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# New Limit Theorems for Increments of Birth-and-Death Processes with Linear Rates

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# NEW LIMIT THEOREMS FOR INCREMENTS OF BIRTH-AND-DEATH PROCESSES WITH LINEAR RATES

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ABSTRACT. We investigate the short-term temporal evolution of the classical birth-and-death process with linear birth and death rates. We prove weak convergence of its increments to those of the Skellam process in the Skorohod space. Similar convergence results are established under a different set of assumptions on the model parameters and the time horizon. We also discuss related short-term approximations in terms of Skellam distributions for the marginals of such birth-and-death processes which start from a growing number of particles. For two special cases of the pure birth and pure death processes, our results yield those on weak convergence to the corresponding Poisson processes.

## 1. Introduction

Analysis of the birth-and-death processes is a classical and well-studied problem which has numerous applications in biology, physics, finance and insurance (see [10], [26], [33]).

One of the most important areas of application is the Mathematical Theory of Epidemics [2], [7], [8], [19], [20], [21]. Traditionally, this theory considers compartmental models which specify the limiting behavior of the pandemic variables. These models are described by the systems of deterministic, non-linear differential equations and do not encapsulate stochastic dynamics. However, in some cases it is desirable to incorporate a stochastic component describing the dynamics of random fluctuations of the pandemic processes.

In [6], an unconventional model, linking pandemic variables to the financial indicators was developed and calibrated to the year 2020 observations of Covid-19 pandemic numbers and some financial indices. This model can be used for a financial scenario generation, in some risk management systems, and for the portfolio risk valuation.

While the initial phase of a pandemic can often be represented by a pure birth process with a high degree of accuracy (compare to [6]), a proper description of subsequent phases of the pandemic necessitates employing more general classes of stochastic processes. The initial phase of growth of a typical pandemic is usually

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of an exponential type. This is consistent with an observation that during a relatively short time period, the number of infected individuals could exceed $10^4$ and continue to grow at a significant rate [23]. (The rate of such exponential growth is often referred to as the *Malthusian parameter*, see also (2.9).)

A similar situation is observed in the case of pure death processes. Specifically, such process starts from a very large initial state with its random fluctuations over a short-term period being important for a short–term forecasting. The analysis of such (Markov) stochastic process per se by using Monte-Carlo methods or by utilizing closed-form representations for the corresponding transition probabilities is difficult, which suggests the need to derive *asymptotic* approximations. In turn, this justifies our interest in studying the limiting behavior of these and similar Markov branching processes under the assumption that the initial state $n \to \infty$.

The stochastic analysis of the temporal evolution of the pure birth and the pure death processes as the initial number of particles $n \to \infty$ leads to simple results which stipulate that the limiting dynamics of the normalized processes over a short period of time is Poisson in both these cases. Weak convergence of the marginal distribution of the increments of the pure death process to a Poisson process and the computation of corresponding parameters of the limiting processes in closed form are considered in [6].

In the present paper, we analyze a similar problem for more general birth-and-death processes with linear rates, denoted by $\mathcal{N}(t) := \mathcal{N}(t; \lambda, \mu)$. The limit theorems considered in this paper deal with the sequences of birth-and-death processes, $\{\mathcal{N}_n(t)\}_{n\geq 1}$, parameterized by the initial state of the process, $\mathcal{N}_n(0) = n$, as $n \to \infty$.

One of our main results is Theorem 3.10 that pertains to studying the temporal behavior of the increments of this process, which are hereinafter denoted by

$$\Delta\mathcal{N}_n(t) := \mathcal{N}_n(t; \lambda_n, \mu_n) - n, \quad \mathcal{N}_n(0, \lambda_n, \mu_n) = n,$$

in the case where the parameters $\lambda_n \sim \lambda/n$ and $\mu_n \sim \mu/n$ as $n \to \infty$ (see also conditions (1.1)–(1.2)). It turns out that under such assumptions, the limit stochastic dynamics of the increments $\Delta\mathcal{N}_n(t)$ is described in terms of the Skellam process, $\mathcal{S}_{\lambda,\mu}(t)$, which is represented as the difference of two independent Poisson processes (see Definition 3.4 for more detail).

An alternative approach towards the derivation of the limiting behavior of the incremental process $\Delta\mathcal{N}_n(t)$ for *small time horizons* $t \sim \tau/n$ as $n \to \infty$ but *fixed* parameters $\lambda_n \equiv \lambda_*$, and $\mu_n \equiv \mu_*$, is also described in terms of the Skellam process $\mathcal{S}_{\lambda_*,\mu_*}(\tau)$. In addition, our "*warm-up*" Proposition 2.2 (which pertains to the marginals and is aimed to present our simplest result as early as possible) stipulates that for such small time horizons, the asymptotic behavior of $\Delta\mathcal{N}(t)$ is given in terms of particular Skellam-distributed r.v.'s. (Note in passing that members of the latter class can be represented as the difference of two independent Poisson r.v.'s, see [12] for more detail.) An intuitive explanation of Proposition 2.2 is provided in Remark 2.3.

Somewhat similar results were obtained in [28] for a fluid-type normalization of the model with many types of interacting organisms. The martingale convergence techniques are used for analysis of the limit dynamics in such complex systems (see, e.g., [9] or [14]).

The limiting dynamics of the branching processes were analyzed in [29] where convergence to a Poisson process and the Brownian motion was proved in the Skorohod space.

The structure of this paper is as follows. In Section 2, we describe the general model of the birth-and-death processes with linear rates. In Section 3, we derive a recursive relation for the generating functions, prove weak convergence of the finite-dimensional distributions and also prove relative compactness of the probability measures describing the distribution of the processes. These results lead to weak convergence of the incremental processes $\Delta\mathcal{N}_n(t)$ in the Skorohod space $\mathbf{D([0,\mathit{T}])}$ (compare to [4], [5], [9], [13] as well as [14] and [31]).

The proofs of the results in this section are given in the case where the birth and death rates are different. In the critical case, the derivation of the corresponding results is almost identical to those for the non-critical case. Hence, we will follow a customary approach to omit such proofs for the sake of brevity.

The asymptotic conditions employed in the proof of limit theorems of this paper are as follows. The time horizon is assumed to be fixed, whereas the parameters $\lambda_n$ and $\mu_n$ depend on $n$ such that

$$\lambda := \lim_{n\to\infty} n\lambda_n < \infty; \tag{1.1}$$

$$\mu := \lim_{n\to\infty} n\mu_n < \infty. \tag{1.2}$$

In contrast, the above-mentioned alternative approach relies on the assumption that $\lambda_n$ and $\mu_n$ do not depend on $n$, but the time horizon is short, namely, $t \sim \tau/n$. The results derived for this case are similar in spirit to those pertaining to the case of a fixed time horizon $T$; their derivation is often omitted.

In Section 4, we consider two important special cases, namely, the pure birth and the pure death processes, in more detail. We simplify the recursive relationship established in Section 3 for the general model and compute the correlation matrix of the incremental process $\Delta\mathcal{N}_n(t)$ per se (i.e., before passing to the limit).

To conclude the Introduction, we summarize some relevant notation and terminology. We will follow the custom of formulating various statements of distribution theory in terms of the properties of random variables (or *r.v.'s*), even when such results pertain only to their *distributions*. Hereinafter, $\mathbb{C}^1$, $\mathbb{R}^1_+$, $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{Z}_-$ and $\mathbb{N}$ stand for the sets of all complex, all positive reals, all integers, all non-negative, all non-positive, and all positive integers, respectively. In what follows, the signs "$\stackrel{\mathrm{d}}{=}$" and "$\stackrel{\mathcal{D}}{=}$" will denote the facts that the distributions of (univariate) r.v.'s and stochastic processes coincide, respectively. We denote the càdlàg or Skorohod space of right-continuous functions on a finite time interval $[0,T]$ with finite left limits which is equipped with the Skorohod $J_1$-topology by $\mathbf{D([0,\mathit{T}])}$. The symbols "$\xrightarrow{\mathrm{d}}$" and "$\xRightarrow{\mathbf{D}}$" are understood in the sense of weak convergence of univariate distributions and in the Skorohod space, respectively. The symbol "$\xRightarrow{w}$" denotes convergence of finite-dimensional distributions.

## 2. The General Model: Birth-and-death Processes With Linear Rates

Our consideration of a generic time-homogeneous birth-and-death Markov process with *linear* birth and death rates follows along the same lines as those in [1], [10], [17], [18], and [22].

Denote by $P_k(t)$ the probability that at time $t$, the process is at state $k$ by

$$P_k(t) = \mathbb{P}\{\mathcal{N}(t) = k\}, \quad k \in \mathbb{Z}_+,\ t > 0.$$

Hereinafter, we denote the (non-negative) birth and death parameters by $\lambda$ and $\mu$, respectively, with $\lambda + \mu > 0$. We assume that the asymptotic behavior of the transition probabilities as the time increment $\Delta t \to 0$ is as follows:

$$\mathbb{P}\left(\mathcal{N}(t+\Delta t) = \mathfrak{i} \,|\, \mathcal{N}(t) = \mathfrak{j}\right) = \begin{cases} \lambda \mathfrak{j} \Delta t + o(\Delta t), & \text{if } \mathfrak{i} = \mathfrak{j}+1, \\ \mu \mathfrak{j} \Delta t + o(\Delta t), & \text{if } \mathfrak{i} = \mathfrak{j}-1, \\ 1 - \mathfrak{j}(\lambda+\mu)\Delta t + o(\Delta t), & \text{if } \mathfrak{i} = \mathfrak{j}, \\ o(\Delta t), & \text{if } |\,\mathfrak{i} - \mathfrak{j}\,| > 1. \end{cases} \tag{2.1}$$

Unless otherwise stated, we impose the condition that $\mathcal{N}(t)$ starts from $n$ particles, i.e., $P_n(0) = 1$. In this case we equip the notation for the process with subindex $n$, writing $\mathcal{N}_n(t)$. A subsequent application of (2.1) yields the following system of the ordinary differential equations:

$$\begin{cases} \frac{dP_k(t)}{dt} = & -(\lambda+\mu)kP_k(t) + \lambda \cdot (k-1)P_{k-1}(t) \\ & \quad +\mu \cdot (k+1)P_{k+1}(t), \quad k = 1, 2, \ldots, \\ \frac{dP_0(t)}{dt} = & \mu P_1(t). \end{cases} \tag{2.2}$$

