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MRM-APPLICABLE ORTHOGONAL POLYNOMIALS FOR CERTAIN HYPERGEOMETRIC FUNCTIONS

IZUMI KUBO* AND HUI-HSIUNG KUO

ABSTRACT. The multiplicative renormalization method (MRM) is introduced to obtain generating functions of orthogonal polynomials of given probability measures. Complete lists of MRM-applicable measures for MRM-factors $h(x) = e^x$ and $(1-x)^{-\kappa}$ were obtained recently. On the other hand, it is known that gamma distributions have at least two types of MRM-factors $h(x) = {}_0F_1(-; \kappa; x)$ and $h(x) = {}_1F_1(c; \kappa; x)$. The usual MRM-factor e^x is a special case of ${}_1F_1(c; \kappa; x)$ when $c = \kappa$. We first determine all MRM-applicable measures for $h(x) = {}_0F_1(-; \kappa; x)$. Then we determine all possible MRM-factors of gamma distributions.

1. MRM-applicability of Orthogonal Polynomials

A probability measure μ on \mathbb{R} with density $f_\mu(x)$ is said to be applicable to the multiplicative renormalization method for $h(x)$ (or simply *MRM-applicable*), if there exists a suitable analytic function $\rho(t)$ around $t = 0$ with $\rho(0) = 0$, $r_1 = \rho'(0) \neq 0$ such that

$$(t, x) = \frac{h(\rho(t)x)}{\varphi(t)} \quad \text{with} \quad \varphi(t) = \theta(\rho(t)), \quad \theta(t) = \int_{\mathbb{R}} h(tx) d\mu(x) \quad (1.1)$$

is a generating function of the orthogonal polynomials $\{P_n(x)\}$ in $L^2(\mu)$ with leading coefficient of one. Then there exist Jacobi-Szegő parameters $\{\alpha_n, \omega_n\}$ satisfying the recursive relation

$$P_{n+1}(x) = (x - \alpha_n)P_n(x) - \omega_n P_{n-1}(x) \quad (1.2)$$

with $\omega_0 = 1, P_{-1}(x) = 0$. Eq. (1.2) is a necessary and sufficient condition for that $\{P_n(x)\}$ are orthogonal polynomials for some signed measure (see Shohat [18]). It is known that

$$\|P_n\|^2 = \lambda_n = \omega_0 \omega_1 \cdots \omega_n \quad \text{for } n \geq 0. \quad (1.3)$$

By Favard's Theorem [8], a set $\{P_n\}$ of polynomials with leading coefficient 1 is orthogonal for a probability measure if and only if they satisfy the recursion relation in Eq. (1.2) with $\omega_n > 0$ for all $n \geq 0$ (see [1] and [5]).

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Let us suppose that

$$h(x) = \sum_{n=0}^{\infty} h_n x^n, \quad h_0 = 1, \quad h_n \neq 0, \quad n \geq 1.$$

Then we have the expansion

$$(t, x) = \sum_{n=0}^{\infty} r_1^n h_n P_n(x) t^n. \quad (1.4)$$

In the previous papers [6] [12] [13] [14] [15], all MRM-applicable measures for $h(x) = e^x$ and $h(x) = (1-x)^{-\kappa}$ are determined.

A set of polynomials is said to be *MRM-applicable for $h(x)$* , if they are orthogonal for a probability measure and are given by the generating function in Eq. (1.1). In this article, we discuss MRM-applicability for $h(x) = {}_0F_1(-; \kappa; x)$. By Favard's Theorem, it is sufficient for the purpose to find ρ - and B -functions such that polynomials given by the following generating function in Eq. (1.5) satisfy Eq. (1.2) with $\omega_0 = 1$ and $\omega_n > 0$ for any $n \geq 0$. For convenience, we write a generating function in the form

$$(t, x) = B(t)h(\rho(t)x), \quad (1.5)$$

where $B(t) = \frac{1}{\theta(\rho(t))}$. This function (t, x) is called the *Boas-Buck generating function* (see §6 of [17]).

