq2\\ -\frac{x^2}{3}+1\ , & -1\leq\lambda<2\\ 0\ , & \lambda<-1\end{array}\right\}=\left(-\frac{x^2}{3}+1\right)H_{-1}(\lambda)+\frac{x^2}{3}H_2(\lambda)\ .$$

Therefore,

$$d\,\langle\Phi,E_\lambda\Phi\rangle=\left(\left(-\frac{x^2}{3}+1\right)\delta_{-1}(\lambda)+\frac{x^2}{3}\delta_2(\lambda)\right)\,d\lambda\ ,$$

and

$$\langle\Phi,e^{itH}\Phi\rangle=\left(-\frac{x^2}{3}+1\right)\int_{\mathbb{R}}e^{it\lambda}\delta_{-1}(\lambda)\,d\lambda+\frac{x^2}{3}\int_{\mathbb{R}}e^{it\lambda}\delta_2(\lambda)\,d\lambda$$
$$=\left(1-\frac{x^2}{3}\right)e^{-it}+\frac{x^2}{3}e^{2it}\ ,$$

in agreement with the result obtained in Section 6 of [4].

5. Errata

The following corrections should be made to [4]:

Formulas (8.5) and (8.7) are correct for positive a, b. For general a, b, with $ab \neq 0$, they should be replaced, respectively, by

$$q(s) = \frac{a}{2\sqrt{ab}} \tanh(2\sqrt{ab}\, s), \tag{5.1}$$

$$r(s) = \frac{b}{2\sqrt{ab}} \tanh(2\sqrt{ab}\, s)\ . \tag{5.2}$$

The formula in Theorem 8.5 is

$$\langle \Phi, e^{it(aX^2+bP^2)}\Phi\rangle = \frac{\sqrt{2}\, e^{\frac{1}{2}p(it)}}{\sqrt{e^{2p(it)} + (2\, q(it) - 1)(2\, r(it) - 1)}}.$$

and, as a result, the formula in Corollary 8.6 is

$$\langle \Phi, e^{itH}\Phi\rangle = e^{\frac{it}{2}} \left(\frac{\operatorname{sech}\left(\frac{t\sqrt{3}}{2}\right)}{1 - \frac{i}{\sqrt{3}} \tanh\left(\frac{t\sqrt{3}}{2}\right)} \right)^{1/2} .$$

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