For a fixed real time instant $t \geq 0$ and the values of the argument $z$ such that $|\,z\,| \leq 1$, consider the probability-generating function (or p.g.f.) of the non-negative integer-valued r.v. $\mathcal{N}(t)$:

$$\mathcal{P}(t,z) := \sum_{k=0}^{\infty} P_k(t) \cdot z^k. \tag{2.3}$$

It can be shown that the system of equations (2.2) implies that for $|\,z\,| \leq 1$,

$$\begin{cases} \frac{\partial \mathcal{P}(t,z)}{\partial t} = \frac{\partial \mathcal{P}(t,z)}{\partial z}(z-1)(\lambda z - \mu), \quad t > 0, \\ \mathcal{P}(0,z) = z^n. \end{cases} \tag{2.4}$$

It is well known (see [1], [17], [18], [22]) that the system of equations (2.4) has the following unique solution:

$$\mathcal{P}(t,z) = \begin{cases} \left( \dfrac{\mu(z-1)e^{(\lambda-\mu)t} - \lambda z + \mu}{\lambda(z-1)e^{(\lambda-\mu)t} - \lambda z + \mu} \right)^n, & \text{if } \lambda \neq \mu, \\ \\ \left( \dfrac{1 + (1-\lambda t)(z-1)}{1 - \lambda t(z-1)} \right)^n, & \text{if } \lambda = \mu. \end{cases} \tag{2.5}$$

Also, the above-described process $\{\mathcal{N}(t),\ t \geq 0\}$ with *linear* rates (which emerge in (2.1)) admits an interpretation as a Markov branching process which undergoes a *binary* branching [1], [17], [18].

It was noted in [18] that in the case where the process $\mathcal{N}(t)$ starts from a single particle then for each real $t > 0$, the p.g.f. $\mathcal{P}(t, z)$ defined by (2.3) (and given by (2.5)) is that of a particular *zero-modified geometric* distribution considered among others in [34]–[35] (see also the references therein).

Subsequently, in the general case where the number $n \geq 1$ of the initial particles is arbitrary but fixed, the p.g.f. (2.5) coincides with that of a certain *Feller-Arley* distribution, obtained by the $n$-fold convolution of a particular zero-modified geometric probability law (see [10], [16]).

The Feller-Arley r.v. can be decomposed into the sum of two independent r.v.'s one of which is negative binomial with the topological support $\mathbb{Z}_+$, and the other one is binomial with $n$ trials.

Next, since the primary object of our study is the incremental process

$$\Delta\mathcal{N}(t) := \mathcal{N}(t) - \mathcal{N}(0) \tag{2.6}$$

that takes on both positive and negative integer values (where $\mathcal{N}(0) = n$ is deterministic and non-negative integer), it might appear to be more convenient to work with the *characteristic* function of this process. The characteristic function of the process $\Delta\mathcal{N}(t)$ can be obtained from its p.g.f. by the following substitution: $z = e^{is}$ with $s \in \mathbb{R}^1$. However, we found it to be more convenient to work with the p.g.f. of the marginals of the stochastic process $\Delta\mathcal{N}(t)$, which is hereinafter denoted by

$$\widehat{\mathcal{P}}(t, z) := \mathbb{E}[z^{\Delta\mathcal{N}(t)}] = \sum_{k=-n}^{+\infty} \mathbb{P}\{\Delta\mathcal{N}(t) = k\} \cdot z^k. \tag{2.7}$$

We stress that since the marginals of the incremental process $\Delta\mathcal{N}(t)$ take on both positive and negative integer values, the expression which emerges on the right-hand side of (2.7) involves not only non-negative, but also *negative* powers of $z$. Note in passing that such use of the p.g.f. for a generic probability distribution on $\mathbb{Z}$ is closely related to the concept of (bilateral) $Z$–transform. The reader is referred to [24, Ch. 4] for more detail on this important connection and on properties of $Z$-transform, respectively. In this respect, we note a difference in the sets of the summation indices employed in (2.3) and (2.7).

Evidently, $\widehat{\mathcal{P}}(t, z)$ must be an analytical function on the unit circle $|\, z \,| = 1$ with $z \in \mathbb{C}^1$ such that

$$\widehat{\mathcal{P}}(t, z) = \begin{cases} \left( \dfrac{\mu(z-1)e^{(\lambda-\mu)t} - \lambda z + \mu}{z \cdot \left(\lambda(z-1)e^{(\lambda-\mu)t} - \lambda z + \mu\right)} \right)^n, & \text{if } \lambda \neq \mu, \\[2ex] \left( \dfrac{1 + (1-\lambda t)(z-1)}{z\,(1 - \lambda t(z-1))} \right)^n, & \text{if } \lambda = \mu, \end{cases} \tag{2.8}$$

*Remark* 2.1. Equation (2.8) demonstrates that the birth-and-death process with linear rates possesses an important scale-invariance property. Namely, its probability distribution is invariant under the scaling transformation of the time variable and the parameters $\lambda$ and $\mu$. More precisely, consider a modified process with new parameters $\lambda' = \theta\lambda$ and $\mu' = \theta\mu$, where $\theta > 0$, at time $t' = t/\theta$. Then the probability distribution of the original process, $\{P_k(t)\}_{k=0}^{\infty}$, is identical to that of the scaled process, denoted by $\{P_k^\theta(t)\}_{k=0}^\infty$:

$$P_k^\theta(t') = P_k(t) \quad \text{for all } t > 0,$$

if $P_k^\theta(0) = P_k(0) \quad$ for all $k \in \mathbb{Z}$.

Relation (2.8) plays an important role in our analysis. For instance, it will allow us to find limiting distributions of the increments of the birth-and-death process.

It is known (and can in fact be also derived from representations (2.8)) that the mean of the r.v. $\mathcal{N}(t)$ equals

$$\mathbb{E}[\mathcal{N}(t)] = ne^{(\lambda-\mu)t}, \tag{2.9}$$

whereas the variance of $\mathcal{N}(t)$ is as follows:

$$\mathbf{Var}\,(\mathcal{N}(t)) = \begin{cases} n(\lambda+\mu)e^{(\lambda-\mu)t} \cdot \dfrac{e^{(\lambda-\mu)t}-1}{\lambda-\mu}, & \text{if } \lambda \neq \mu, \\ 2n\lambda t, & \text{if } \lambda = \mu. \end{cases}$$

Now, fix $\tau \in \mathbb{R}_+^1$ and consider short-term changes in the size of population, which are given by the increments of the birth-and-death process $\mathcal{N}(t)$ which starts from a large number $n$ of initial particles. Specifically, assume that as $n \to \infty$,

$$t_n \sim \tau/n. \tag{2.10}$$

The following assertion motivated our main result, Theorem 3.8 which pertains to establishing weak convergence in the Skorohod space.

**Proposition 2.2.** *Suppose that the birth-and-death process $\mathcal{N}(t)$ with linear rates is such that* (2.1) *is met, and $\mathcal{N}(0) = n$, $n \to +\infty$, and the sequence $t_n$ satisfies* (2.10). *Then*

$$\mathcal{N}(t_n) - n \xrightarrow{\text{d}} \xi_{\lambda\tau} - \eta_{\mu\tau}, \tag{2.11}$$

*where $\xi_a$ and $\eta_b$ are independent r.v.'s having their own Poisson distributions with parameters $a$ and $b$, respectively.*

It is relevant that the difference $\xi_{\lambda\tau} - \eta_{\mu\tau}$ constitutes the corresponding *Skellam-distributed* r.v. (see (3.9) below and [30]). If instead of r.v.'s $\xi_{\lambda\tau}$ and $\eta_{\mu\tau}$ we consider the corresponding independent Poisson processes with intensities $\lambda$ and $\mu$, respectively, we obtain a stochastic process (3.8) which is the limit of the incremental processes, $\mathcal{N}_n(t) - n$, as $n \to \infty$, in the sense of weak convergence in the Skorohod space. This convergence result, proved in Theorem 3.10, significantly generalizes Proposition 2.2.

Hence, the proof of the above Proposition 2.2 is unnecessary and skipped. Instead, we will now provide an intuitive explanation of the validity of (2.11) by virtue of the distribution theory and some well-known limit theorems, which is isolated into the following

*Remark* 2.3. The r.v. $\mathcal{N}(t_n) - n$ which emerges on the left-hand side of (2.11) can be decomposed into the *difference* of two independent negative binomial and binomial r.v.'s. Subsequently, it can be shown that the former r.v. converges to a Poisson r.v. by [15, Sec. 3.3.5], whereas the latter one is approximated by its own Poisson r.v. in view of the classical Poisson law of small numbers. In fact, these intuitive arguments can be made rigorous to derive a probabilistic proof of the weak convergence result (2.11) of Proposition 2.2.

To conclude this section, note that for the theory of branching processes it is customary to isolate the following three cases:

1. Supercritical dynamics: $0 < \mu < \lambda$.
2. Subcritical dynamics: $\mu > \lambda > 0$.
3. Critical dynamics: $\mu = \lambda$, $\lambda > 0$.

Indeed, for some results of this paper including the proof of Theorem 3.8, certain technical details depend on relationships between $\lambda$ and $\mu$. However, the limiting process remains invariant with respect to such relations, as we will see in Section 3.

## 3. Weak Convergence of Birth-and-death Processes

In this section, we explore analytical properties of the function $\widehat{\mathcal{P}}(t,z)$ and prove a limit theorem for the increments of the birth and death process, $\Delta\mathcal{N}_n(t)$, in the Skorohod space [9], [31], as $n = \mathcal{N}_n(0)$ and $n \to \infty$. We start with weak convergence of the marginal distributions motivating appearance of the Skellam process in the limit Theorem 3.8.

**3.1. Convergence of marginal distributions.** Recall that the p.g.f. of the process $\mathcal{N}(t) - n$ satisfies Equation (2.8). The latter equation has the following probabilistic interpretation: the r.v. $\mathcal{N}(t) - n$ can be represented as the following sum of $n$ independent and identically distributed r.v.'s (each of them being interpreted as the change in size of the cluster of descendents of the corresponding initial particle which are alive at time $t$):

$$\mathcal{N}(t) - n = \sum_{j=1}^{n} \xi_j(t)$$

such that for $z \neq 0$, the p.g.f. $\varphi(t,z) := \mathbb{E}\big[z^{\xi(t)}\big]$ admits the following representation:

$$\varphi(t,z) = \begin{cases} \dfrac{\mu(z-1)e^{(\lambda-\mu)t} - \lambda z + \mu}{z\cdot\big(\lambda(z-1)e^{(\lambda-\mu)t} - \lambda z + \mu\big)}, & \text{if } \lambda \neq \mu, \\[2ex] \dfrac{1 + (1-\lambda t)(z-1)}{z\,(1 - \lambda t(z-1))}, & \text{if } \lambda = \mu, \end{cases} \tag{3.1}$$

In this case, for all $t > 0$ and integer $n \geq 1$,

$$\widehat{\mathcal{P}}_n(t,z) = \varphi^n(t,z), \quad z \in \mathbb{C}^1. \tag{3.2}$$

We will need some properties of the function $\varphi(t,z)$ used in what follows.