2. Hypergeometric Functions as MRM-factors

Hypergeometric function ${}_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; x)$ is defined by

$${}_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; x) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n n!} x^n,$$

where

$$(a)_n = a(a+1)(a+2) \cdots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

In the special case of $p = 0$ (or $p = 1, q = 0$), we denote as

$${}_0F_q(-; b_1, b_2, \dots, b_q; x) \quad (\text{respectively, } {}_1F_0(a_1; -; x)).$$

Let μ_κ be the gamma distribution with the density

$$f_\kappa(x) = \frac{1}{\Gamma(\kappa)} x^{\kappa-1} e^{-x}, \quad \text{for } x \geq 0. \quad (2.1)$$

Laguerre polynomials

$$L_n^{\kappa-1}(x) = \sum_{m=0}^n (-1)^m \frac{\Gamma(\kappa+n)}{\Gamma(m+\kappa)(n-m)!m!} x^m \quad (2.2)$$

are orthogonal for gamma distribution μ_κ and their Jacobi-Szegő parameters are

$$\alpha_n = \kappa + 2n \quad \text{and} \quad \omega_n = n(n + \kappa - 1), \quad \omega_0 = 1. \quad (2.3)$$

A generating function is given as

$$(1+t)^{-\kappa} e^{\frac{t}{1+t}x} = \sum_{n=0}^{\infty} (-1)^n L_n^{\kappa-1}(x) t^n \quad (2.4)$$

(see (7) and (17) in §10.12 of [7].) There are other generating functions given by hypergeometric functions as

$$e^{-t} {}_0F_1(-; \kappa; tx) = \sum_{n=0}^{\infty} (-1)^n \frac{L_n^{\kappa-1}(x)}{(\kappa)_n} t^n \tag{2.5}$$

(see Eq. (1) in §113 of [17] and Eq. (8) in §19.9 of [7]) and

$$\frac{1}{(1+t)^c} {}_1F_1\left(c; \kappa; \frac{t}{1+t}x\right) = \sum_{n=0}^{\infty} \frac{(c)_n (-1)^n L_n^{(\kappa-1)}(x)}{(\kappa)_n} t^n \quad \text{for any } c > 0 \tag{2.6}$$

(see Eq. (3) in §113 of [17]). Special choices of c as $c = 1$ and $c = \kappa$ give examples

$$\begin{aligned} {}_1F_1(1; \kappa; x) &= \sum_{n=0}^{\infty} \frac{\Gamma(\kappa)}{\Gamma(\kappa+n)} x^n, \\ {}_1F_1(\kappa; \kappa; x) &= \sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x. \end{aligned}$$

For convenience, we say that $h(x)$ is an *MRM-factor* of μ if μ is MRM-applicable for $h(x)$. We have the following chart.

Gamma distribution μ_κ and MRM-factor $h(x)$'s				
μ	$h(x)$	$\rho(t)$	$\varphi(t)$	(t, x)
μ_κ	e^x	$\frac{t}{1+t}$	$(1+t)^\kappa$	$(1+t)^{-\kappa} e^{\frac{tx}{1+t}}$
μ_κ	${}_0F_1(-; \kappa; x)$	t	e^t	$e^{-t} {}_0F_1(-; \kappa; tx)$
μ_κ	${}_1F_1(c; \kappa; x)$	$\frac{t}{1+t}$	$(1+t)^c$	$(1+t)^{-c} {}_1F_1(c; \kappa; \frac{tx}{1+t})$

Our purpose of this article is to find all possible probability measures (equivalently Jacobi-Szegö parameters), which are MRM-applicable for hypergeometric functions $h(x) = {}_0F_1(-; \kappa; x)$. Of course, gamma distribution is one of them as seen above.