**Lemma 3.1.** *The function $\varphi(t,z)$ has the following two properties:*

1. *The function $\varphi(t,z)$ is analytic in the punctured unit disk*

$$\mathcal{D}_z = \{z \in \mathbb{C}^1 : 0 < |\,z\,| \leq 1\}.$$

2. *The equation $\varphi(t,z)=0$ has a single real root $z^*(t)=-1$ on the unit circle, for $t=t_*$ where*

$$t_* = \begin{cases} \dfrac{1}{\lambda-\mu}\log\left(\dfrac{\lambda+\mu}{2\mu}\right) & \text{if } \lambda\neq\mu, \\ \dfrac{1}{2\lambda} & \text{if } \lambda=\mu. \end{cases} \tag{3.3}$$

*Proof.* From (3.1) we have $\varphi(t,1)\equiv 1$ for all $t\geq 0$ and $\varphi(0,z)\equiv 1$. Consider the case $0<\mu<\lambda$. The function $\varphi(t,z)$ has two simple poles, $z_1=0$ and $z_2=\frac{\lambda e^{(\lambda-\mu)t}-\mu}{\lambda\left(e^{(\lambda-\mu)t}-1\right)}$, but $|\,z_2\,|>1$ and, therefore, neither $z_1$ nor $z_2$ belong to $\mathcal{D}_z$.

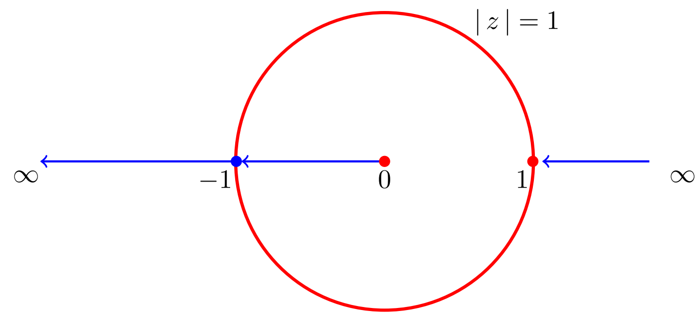


FIGURE 1. Root $z^*(t)$ of the equation $\varphi(t,z)=0$ in the complex domain $z\in\mathbb{C}^1$: supercritical dynamics, $\lambda>\mu$

The first statement is proved. Let us now prove the second statement. The root $z^*$, of the equation

$$\varphi(t,z)=0 \tag{3.4}$$

is a real-valued function of time $t$:

$$z^*(t)=\frac{\mu\left(e^{(\lambda-\mu)t}-1\right)}{\mu e^{(\lambda-\mu)t}-\lambda},\quad t>0. \tag{3.5}$$

The limit $\hat{z}=\lim\limits_{t\to 0} z^*(t)$ is not a root of the equation (3.4), since the function $\varphi(0,z)\equiv 1$. As $t$ increases from 0 to $t_*=\frac{1}{\lambda-\mu}\log\left(\frac{\lambda+\mu}{2\mu}\right)$, the root $z^*$ runs from 0 to $-1$ (see Figure 1). As $t$ further increases from $t_*$ to $t^*=\frac{1}{\lambda-\mu}\log\left(\frac{\lambda}{\mu}\right)$, the root, $z^*$, runs from $-1$ to $\infty$, remaining negative. Finally, as $t$ increases from $t^*$ to $\infty$, the root, $z^*$ moves from the point $\infty$ to 1 remaining a real positive number. Therefore, only for $t=t_*$ the root $z^*(t)$ belongs to the unit circle, $\{z:|\,z\,|=1\}$. In this case $z^*(t_*)=-1$.

In the case $\mu>\lambda>0$, the proof is analogous. The poles of the function $\varphi(t,z)$ are simple, $z_1=0$ and $z_2=\frac{\mu-\lambda e^{-(\mu-\lambda)t}}{\lambda\left(1-e^{-(\mu-\lambda)t}\right)}$, but $|\,z_2\,|>1$ and, again, neither $z_1$ nor $z_2$ belong to $\mathcal{D}_z$.

The root $z^*(t)$ of Equation (3.4) satisfies (3.5) and follows along the path which is topologically equivalent to that in the supercritical case, as time $t$ changes from 0 to $+\infty$. The only difference is that in the limit as $t\to\infty$, we have that $z^*(t)\to\frac{\mu}{\lambda}$

(see Figure 2). The critical times, $t_*$ and $t^*$, have the same values and are positive real numbers, as in the supercritical case. In the critical case, $\mu = \lambda$, the pole

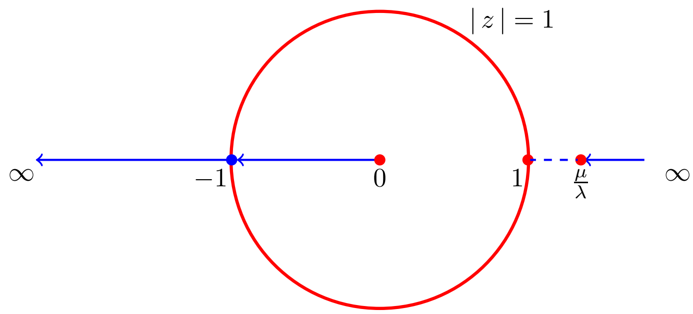


FIGURE 2. Root $z^*(t)$ of the equation $\varphi(t,z) = 0$ in the complex domain $z \in \mathbb{C}^1$: subcritical dynamics, $\mu > \lambda$.

$z_1 = 0$ and the second pole is real satisfying the inequality $z_2 = 1 + \frac{1}{\lambda t} > 1$ for $t > 0$. Thus, the first statement of the lemma is proved. The root of Equation (3.4) in this case, $z^*(t) = \frac{\lambda t}{\lambda t - 1}$, follows along the same path as in the supercritical case. The critical times are given by $t_* = \frac{1}{2\lambda}$ and $t^* = \frac{1}{\lambda}$, and $z^*(t_*) = -1$. In the limit as $t \to \infty$, we get that

$$\lim_{t\to\infty} z^*(t) = 1.$$

Thus, the lemma is proved. □

*Remark* 3.2. Since the function $\varphi(t,z)$ has a pole at $z = 0$, the superposition $\varphi(t_1, \varphi(t_2, z))$ will have poles at $z = z^*(t_2)$. Thus, the superposition operation contains an implicit mechanism of adding new singular points in the series of iterations. This feature must be taken into account in the analysis of the p.g.f of interest, as it is applied in Lemma 3.9.

Next, we obtain the limiting marginal distribution in the case where the parameters satisfy (1.1) and (1.2) and $\mu_n \neq \lambda_n$. It follows from Equation (2.8) that

$$\begin{aligned}
\widehat{\mathcal{P}}_n(t,z) &= \left(\frac{\mu_n(z-1)e^{(\lambda_n-\mu_n)t} - \lambda_n z + \mu_n}{z\left(\lambda_n(z-1)e^{(\lambda_n-\mu_n)t} - \lambda_n z + \mu_n\right)}\right)^n \\
&= \left(\frac{\mu(z-1)\left(1 + (\lambda-\mu)\frac{t}{n} + o(n^{-1})\right) - \lambda z + \mu}{z\left(\lambda(z-1)\left(1 + (\lambda-\mu)\frac{t}{n} + o(n^{-1})\right) - \lambda z + \mu\right)}\right)^n \\
&= \left(\frac{(\mu-\lambda)z - \mu(\mu-\lambda)(z-1)\frac{t}{n} + o(n^{-1})}{z\left(\mu - \lambda - (\mu-\lambda)\lambda(z-1)\frac{t}{n} + o(n^{-1})\right)}\right)^n. \\
&= \left(\frac{z\left(1 - \mu(1-z^{-1})\frac{t}{n} + o(n^{-1})\right)}{z\left(1 - \lambda(z-1)\frac{t}{n} + o(n^{-1})\right)}\right)^n.
\end{aligned}$$

Passing to the limit in the latter equation as $n \to \infty$ we ascertain that

$$\lim_{n\to\infty} \widehat{\mathcal{P}}_n(t,z) = \lim_{n\to\infty} \left( \frac{1-\mu(1-z^{-1})\frac{t}{n}+o(n^{-1})}{1-\lambda(z-1)\frac{t}{n}+o(n^{-1})} \right)^n = e^{\lambda t(z-1)} e^{\mu t(z^{-1}-1)} \quad \text{for all } z \neq 0. \tag{3.6}$$

Note that if $\hat{g}(z) = \mathbb{E}[z^{\xi}]$, where $\xi$ is a generic non-negative r.v., then the p.g.f. of it negative, $-\xi$ equals $\hat{g}\,(1/z)$. Therefore, the second exponential factor in (3.6) constitutes the p.g.f. of the r.v. $\eta \in \mathbb{Z}_-$ whose distribution is as follows:

$$\mathbb{P}\,(\eta = -k) = e^{-\mu t}\frac{(\mu t)^k}{k!}, \quad k = 0, 1, 2, \ldots,$$

which is that of the negative of a Poisson r.v. with mean $\mu$. Thus, we conclude that for all $t > 0$, the limit of the r.v. $\mathcal{N}_n(t) - n$ as $n \to \infty$ can be represented as the difference of two independent r.v.'s having their own Poisson distributions.

*Remark* 3.3. In the critical case $\lambda = \mu$, we obtain weak convergence of the marginal distributions to that of the r.v. having a symmetric Skellam distribution. In this case, equation (3.6) simplifies as follows:

$$\lim_{n\to\infty} \widehat{\mathcal{P}}_n(t,z) = e^{\lambda t(z+z^{-1}-2)}, \quad z \neq 0.$$

The proof is identical to that of (3.6).

Similar convergence results can be obtained if the parameters $\lambda_n = \lambda_0 > 0$ and $\mu_n = \mu_0 > 0$ are fixed but the time horizon is small such that

$$t = \tau/n + o(n^{-1}) \quad \text{as } n \to \infty.$$

In this case, one gets that

$$\lim_{n\to\infty} \widehat{\mathcal{P}}_n(t,z) = e^{\lambda_0 \tau(z-1)} e^{\mu_0 \tau(z^{-1}-1)}, \quad z \neq 0. \tag{3.7}$$

In view of (3.7), we obtain the factorization of the limit distribution for the marginals of the incremental process which is identical to (3.6), but on the new time scale. Also, the factorization given on the right-hand side of (3.7) is consistent with the difference $\xi_{\lambda\tau} - \eta_{\mu\tau}$ which emerges on the right-hand side of formula (2.11) of Proposition 2.2.

**3.2. Skellam process.** Once we established weak convergence of the marginal distributions of the sequence of birth-and-death processes, it is natural to generalize this result and to prove weak convergence to a limit process. Intuitively, this process should be described as a difference of two independent Poisson processes.

**Definition 3.4.** Let $\pi_\lambda(t)$ and $\pi_\mu(t)$ be two independent time–homogeneous Poisson processes with constant parameters $\lambda$ and $\mu$ such that $\pi_\lambda(0) = \pi_\mu(0) = 0$. Then the process

$$\mathcal{S}_{\lambda,\mu}(t) := \pi_\lambda(t) - \pi_\mu(t), \quad t \geq 0, \tag{3.8}$$

is called the time–homogeneous Skellam process indexed by parameters $\lambda$ and $\mu$.

It follows from [11][Chapter 2, Section 7] that for each real $t > 0$, the r.v. $\mathcal{S}_{\lambda,\mu}(t)$ can take on an arbitrary integer value with a positive probability which is expressed in terms of the modified Bessel function of the first kind,

$$\mathrm{I}_k\left(x\right) := \sum_{n=0}^{\infty} \frac{\left(x/2\right)^{2n+k}}{n! \cdot (n+k)!}.$$

**Proposition 3.5.** *The probability mass function* $\mathbb{P}\left(\mathcal{S}_{\lambda,\mu}(t) = k\right)$ *of the Skellam r.v.* $\mathcal{S}_{\lambda,\mu}(t)$ *admits the following closed-form representation:*

$$\mathbb{P}\left(\mathcal{S}_{\lambda,\mu}(t) = k\right) = e^{-(\lambda+\mu)t}\left(\frac{\lambda}{\mu}\right)^{k/2} \mathrm{I}_{|\,k\,|}\left(2\sqrt{\lambda\mu t}\right), \quad k \in \mathbb{Z}. \tag{3.9}$$

*Proof.* Follows from [11][Chapter 2, Section 7]. □

The p.g.f. $\Psi_{\mathcal{S}}(t,z) := \mathbb{E}\left[z^{\mathcal{S}_{\lambda,\mu}(t)}\right]$ of the r.v. $\mathcal{S}_{\lambda,\mu}(t)$ acquires the following form:

$$\Psi_{\mathcal{S}}(t,z) = e^{\lambda t(z-1)+\mu t(z^{-1}-1)}, \quad z \neq 0.$$

*Remark* 3.6. If $X(\cdot)$ is a Skellam process, $X(t) = \mathcal{S}_{\lambda,\mu}(t)$ then $-X(t)$ also is a Skellam process such that $-X(t) = \mathcal{S}_{\mu,\lambda}(t)$.

*Remark* 3.7. By (3.8), a generic Skellam process $\mathcal{S}_{\lambda,\mu}(t)$ can be decomposed into the following sum of two degenerate independent Skellam processes:

$$\mathcal{S}_{\lambda,\mu}(t) = \mathcal{S}_{\lambda,0}(t) + \mathcal{S}_{0,\mu}(t).$$

**3.3. Convergence to the Skellam process in the Skorohod space.** Consider the space $\mathbf{D([0,\boldsymbol{T}])}$ of right-continuous functions with left-hand limits defined on a finite interval $[0,T]$ equipped with the Skorohod $J_1$-topology [3], [4], [9], [28], [29], [31]. This space is equipped with the Borel $\sigma$-algebra, $\boldsymbol{\mathcal{B}(D[0,T])}$. All the stochastic processes employed in this paper are considered as random elements of the space $\boldsymbol{(D[0,T],\mathcal{B}(D[0,T]))}$ defined on the probability space $\boldsymbol{(\Omega,\mathfrak{B},\mathbb{P})}$. Convergence in distribution of stochastic processes is understood with respect to the Skorohod $J_1$-topology (see [4], [9], [14], [31]).

It is easily seen that all the trajectories of birth-and-death processes with linear rates belong to $\mathbf{D([0,\boldsymbol{T}])}$ since the jump size are either $-1$ or $1$ almost surely. The initial distribution of the processes $\mathcal{N}_n(t)$ diverges as $n \to \infty$. At the same time, the initial state of the incremental process $\mathcal{N}_n(t) - n$ is 0 with probability one for all $n \in \mathbb{Z}_+$.

Let $\boldsymbol{\mathcal{A}}$ be a measurable set, $\boldsymbol{\mathcal{A}} \in \boldsymbol{\mathcal{B}}(\mathrm{D}([0,\boldsymbol{T}]))$. Denote by $\boldsymbol{\mathfrak{m}_n(\mathcal{A})}$ the probability measure $\boldsymbol{\mathfrak{m}_n(\mathcal{A}) = \mathbb{P}(\Delta\mathcal{N}_n \in \mathcal{A})}$ induced by the process $\Delta\mathcal{N}_n$, $(n = 1, 2, \dots)$.

**Theorem 3.8.** *Suppose that the sequence of birth-and-death processes,* $\mathcal{N}_n(t)$ *satisfies* (1.1) and (1.2)*, and* $\mathcal{N}_n(0) = n$. *Then the sequence of the incremental processes* $\mathcal{N}_n(t) - n$ *converges weakly in the Skorohod space* $\mathbf{D([0,\boldsymbol{T}])}$ *as* $n \to \infty$ *to the Skellam process:*

$$\mathcal{N}_n(t) - n \overset{\mathbf{D}}{\Longrightarrow} \mathcal{S}_{\lambda,\mu}(t).$$

Since convergence of the initial distributions is trivially fulfilled, the remainder of proof of weak convergence of the processes in the Skorohod space should be split into the following two steps: the proof of convergence of the finite-dimensional distributions (see Theorem 3.10), and the proof of tightness of the family of probability measures $\{\mathfrak{m}_n(\cdot)\}_{n\geq 1}$ (see Theorem 3.11).

**Convergence of the finite-dimensional distributions.** We shall start with the proof of convergence of the finite-dimensional distributions. To this end, we will need to establish a recursive relation for the p.g.f. of the incremental process $\Delta\mathcal{N}(t)$. The key role in the derivation of this result (given as Lemma 3.9) is played by the structure of the expression for the generating function $\widehat{\mathcal{P}}(t,s)$.

For the sake of brevity and notational convenience we define for a complex or real vector $\vec{x} = (x_1, x_2, \dots, x_{m-1}, x_m) \in \mathbb{C}^m$ its $k$-dimensional projection,

$$(\vec{x})_{(k)} := (x_1, x_2, \dots, x_k) \in \mathbb{C}^k, \quad 1 \leq k < m.$$

Consider a finite partition, $\{T_k\}_{k=1}^m$, of the time interval $[0, T_m]$,

$$T_k = T_{k-1} + \Delta t_k, \quad k = 1, 2, \dots, m; \quad T_0 = 0,$$

where $\Delta t_k > 0$. Let $\vec{\Delta t}$ be the vector $(\Delta t_1, \Delta t_2, \dots, \Delta t_{m-1}, \Delta t_m)$. Then $(\vec{\Delta t})_{(m-1)}$ denotes the vector $(\Delta t_1, \Delta t_2, \dots, \Delta t_{m-1})$ of the first $(m-1)$ coordinates of $\Delta\vec{t}$.

In the following lemma, we consider the case of the birth-and-death process $\mathcal{N}(t)$ which starts from $\ell$ particles. The recursion for the p.g.f. will be derived with respect to parameter $m$, $(m = \dim \Delta\vec{t})$.

**Lemma 3.9.** *Consider a birth-and-death process $\mathcal{N}(t)$, $\mathcal{N}(0) = \ell$ with linear birth and death rates. Denote by $\widehat{\mathcal{P}}_\ell^{(m)}(\Delta\vec{t}, \vec{z})$ the joint p.g.f. such that*

$$\widehat{\mathcal{P}}_\ell^{(m)}(\Delta\vec{t}, \vec{z}) := \mathbb{E}\left[\prod_{j=1}^m z_j^{\Delta\mathcal{N}_j}\right],$$

*where $\Delta\mathcal{N}_j = \mathcal{N}(T_j) - \mathcal{N}(T_{j-1})$, and $\vec{z} = (z_1, z_2, \dots, z_m) \in \mathbb{C}^m$. Then*

$$\widehat{\mathcal{P}}_\ell^{(m)}(\Delta\vec{t}; \vec{z}) = \varphi^\ell(\Delta t_m, z_m) \cdot \widehat{\mathcal{P}}_\ell^{(m-1)}\big((\Delta\vec{t})_{(m-1)}; \varphi(\Delta t_m, z_m) \cdot (\vec{z})_{(m-1)}\big), \quad (3.10)$$

*where the function $\varphi(t,z)$ is defined by* (3.1)*. The function $\widehat{\mathcal{P}}_\ell^{(m)}(\Delta\vec{t}; \vec{z})$ is analytic in the domain $\boldsymbol{C_m} = \{\vec{z} \in \mathbb{C}^m : 0 < |\, z_j\,| \leq 1, j = 1, 2, \dots, m\}$.*

*Proof.* Let $k_j, (j = 1, 2, \dots, m)$ be integers and $K_j = \ell + \sum_{i=1}^{j} k_i$ . The process $\mathcal{N}(t)$ is Markov. The probability

$$\mathbb{P}\left(\bigcap_{j=1}^m \{\Delta\mathcal{N}_j = k_j\}\right) = \mathbb{P}\left(\bigcap_{j=1}^m \{\mathcal{N}(T_j) = \ell + K_j\}\right)$$

can be written as

$$\mathbb{P}\left(\bigcap_{j=1}^m \{\Delta\mathcal{N}_j = k_j\}\right) = \mathbb{P}\left(\Delta\mathcal{N}_1 = k_1\right) \cdot \mathbb{P}\left(\Delta\mathcal{N}_2 = k_2 \,|\, \mathcal{N}_1 = \ell + k_1\right) \cdot \dots$$
$$\times \mathbb{P}\left(\Delta\mathcal{N}_m = k_m \,|\, \mathcal{N}_{m-1} = \ell + K_{m-1}\right).$$