Put

$$h(x) = {}_0F_1(-; \kappa; x) = \sum_{n=0}^{\infty} h_n x^n, \quad h_n = \frac{1}{n! (\kappa)_n}, \tag{2.7}$$

$$\rho(t) = \sum_{n=1}^{\infty} r_n t^n, \quad r_1 = 1; \quad B(t) = \sum_{n=0}^{\infty} b_n t^n, \quad b_0 = 1. \tag{2.8}$$

Let $\{P_n(x)\}$ be the polynomials given by

$$B(t)h(\rho(t)x) = \sum_{n=0}^{\infty} h_n P_n(x)t^n. \tag{2.9}$$

If P_n 's satisfy Eq. (1.2), then parameters $\{h_n, b_n, r_n, \alpha_n, \omega_n\}$ are related. We will see relationships between them in §3. Moreover it is shown in §3 that Jacobi-Szegő parameters $\{\alpha_n, \omega_n\}$, $B(t)$ and $\rho(t)$ satisfying Eq. (1.2) are parametrized by constants b_1, b_2, r_2 and r_3 , which are not independent. We give below some examples by using the results to be proved in §4.

We introduce a function $E(a, b; t)$ defined by

$$E(a, b; t) = \int_0^t \frac{1}{\sqrt{u(1+bu+au^2)}} du, \tag{2.10}$$

which can be obtained by the elliptic integral of the first kind. Here for u with $u(1+bu+au^2) < 0$, we use the convention $\sqrt{u(1+bu+au^2)} = i\sqrt{-u(1+bu+au^2)}$. Then it is very important to see that the singularity of $(E(a, b; t))^2$ at $t = 0$ is removable and it can be regarded analytic in t around $t = 0$. In this sense, we say that the function is analytic in t around $t = 0$ simply.

Example 2.1. If $b_2 = b_1^2, r_2 = 0, r_3 = 0$, then

$$\alpha_n = -b_1(2n + \kappa), \quad \omega_n = b_1^2 n(n + \kappa - 1), \quad B(t) = e^{b_1 t}, \quad \rho(t) = t.$$

In particular, if $b_1 = -1$, then this corresponds to Eq. (2.5).

Example 2.2. For the case of $\kappa = \frac{1}{2}$, we see

$$h(x) = {}_0F_1\left(-; \frac{1}{2}; x\right) = \begin{cases} \cosh 2\sqrt{x}, & \text{if } x \geq 0, \\ \cos 2\sqrt{-x}, & \text{if } x < 0. \end{cases}$$

By the convention $\sqrt{x} = i\sqrt{-x}$, $h(x)$ is written as $\cosh \sqrt{x}$ for both cases and it is analytic around $x = 0$. If we take as $b_1 = -\frac{1}{4}, b_2 = \frac{1}{8}, r_2 = -\frac{1}{3}, r_3 = \frac{23}{120}$, then

$$\alpha_n = \frac{1}{8}(2n + 1)^2, \quad \omega_n = \frac{1}{64}n^2(2n - 1)(2n + 1),$$

$$B(t) = \frac{4}{4+t}, \quad \rho(t) = 4 \left(\tan^{-1} \frac{\sqrt{t}}{2} \right)^2.$$

The corresponding generating function $(t, x) = \frac{4}{4+t} \cosh \left(4\sqrt{x} \tan^{-1} \frac{1}{2} \sqrt{t} \right)$ gives Wilson polynomials $\{P_n(x) = \frac{(-1)^n n!}{(2n)!} W_n(x; 0, \frac{1}{2}, \frac{1}{2}, 1)\}$, which are orthogonal with respect to the probability measure with the density $\frac{16}{\pi} \sqrt{x} |\Gamma(2ix)|^2$ (cf. §3.8 of [1]). In this case such a measure is not necessarily unique (see [19]).