The p.g.f.

$$\widehat{\mathcal{P}}_\ell^{(m)}(\Delta\vec{t};\vec{z}) = \sum_{k_1}\mathbb{P}\left(\Delta\mathcal{N}_1 = k_1\right) z_1^{k_1}\sum_{k_2}\mathbb{P}\left(\Delta\mathcal{N}_2 = k_2\,|\,\mathcal{N}_1 = \ell + k_1\right) z_2^{k_2}\cdot\ldots$$
$$\times\sum_{k_m}\mathbb{P}\left(\Delta\mathcal{N}_m = k_m\,|\,\mathcal{N}_{m-1} = \ell + K_{m-1}\right) z_m^{k_m}.$$

The latter sum

$$\sum_{k_m}\mathbb{P}\left(\Delta\mathcal{N}_m = k_m\,|\,\mathcal{N}_{m-1} = \ell + K_{m-1}\right) z_m^{k_m} = \varphi^{\ell+K_{m-1}}(\Delta t_m, z_m).$$

Then we obtain that

$$\widehat{\mathcal{P}}_\ell^{(m)}(\Delta\vec{t};\vec{z}) = \varphi^{\ell}(\Delta t_m, z_m)\cdot\sum_{k_1}\mathbb{P}\left(\Delta\mathcal{N}_1 = k_1\right) z_1^{k_1}\varphi^{k_1}(\Delta t_m, s_m)$$
$$\times\sum_{k_2}\mathbb{P}\left(\Delta\mathcal{N}_2 = k_2\,|\,\mathcal{N}_1 = n + k_1\right) z_2^{k_2}\varphi^{k_2}(\Delta t_m, z_m)\cdot\ldots$$
$$\times\sum_{k_{m-1}}\mathbb{P}\left(\Delta\mathcal{N}_{m-1} = k_{m-1}\,|\,\mathcal{N}_{m-2} = n + K_{m-2}\right)\left(z_{m-1}\varphi(\Delta t_m, z_m)\right)^{k_{m-1}}.$$

The latter relation is equivalent to (3.10).

Next, let us show that the recursive relationship (3.10) does not create new singularities as a result of substitution of the function $z_k\varphi(\Delta t_m, z_m)$ for $z_k$, ($k = 1, 2, \ldots, m-1$). Indeed, by inspection of the function $\varphi$ we find that the function $\varphi^\ell(\Delta t_m, z_m)\cdot\varphi^\ell\left(\Delta t_k, z_k\cdot\varphi(\Delta t_m, z_m)\right)$ has a pole only at $z_k = 0$ because the other zeros of the function $\varphi(\Delta t_m, z_m)$ disappear after cancellation of the factor $\varphi^\ell(\Delta t_m, z_m)$. Thus, the p.g.f. $\widehat{\mathcal{P}}_\ell^{(m)}(\Delta\vec{t};\vec{z})$ is analytic in the domain $\boldsymbol{C_m}$. □

**Theorem 3.10.** *Let $\mathcal{N}_n(t)$ be a birth-and-death process such that $\mathcal{N}_n(0) = n$, and the birth and death parameters are equal to $\lambda_n$ and $\mu_n$, respectively. Assume that conditions* (1.1) *and* (1.2) *are fulfilled, and consider a finite partition*

$$0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = T, \quad T > 0, \tag{3.11}$$

*of the finite time interval* $[0, T]$.

*Then there exists the limit as $n\to\infty$ of the joint p.g.f. $\widehat{\mathcal{P}}_n^{(m)}(\Delta\vec{t};\vec{z})$ of the increments of the processes $\mathcal{N}_n(t) - n$ which coincides with that of the Skellam process $\mathcal{S}_{\lambda,\mu}(t)$ whose specific values of parameters $\lambda$ and $\mu$ satisfy conditions* (1.1) *and* (1.2)*. Namely,*

$$\lim_{n\to\infty}\widehat{\mathcal{P}}_n^{(m)}(\Delta\vec{t};\vec{z}) = \prod_{j=1}^{m} e^{\lambda\Delta t_j(z_j-1)}\cdot\prod_{j=1}^{m} e^{\mu\Delta t_j(z_j^{-1}-1)}. \tag{3.12}$$

*Proof.* We shall prove Theorem 3.10 by induction in $m$ in the non-critical case for which $\lambda_n \neq \mu_n$. (The proof for the critical case is almost identical to that discussed below and hence is skipped.)

The induction base. For $m = 1$ the validity of (3.12) easily follows from (3.6). The induction hypothesis. Suppose that convergence in (3.12) holds for $m = k$:

$$\lim_{n\to\infty} \widehat{\mathcal{P}}_n^{(k)}(\Delta\vec{t};\vec{z}) = \prod_{j=1}^{k} e^{\lambda\Delta t_j(z_j-1)} \cdot \prod_{j=1}^{k} e^{\mu\Delta t_j(z_j^{-1}-1)}.$$

The induction step. Let us prove the validity of (3.12) for $m = k+1$. To this end, observe that Lemma 3.9 stipulates that

$$\lim_{n\to\infty} \widehat{\mathcal{P}}_n^{(k+1)}(\Delta\vec{t};\vec{z}) = \lim_{n\to\infty} \varphi^n(\Delta t_{k+1}, z_{k+1}) \times \lim_{n\to\infty} \widehat{\mathcal{P}}_n^{(k)}\big((\Delta\vec{t})_{(k)}; \varphi(\Delta t_{k+1}, z_{k+1}) \cdot (\vec{z})_{(k)}\big).$$

The first limit which emerges on the right-hand side of the above equation is as follows:

$$\lim_{n\to\infty} \varphi^n(\Delta t_{k+1}, z_{k+1}) = e^{\lambda\Delta t_{k+1}(z_{k+1}-1)+\mu\Delta t_{k+1}(z_{k+1}^{-1}-1)}.$$

Also, in view of (1.1) and (1.2), we get that

$$\lim_{n\to\infty} \varphi(\Delta t_{k+1}, z_{k+1}) = 1.$$

Therefore,

$$\lim_{n\to\infty} \widehat{\mathcal{P}}_n^{(k)}\big((\Delta\vec{t})_{(k)}; \varphi(\Delta t_{k+1}, z_{k+1}) \cdot (\vec{z})_{(k)}\big) = \prod_{j=1}^{k} e^{\lambda\Delta t_j(z_j-1)} \cdot \prod_{j=1}^{k} e^{\mu\Delta t_j(z_j^{-1}-1)}.$$

Finally, we obtain that

$$\lim_{n\to\infty} \widehat{\mathcal{P}}_n^{(k+1)}(\Delta\vec{t};\vec{z}) = \prod_{j=1}^{k+1} e^{\lambda\Delta t_j(z_j-1)} \cdot \prod_{j=1}^{k+1} e^{\mu\Delta t_j(z_j^{-1}-1)},$$

which completes the proof of the induction step. □

**Tightness of the family of the probability measures.** Our next step is to prove tightness of the family of probability measures $\{\mathfrak{m}_n(\cdot)\}_{n\geq 1}$, which is stipulated by the following assertion.

**Theorem 3.11.** *Suppose that parameters $\lambda_n$ and $\mu_n$ satisfy conditions* (1.1) *and* (1.2)*. Then the family $\{\mathfrak{m}_n(\cdot)\}_{n\geq 1}$ of the probability measures is tight.*

*Proof.* The idea of the proof is as follows. We will construct a sequence of pure death processes, $\mathcal{N}_n^D(t)$ and a sequence of pure birth processes, $\mathcal{N}_n^B(t)$, such that the birth-and-death process, $\mathcal{N}_n(t)$ is bounded from below by $\mathcal{N}_n^D(t)$ and also bounded from above by $\mathcal{N}_n^B(t)$. After that we prove that the family of probability measures, $\mathfrak{m}_n^D$ and $\mathfrak{m}_n^B$, induced by the processes $\mathcal{N}_n^D(t) - n$ and $\mathcal{N}_n^B(t) - n$ respectively, are tight if $\lambda_n$ and $\mu_n$ satisfy conditions (1.1) and (1.2).

Let $\mathcal{N}_n^D(t)$ be the pure death process, such that $\mathcal{N}_n^D(0) = n$ and death rate, $\mu_n^* = \lambda_n + \mu_n$. Let us also introduce the pure birth process, $\mathcal{N}_n^B(t)$, such that $\mathcal{N}_n^B(0) = n$ and birth rate, $\lambda_n^* = \mu_n^*$. Clearly,

$$\lim_{n\to\infty} n \cdot \mu_n^* = \lim_{n\to\infty} n \cdot \lambda_n^* = \lambda + \mu.$$

Using the coupling argument, we obtain the following result:

**Lemma 3.12.** *The processes* $\mathcal{N}_n^D(t)$, $\mathcal{N}_n^B(t)$ *and* $\mathcal{N}_n(t)$ *can be defined on the same probability space such that for almost all* $\omega$ *and all* $t \in [0,T]$,

$$\mathcal{N}_n^D(t) \le \mathcal{N}_n(t) \le \mathcal{N}_n^B(t), \quad n = 1, 2, \dots. \tag{3.13}$$

*Proof.* At time $t = 0$, all the three processes have a common starting point:

$$\mathcal{N}_n^D(t) = \mathcal{N}_n(t) = \mathcal{N}_n^B(t) = n$$

and, obviously,

$$\mathcal{N}_n^D(t) - n = \mathcal{N}_n(t) - n = \mathcal{N}_n^B(t) - n = 0.$$

Next, denote the jump epochs of the process $\mathcal{N}_n(t)$ by $\mathfrak{t}_k^{(n)}$, $(k = 1, 2, \dots)$. If at time $\mathfrak{t}_k^{(n)}$

$$\mathcal{N}_n^D(\mathfrak{t}_k^{(n)}) < \mathcal{N}_n(\mathfrak{t}_k^{(n)}) < \mathcal{N}_n^B(\mathfrak{t}_k^{(n)}),$$

the next jump epochs of the processes $\mathcal{N}_n^D$ and $\mathcal{N}_n^B$ are determined independently of $\mathcal{N}_n$ until either $\mathcal{N}_n(\mathfrak{t}_{k+\mathfrak{j}}^{(n)}) = \mathcal{N}_n^D(\mathfrak{t}_{k+\mathfrak{j}}^{(n)})$ or $\mathcal{N}_n(\mathfrak{t}_{k+\mathfrak{j}}^{(n)}) = \mathcal{N}_n^B(\mathfrak{t}_{k+\mathfrak{j}}^{(n)})$, where $\mathfrak{t}_{k+\mathfrak{j}}^{(n)} \le T$, $\mathfrak{j} \ge 1$. If such $\mathfrak{j}$ does not exist, inequality (3.13) is satisfied for all $t \in [0,T]$.

If $\mathcal{N}_n(\mathfrak{t}_k^{(n)}) = \mathcal{N}_n^B(\mathfrak{t}_k^{(n)})$ then we define $\mathcal{N}_n^B(\mathfrak{t}_{k+1}^{(n)}) := \mathcal{N}_n^B(\mathfrak{t}_k^{(n)}) + 1$ and $\mathfrak{t}_{k+1}^{(n)}$ becomes the jump epoch of the process $\mathcal{N}_n^B$. The jump epochs of the process $\mathcal{N}_n^D$ are defined independently of the process $\mathcal{N}_n$ in the interval $[\mathfrak{t}_k^{(n)}, \mathfrak{t}_{k+1}^{(n)}]$. Evidently, inequality (3.13) is satisfied if $t \in [\mathfrak{t}_k^{(n)}, \mathfrak{t}_{k+1}^{(n)}]$.

Analogously, if $\mathcal{N}_n(\mathfrak{t}_k^{(n)}) = \mathcal{N}_n^D(\mathfrak{t}_k^{(n)})$ then we define $\mathcal{N}_n^D(\mathfrak{t}_{k+1}^{(n)}) := \mathcal{N}_n^D(\mathfrak{t}_k^{(n)}) - 1$, and $\mathfrak{t}_{k+1}^{(n)}$ becomes the jump epoch of the process $\mathcal{N}_n^D$. In this case, the jump epochs of the process $\mathcal{N}_n^B$ are defined independently of the process $\mathcal{N}_n$ and again, inequality (3.13) is satisfied if $t \in [\mathfrak{t}_k^{(n)}, \mathfrak{t}_{k+1}^{(n)}]$.

It is not difficult to demonstrate that the sequence $\mathfrak{t}_k^{(n)} \xrightarrow{p} \infty$, as $k \to \infty$, almost surely for all $n$ and, therefore, $\mathbb{P}\left(\sup_k \mathfrak{t}_k^{(n)} > T\right) = 1, \quad n \in \mathbb{N}$. Thus, the processes $\mathcal{N}_n^D$, $\mathcal{N}_n$ and $\mathcal{N}_n^B$ satisfy inequality (3.13) for all $t \in [0,T]$ by construction. □

Now, consider the sequence of the incremental processes $\mathcal{N}_n^B(t) - n$. Let us prove that the family of the corresponding probability measures is tight. To this end, it suffices to show that for any $\varepsilon > 0$ there exists $L \ge 1$ such that for each $n \ge 1$,

$$\mathbb{P}\left(\mathcal{N}_n^B(t) - n > L\right) < \varepsilon, \quad t \in [0,T]. \tag{3.14}$$

Next, it is known that a pure birth process with linear rates can be represented as a time-changed Poisson process (with a *random* time change). The following lemma characterizes such random time change that helps to overcome the above-described technical obstacle (which is due to the fact that the rates of our pure birth process are linear).

**Lemma 3.13** (Compare to [27], [36])**.** *Suppose that* $X(t)$ *is a pure birth process with birth rate* $\lambda$, *and* $\mathbb{P}(X(0) = n_0) = 1$ *for some integer* $n_0 \in \mathbb{N}$. *Then there exists a gamma-distributed r.v.* $\gamma$, *with the probability density function*

$$f_\gamma(x) = e^{-x} \frac{x^{n_0 - 1}}{(n_0 - 1)!}, \quad x \ge 0,$$

*such that*

$$X(t)\overset{\mathcal{D}}{=}n_0+\pi(\gamma\tau(t)).$$

*Here, $\pi(t)$ is a unit-intensity Poisson process which does not depend on the r.v. $\gamma$, and $\tau(t)=\exp(\lambda t)-1$.*

In turn, Lemma 3.13 implies the following corollary, whose proof is omitted.

**Corollary 3.14.** *Let $\pi(t)$ be a Poisson process with unit rate, $\tau_n(t)=e^{(\lambda_n+\mu_n)t}-1$ and $\gamma_n$ be a gamma-distributed r.v. with $\mathbb{E}[\gamma_n]=\mathbf{Var}[\gamma_n]=n$, which does not depend on $\pi(t)$. Then the pure birth process $\mathcal{N}_n^B(t)$ admits the following representation:*

$$\mathcal{N}_n^B(t)\overset{\mathcal{D}}{=}n+\pi(\gamma_n\tau_n(t)),\quad t\geq 0. \tag{3.15}$$

Let us now prove inequality (3.14). Since $\sup\limits_{t\in[0,T]}\mathcal{N}_n^B(t)=\mathcal{N}_n^B(T)$ and $\mathcal{N}_n^B(T)-n=\pi(\gamma_n\tau_n(T))$, it suffices to find $L$ such that $\mathbb{P}\left(\pi(\gamma_n\tau_n(T))>L\right)<\varepsilon$ for a given $\varepsilon>0$. Denote $\zeta_n:=\gamma_n\tau_n(T)$. Then we have

$$\begin{aligned}\mathbb{E}[\zeta_n] &= \mathbb{E}[\gamma_n]\cdot\left(e^{(\lambda_n+\mu_n)T}-1\right)\\ &= n\cdot\left(\frac{(\lambda+\mu)T}{n}+o(n^{-1}\right)\\ &= (\lambda+\mu)T+o(1),\quad \text{as } n\to\infty.\end{aligned}$$

The variance

$$\sigma^2(\zeta_n)=\frac{((\lambda+\mu)T)^2}{n}+o(n^{-1}).$$

Therefore, the sequence $\zeta_n$ converges in probability: $\zeta_n\xrightarrow{p}(\lambda+\mu)T$ as $n\to\infty$. Then

$$\pi\left(\gamma_n\cdot(e^{(\lambda_n+\mu_n)T}-1)\right)\xrightarrow{p}\pi((\lambda+\mu)T). \tag{3.16}$$

Next, an application of Chebyshev inequality to $\pi((\lambda+\mu)T)$ for $L>(\lambda+\mu)T$ ascertains that

$$\begin{aligned}\mathbb{P}\left(\pi((\lambda+\mu)T)>L\right) &\leq \mathbb{P}\left(|\,\pi((\lambda+\mu)T)-(\lambda+\mu)T\,|>L-(\lambda+\mu)T\right)\\ &\leq \frac{((\lambda+\mu)T)^2}{n\cdot(L-(\lambda+\mu)T)^2}.\end{aligned}$$

A subsequent combination of the latter inequality, (3.16) and Corollary 3.14 implies tightness of the probability measures $\mathfrak{m}_n^B(\cdot)$.

Let us now prove tightness of the measures $\mathfrak{m}_n^D(\cdot)$. Note that the incremental process, $\mathcal{B}_n(t)=n-\mathcal{N}_n^D(t)$, In the case of the pure death processes $\mathcal{N}_n^D(t)$, the increments, $\mathcal{B}_n(t)=n-\mathcal{N}_n^D(t)$, form a binomial process with the time-dependent probability of success, $p_n=e^{-\mu_n^*t}$. Note that a Poisson process $\pi_{\mu_n^*}(t)$ with intensity $\mu_n^*$ and the process $\mathcal{B}_n(t)$ can be defined on the same probability space $(\Omega,\mathfrak{B},\mathbb{P})$ such that for all $t\geq 0$ and all $n\in Z_+$

$$0\leq\mathcal{B}_n(t)\leq\pi_{\mu_n^*}(t)\quad\text{for almost all } \omega. \tag{3.17}$$

Recall that $\lim_{n\to\infty} n\mu_n^* = \lambda + \mu$. Then by repeating the same line of arguments as used in the derivation of inequality (3.14), we obtain that the family of probability measures, $\mathfrak{m}_n^\pi$, corresponding to the processes $\pi_{\mu_n^*}(t)$, is tight. Inequality (3.17) implies now tightness of the family of measures $\mathfrak{m}_n^{\mathcal{B}}(\cdot)$, corresponding to the processes $\mathcal{B}_n(t)$. In turn, this results in the tightness of the family $\{\mathfrak{m}_n^D(\cdot)\}_{n\geq 1}$. Finally, it remains to apply Lemma 3.12 to ascertain that $\{\mathfrak{m}_n(\cdot)\}_{n\geq 1}$ is tight. □

Evidently, the proof of tightness of the family of probability measures $\{\mathfrak{m}_n\}_{n\geq 1}$ concludes the proof of Theorem 3.8.

*Remark* 3.15. If instead of conditions (1.1) and (1.2) we impose the following assumptions:

$$\Delta t_m = \frac{\tau_m}{n} + o(n^{-1}), \quad m = 1, 2, \ldots, M,$$

and

$$\lambda_n = \lambda_*, \quad \mu_n = \mu_*, \quad n = 1, 2, \ldots,$$

then the next modification of Theorem 3.8 is valid:

$$\mathcal{N}_n(t) - n \overset{\mathbf{D}}{\Longrightarrow} \mathcal{S}_{\lambda_*,\mu_*}(\tau)$$

in the Skorohod space $\mathbf{D}\Big(\Big[\mathbf{0}, \boldsymbol{T_M}\Big]\Big)$, where $T_M = \sum_{m=1}^{M} \tau_m$.

## 4. Two Special Cases: Pure Birth and Pure Death Processes

In this section, we present two corollaries from Theorem 3.10 on convergence of finite-dimensional distributions of the pure birth and pure death processes (see Corollaries 4.1 and 4.11, respectively). We also discuss the explicit closed-form expression for the joint generating function of the incremental process $\Delta\mathcal{N}(t)$ and compute the correlation coefficients of its increments. Both the processes considered in this section have monotone trajectories. The increments of the pure birth process are always non-negative. In the case of the pure death process, we consider the adjusted incremental process, $n - \mathcal{N}(t)$, which also has non-negative increments.

**4.1. Pure birth processes.** It follows from [10] that in the case where $\mathcal{N}(t)$ is a pure birth process which originates from $n$ particles (i.e., $\mathcal{N}(0) = n$), the distribution $P_k(t) = \mathbb{P}(\mathcal{N}(t) = k)$ is negative binomial (with the topological support $\{n, n+1, \ldots\}$) whose probability mass function is as follows:

$$P_{n+j}(t) = \binom{n+j-1}{j} e^{-n\lambda t} \left(1 - e^{-\lambda t}\right)^j, \quad j = 0, 1, \ldots. \tag{4.1}$$

It is well known that in this case, the stochastic process $\mathcal{N}(t)$ is a time-homogeneous Markov process whose transition probabilities are given by

$$\begin{aligned} P_{\Delta t}(k, k+m) &:= \mathbb{P}(\mathcal{N}(t + \Delta t) = k + m \,|\, \mathcal{N}(t) = k) \\ &= \binom{k+m-1}{m} e^{-\lambda \Delta t k} \cdot (1 - e^{-\lambda \Delta t})^m, \quad m = 0, 1, \ldots. \end{aligned}$$

Consider a finite partition (3.11) of the time interval $[0,T]$. As before, we denote by $\Delta\mathcal{N}_m := \mathcal{N}(t_m) - \mathcal{N}(t_{m-1})$ the increments of the process $\mathcal{N}(\cdot)$ and introduce the joint p.g.f. of the increments:

$$\widehat{\mathcal{P}}_n^{(k)}(\Delta\vec{t};\vec{z}) := \mathbb{E}\Big[\prod_{m=1}^{k} z_m^{\Delta\mathcal{N}_m}\Big]. \tag{4.2}$$

Since the above-described pure birth process is a degenerate case of the birth-and-death process considered in the previous section, the limit under the fulfillment of condition (1.1) should be a degenerate Skellam process with parameter $\mu = 0$ (compare to Remark 3.7). Evidently, the latter constitutes a time-homogeneous Poisson process $\pi_\lambda$ with parameter $\lambda$. From Theorem 3.10 we obtain

**Corollary 4.1.** *Suppose that condition* (1.1) *is fulfilled. Then*

$$\mathcal{N}_n(t) - n \overset{\mathbf{D}}{\Longrightarrow} \pi_\lambda(t)$$

*in* $\mathbf{D([0,}\boldsymbol{T}\mathbf{])}$ *as* $n \to \infty$, *for any fixed real* $T > 0$. *In particular, the finite–dimensional distributions of the incremental process* $\mathcal{N}_n(t) - n$ *converge to those of the Poisson process with parameter* $\lambda$ *such that*

$$\exists \lim_{n\to\infty} \widehat{\mathcal{P}}_n^{(m)}(\Delta\vec{t};\vec{z}) = \prod_{j=1}^{m} e^{\lambda\Delta t_j(z_j-1)}, \tag{4.3}$$

*where* $\max\limits_{1\le j\le k} |\, z_j \,| \le 1$.

*Remark* 4.2. If instead of the condition (1.1) we assume that $\lambda_n \equiv \lambda_*$ is constant (i.e., independent of $n$), but impose the following time-scaling condition: $\Delta t_m \sim \Delta\tau_m/n$ as $n \to \infty$ with $m = 1, 2, \ldots, k$, we will derive the following result on convergence of the corresponding finite-dimensional distributions to those of the Poisson process with intensity $\lambda_*$:

$$\lim_{n\to\infty} \widehat{\mathcal{P}}_n^{(k)}(\Delta\vec{t};\vec{z}) = \prod_{j=1}^{k} e^{\lambda_*\tau_j(z_j-1)}.$$

In principle, the recursive equation derived in Lemma 3.9 makes it possible to obtain the following elegant formula for the joint p.g.f. of the increments. However, we elected to provide the probabilistic proof of this result which relies on Markov property of the pure birth process which is not that routine. To this end, denote by $p_j = e^{-\lambda\Delta t_j}$, and $q_j = 1 - p_j$, $(j = 1, 2, \ldots k)$.

**Proposition 4.3.** *The joint p.g.f. of the increments of the above-described pure birth process is as follows:*

$$\widehat{\mathcal{P}}_n^{(k)}\left(\Delta\vec{t};\vec{z}\right) = \left(\frac{\prod\limits_{j=1}^{k} p_j}{1 - \sum\limits_{j=1}^{k} q_j z_j \prod\limits_{i=j+1}^{k} p_i}\right)^n. \tag{4.4}$$

*Proof.* Clearly, the process $\mathcal{N}_n(t)$ is Markov. Also, it can be shown with some effort that that the probability of intersection

$$\mathbb{P}\left(\bigcap_{j=0}^{k}\{\mathcal{N}_n(t_j)=n+M_j\}\right)=\prod_{j=1}^{k}\binom{n+M_j-1}{m_j}p_j^{n+M_{j-1}}q_j^{m_j}, \tag{4.5}$$

where $M_j=\sum_{i=1}^{j}m_i$, and $q_j=1-p_j$. Observe that the joint p.g.f. (4.2) admits the following representation:

$$\widehat{\mathcal{P}}_n^{(k)}(\Delta\vec{t};\vec{z}):=\sum_{m_1=0}^{\infty}\cdots\sum_{m_k=0}^{\infty}\prod_{j=1}^{k}\binom{n+M_j-1}{m_j}p_j^{n+M_{j-1}}(q_jz_j)^{m_j}. \tag{4.6}$$

It is now convenient to write down explicitly the full list of the arguments of the generating function $\widehat{\mathcal{P}}_n^{(k)}\left(\Delta\vec{t};\vec{z}\right)$ in terms of parameters $p_k$:

$$\widehat{\mathcal{P}}_n^{(k)}\left(\Delta\vec{t};\vec{z}\right)=\widehat{\mathcal{P}}_n^{(k)}(p_1,\dots,p_k;z_1,\dots,z_k).$$

A combination of Lemma 3.9 with Equation (4.6) yields the following recursive relationship:

$$\begin{aligned}&\widehat{\mathcal{P}}_n^{(k)}(z_1,\dots,z_k;p_1,\dots,p_k)\\&=\widehat{\mathcal{P}}_n^{(k-1)}\left(z_1,\dots,z_{k-2},z_{k-1}\frac{p_k}{1-q_kz_k};p_1,\dots,p_{k-2},p_{k-1}\frac{p_k}{1-q_kz_k}\right),\end{aligned} \tag{4.7}$$

which can be interpreted as follows: in order to obtain the $k$th generating function $\widehat{\mathcal{P}}_n^{(k)}(\vec{p};\vec{z})$ from $\widehat{\mathcal{P}}_n^{(k-1)}$, one should substitute $p_{k-1}\cdot\frac{p_k}{1-q_kz_k}$ for $p_{k-1}$ and $z_{k-1}\cdot\frac{p_k}{1-q_kz_k}$ for $z_{k-1}$. The induction base, i.e., the case of $k=1$, follows from (2.8). Namely,

$$\widehat{\mathcal{P}}_n^{(1)}(z_1;p_1)=\left(\frac{p_1}{1-q_1z_1}\right)^n.$$

A subsequent substitution of the recursive formula (4.7) with $k=2$ to the above equation yields that

$$\widehat{\mathcal{P}}_n^{(2)}(p_1,p_2;z_1,z_2)=\left(\frac{p_1p_2}{1-q_2z_2-q_1z_1p_2}\right)^n,$$

and finally, after $(k-1)$ iterations we obtain the validity of (4.4). □

Equation (4.4) allows us to compute all the moments of the random vector comprised of the increments $(\Delta\mathcal{N}_1,\dots\Delta\mathcal{N}_k)$. Here, we will present only one result for the correlation coefficients in the case where all the time increments are equal to each other.

**Corollary 4.4.** *Suppose that $\Delta t_j=\Delta t$, $(j=1,2,\dots,k)$. Let $p:=\exp(-\lambda\Delta t)$, $q:=1-p$, and parameter $n$ be fixed. Then the correlation coefficient of the increments $\Delta\mathcal{N}_i$ and $\Delta\mathcal{N}_j$, $(i\neq j)$, is as follows:*

$$\rho_{i,j}=\frac{q}{\sqrt{(q+p^i)(q+p^j)}}. \tag{4.8}$$

If the time interval $\Delta t\to 0$ then $q\to 0$ and we have asymptotically uncorrelated increments of the process $\mathcal{N}(t)$.

**4.2. Pure death process.** Consider the pure death process $\mathcal{N}(t)$ with death rate $\mu > 0$ and the initial state $\mathcal{N}(0) = n$. By [10], for each real $t > 0$, the r.v. $\mathcal{N}(t)$ has a binomial distribution such that $\mathcal{N}(t) \stackrel{\mathrm{d}}{=} \mathbf{B}(n, e^{-\mu t})$ and hence, the probabilities $P_k(t) := \mathbb{P}(\mathcal{N}(t) = k)$ for $t \geq 0$ are as follows:

$$P_k(t) = \binom{n}{k} e^{-k\mu t} \left(1 - e^{-\mu t}\right)^{n-k}, \quad k = 0, 1, \ldots, n, \tag{4.9}$$
$$P_n(0) = 1.$$

*Remark* 4.5. In view of [10], the random time $T_n$ of transition from state $n$ to state 0 is characterized by the following survival function:

$$\mathbb{P}\left(T_n > t\right) = 1 - (1 - e^{-\mu t})^n.$$

Subsequently, since its expected value $\mathbb{E}[T_n] = \int_0^\infty \mathbb{P}\left(T_n > t\right)\mathrm{d}t$, we derive that

$$\mathbb{E}[T_n] = \frac{1}{\mu} \sum_{m=1}^{n} \frac{1}{m} = \frac{1}{\mu} \cdot (-\Gamma'(1) + \log n) + \mathcal{O}(1/n) \sim \frac{1}{\mu} \log n$$

as $n \to +\infty$. Here, $-\Gamma'(1) \approx 0.577$ denotes the Euler–Mascheroni constant. Some results on the limit distribution of $T_n$ for the subcritical birth-and-death processes can be found in [25], see also [32].

Observe that for an arbitrary fixed $t \geq 0$ and the values of argument $|\, z\, | \leq 1$, the p.g.f. $\hat{G}(t, s) := \mathbb{E}[z^{\mathcal{N}(t)}]$ of a binomial r.v. $\mathcal{N}(t)$ admits the following representation:

$$\hat{G}(t, s) = \left(z e^{-\mu t} + 1 - e^{-\mu t}\right)^n. \tag{4.10}$$

**Corollary 4.6.** *Consider the sequence of the pure death processes $\mathcal{N}_n$ with $\mathcal{N}(0) = n$, and suppose that condition* (1.2) *is met. Then all the finite-dimensional distributions of the incremental process $\mathcal{N}_n - n$ converge to those of the Skellam process $\mathcal{S}_{0,\mu}$ as $n \to \infty$:*

$$\lim_{n\to\infty} \widehat{\mathcal{P}}_n^{(m)}\left(\Delta\vec{t}; \vec{z}\right) = \prod_{j=1}^{m} e^{\mu \Delta t_j (z_j^{-1} - 1)}. \tag{4.11}$$

*Proof.* It follows with some effort from Theorem 3.10. □

Now, consider the following family of stochastic processes

$$\mathcal{M}_n(t) := n - \mathcal{N}_n(t),$$

where $\mathcal{N}_n(t)$ is the pure death process starting from $\mathcal{N}_n(0) = n$.
Since $n - \mathbf{B}(n, p) \stackrel{\mathrm{d}}{=} \mathbf{B}(n, 1 - p)$, we easily obtain that

$$\mathbb{E}\left[z^{\mathcal{M}_n(t)}\right] = \left(e^{-\mu t} + z \cdot (1 - e^{-\mu t})\right)^n. \tag{4.12}$$

Let us now assume the fulfillment of condition (1.2) on death rates. Since the incremental process $\mathcal{N}_n - n \stackrel{\mathbf{D}}{\Longrightarrow} \mathcal{S}_{0,\mu}$, we obtain convergence $\mathcal{M}_n(t) \stackrel{\mathbf{D}}{\Longrightarrow} \mathcal{S}_{\mu,0}$. Thus, the limiting processes is a time-homogeneous Poisson process with parameter $\mu$.

Let us now derive the joint p.g.f. for the process $\mathcal{M}(t) = \mathcal{N}(0) - \mathcal{N}(t)$, where $\mathcal{N}(t)$ is a pure death process. To this end, we make a partition (3.11) of the time interval $[0, T]$ and consider the increments of the process $\mathcal{M}(t)$, $\Delta\mathcal{M}_j =$

$\mathcal{M}(t_j) - \mathcal{M}(t_{j-1})$, $(j = 1, 2, \dots, k)$. Consider the joint probability-generating function

$$\mathfrak{g}_k(\Delta\vec{t}; \vec{z}) := \mathbb{E}\left[\prod_{j=1}^{k} z_j^{\Delta\mathcal{M}_j}\right], \quad \max_{1\le j\le k} |\, z_j \,| \le 1,$$

The following assertion provides a closed-form representation for the joint generating function $\mathfrak{g}_k\left(\Delta\vec{t}; \vec{z}\right)$ of the pure death process.

**Proposition 4.7.** *Let the vector* $\vec{\Delta t} = (\Delta t_1, \Delta t_2, \dots, \Delta t_k)$ *and* $\vec{z} = (z_1, z_2, \dots z_k)$. *Then joint p.g.f.* $\mathfrak{g}_k(\Delta\vec{t}; \vec{z})$ *of the increments of the above-described pure death process is as follows:*

$$\mathfrak{g}_k(\Delta\vec{t}; \vec{z}) = \left(\sum_{j=1}^{k} q_j z_j \prod_{i=1}^{j-1} p_i + \prod_{j=1}^{k} p_j\right)^n, \tag{4.13}$$

*where* $p_j = e^{-\mu(t_j - t_{j-1})}$ *and* $q_j = 1 - p_j$.

*Proof.* Note that the function $\mathfrak{g}_k(\vec{z}) = \mathfrak{g}_k(\Delta\vec{t}; \vec{z})$ in Equation (4.13) is a polinomial function of $\vec{z}$ with the coefficients depending on $\vec{p} = (p_1, \dots, p_k)$ and $\vec{q} = (q_1, \dots, q_k)$. We shall prove Proposition 4.7 by induction in $k$ using a simple recursion for the function $\mathfrak{g}_k(\vec{z})$ derived below. We start from the induction base, i.e., $k = 1$. For $m_1 = 0, 1, 2, \dots n$, we have that

$$\begin{aligned}\mathbb{P}\left(\Delta\mathcal{M}_{n,1} = m_1\right) &= \mathbb{P}\left(\mathcal{N}_n(t_1) = n - m_1\right) \\ &= \binom{n}{m_1} e^{-\mu(t_1 - t_0)(n - m_1)} \left(1 - e^{-\mu(t_1 - t_0)}\right)^{m_1}.\end{aligned}$$

It is obvious that $\mathfrak{g}_1(z_1) = (p_1 + q_1 z_1)^n$. In our case, the induction hypothesis states that the p.g.f. $\mathfrak{g}_{k-1}(\vec{z}_{(k-1)})$ is as follows:

$$\mathfrak{g}_{k-1}(\vec{z}_{(k-1)}) = \left(\sum_{j=1}^{k-1} q_j z_j \prod_{i=1}^{j-1} p_i + \prod_{j=1}^{k-1} p_j\right)^n. \tag{4.14}$$

In order to prove the validity of (4.13) it remains to perform the induction step. To this end, denote $\hat{m}_\ell := \sum_{j=1}^{\ell} m_j$, $\ell = 1, 2, \dots k$, and $n_\ell := n - \hat{m}_\ell$. Then the joint p.g.f. $\mathfrak{g}_k(\vec{z})$ can be written as follows:

$$\begin{aligned}\mathfrak{g}_k(\vec{z}) &= \sum_{m_1=0}^{n} \binom{n}{m_1} p_1^{n-m_1} (q_1 z_1)^{m_1} \sum_{m_2=0}^{n_1} \binom{n - \hat{m}_1}{m_2} p_2^{n-\hat{m}_2} (q_2 z_2)^{m_2} \cdot \ldots \\ &\times \sum_{m_{k-1}=0}^{n_{k-2}} \binom{n - \hat{m}_{k-2}}{m_{k-1}} p_{k-1}^{n-\hat{m}_{k-1}} (q_{k-1} z_{k-1})^{m_{k-1}} \cdot \\ &\times \sum_{m_k=0}^{n_{k-1}} \binom{n - \hat{m}_{k-1}}{m_k} p_k^{n-\hat{m}_k} (q_k z_k)^{m_k}.\end{aligned} \tag{4.15}$$

The latter sum in (4.15) equals $(p_k+q_kz_k)^{n_{k-1}}$. Thus, we obtain that $\mathfrak{g}_k(\vec{z})$ satisfies the following recursive relationship:

$$\begin{aligned}\mathfrak{g}_k(\vec{z}) &= \sum_{m_1=0}^{n}\binom{n}{m_1}p_1^{n-m_1}(q_1z_1)^{m_1}\sum_{m_2=0}^{n_1}\binom{n-\hat{m}_1}{m_2}p_2^{n-\hat{m}_2}(q_2z_2)^{m_2}\cdot\ldots\\ &\quad\times\sum_{m_{k-1}=0}^{n_{k-2}}\binom{n-\hat{m}_{k-2}}{m_{k-1}}p_{k-1}^{n-\hat{m}_{k-1}}\cdot(q_{k-1}z_{k-1})^{m_{k-1}}\cdot(p_k+q_kz_k)^{n_{k-1}}\\ &= \sum_{m_1=0}^{n}\binom{n}{m_1}p_1^{n-m_1}(q_1z_1)^{m_1}\sum_{m_2=0}^{n_1}\binom{n-\hat{m}_1}{m_2}p_2^{n-\hat{m}_2}(q_2z_2)^{m_2}\cdot\ldots\\ &\quad\times\sum_{m_{k-1}=0}^{n_{k-2}}\binom{n-\hat{m}_{k-2}}{m_{k-1}}\left(p_{k-1}(p_k+q_kz_k)\right)^{n-\hat{m}_{k-1}}\cdot(q_{k-1}z_{k-1})^{m_{k-1}}\\ &= \widehat{\mathfrak{g}}_{k-1}(\vec{z}_{(k-1)}),\end{aligned}$$

where the coefficients of the polynomial function $\widehat{\mathfrak{g}}_{k-1}(\vec{z}_{(k-1)})$ are defined by the vectors $\vec{p}^{*}=(p_1^*,p_2^*,\ldots,p_{k-1}^*)$ and $\vec{q}_{(k-1)}=(q_1,q_2,\ldots,q_{k-1})$ with the coordinates

$$p_1^*=p_1,\ p_2^*=p_2,\ldots,p_{k-2}^*=p_{k-2},\ p_{k-1}^*=p_{k-1}\cdot(p_k+q_kz_k).$$

Then from (4.14) we derive that

$$\begin{aligned}\mathfrak{g}_k(\vec{z}) &= \left(\sum_{j=1}^{k-1}q_jz_j\prod_{i=1}^{j-1}p_i+\prod_{j=1}^{k-1}p_j(p_k+q_kz_k)\right)^n\\ &= \left(\sum_{j=1}^{k}q_jz_j\prod_{i=1}^{j-1}p_i+\prod_{j=1}^{k}p_j\right)^n,\end{aligned}$$

which coincides with (4.13) thus completing the proof of the induction step. □

Lemma 4.7 allows us to compute all the moments of the random vector of increments of the process $\mathcal{M}(t)$. We mention here only one such result on the correlation coefficients in the case where all the time increments are equal to each other such that $\Delta t_j=\Delta t$, $(j=1,2,\ldots,k)$. Let $p:=\exp(-\mu\Delta t)$, $q:=1-p$, and parameter $n$ be fixed.

**Corollary 4.8.** *The correlation coefficient $\rho_{m,m+\ell}$ of the increments $\Delta\mathcal{M}_{n,m}$ and $\Delta\mathcal{M}_{n,m+\ell}$ admits the following representation:*

$$\rho_{m,m+\ell}=-q\frac{p^{m+\ell/2-1}}{\sqrt{(1-qp^{m-1})(1-qp^{m+\ell-1})}}<0,\quad \ell=1,2,\ldots,k-m. \tag{4.16}$$

Note that as parameter $k\to\infty$ and $\Delta t\to 0$, the probability $q\to 0$. Hence, we obtain that in the limit, the increments of the process are uncorrelated.

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