Example 2.3. For the case of $\kappa = \frac{3}{2}$, we see

$$h(x) = {}_0F_1\left(-; \frac{3}{2}; x\right) = \begin{cases} \frac{1}{2\sqrt{x}} \sinh 2\sqrt{x}, & \text{if } x > 0, \\ 1, & \text{if } x = 0, \\ \frac{1}{2\sqrt{-x}} \sin 2\sqrt{-x}, & \text{if } x < 0. \end{cases}$$

If we take as $b_1 = -\frac{1}{12}$, $b_2 = \frac{1}{40}$, $r_2 = -\frac{1}{3}$, $r_3 = \frac{23}{120}$, then

$$\alpha_n = \frac{1}{8}(2n + 1)^2, \quad \omega_n = \frac{1}{64}n^2(2n - 1)(2n + 1),$$

$$B(t) = \frac{2}{\sqrt{t}} \tan^{-1} \frac{\sqrt{t}}{2}, \quad \rho(t) = 4 \left(\tan^{-1} \frac{\sqrt{t}}{2} \right)^2.$$

The corresponding generating function $(t, x) = \frac{1}{2\sqrt{tx}} \sinh(4\sqrt{x} \tan^{-1} \frac{1}{2}\sqrt{t})$ gives the same Wilson polynomials $P_n(x) = \frac{(-1)^n n!}{(2n)!} W_n(x; 0, \frac{1}{2}, \frac{1}{2}, 1)$ as in Example 2.2.

3. Lemmas

By Favard’s Theorem [8], a set $\{P_n\}$ of polynomials with leading coefficient 1 satisfies the recursion relation in Eq. (1.2)

$$P_{n+1}(x) = (x - \alpha_n)P_n(x) - \omega_n P_{n-1}(x), \quad n \geq 0,$$

with Jacobi-Szegö parameters $\{\alpha_n, \omega_n\}$ satisfying $P_{-1}(x) = 0, \alpha_{-1} = 0, \omega_0 = 1, \omega_n > 0$ for any $n \geq 0$, if and only if they are orthogonal polynomials with respect to a probability measure μ .

Through out this paper, we will assume that

$$h(x) = \sum_{n=0}^{\infty} h_n x^n, \quad \rho(t) = \sum_{n=0}^{\infty} r_n t^n, \quad B(t) = \sum_{n=0}^{\infty} b_n t^n. \tag{3.1}$$

We may normalize these functions so that $h(0) = h_0 = 1, B(0) = b_0 = 1, \rho(0) = r_0 = 0$, and $\rho'(0) = r_1 = 1$. For convenience, we put $h_{-1} = b_{-1} = 0$. Suppose that $(t, x) = B(t)h(\rho(t)x)$ is a generating function of $\{P_n\}$, i.e.,

$$(t, x) = B(t)h(\rho(t)x) = \sum_{n=0}^{\infty} h_n P_n(x) t^n. \tag{3.2}$$

Put

$$\theta(t) = \int_{\mathbb{R}} h(tx) d\mu(x), \quad \varphi(t) = \theta(\rho(t)).$$

Obviously, we have

$$B(t) = \frac{1}{\varphi(t)}. \tag{3.3}$$

Define

$$W_n(x) = P_n(x) - (x - \alpha_{n-1})P_{n-1}(x) + \omega_{n-1}P_{n-2}(x) \tag{3.4}$$

and let $W_{n,m}$ be the m -th coefficient of $W_n(x)$ for $n \geq m \geq 0$. Then

$$W_{n+1,m} = c_{n+1,m} - c_{n,m-1} + \alpha_n c_{n,m} + \omega_n c_{n-1,m} \tag{3.5}$$

for $n \geq m \geq 0$. Since $W_n(x) = 0$ must hold, all $W_{n,m}$ must vanish. Let $B_{m,k}$ be the coefficient of $B(t)\rho^m(t)$, $m \geq 0$. Then $B_{m,k} = 0$ for $m > k$. Hence

$$B(t)\rho^m(t) = \sum_{k=0}^{\infty} B_{m,k} t^k = \sum_{k=m}^{\infty} B_{m,k} t^k. \tag{3.6}$$

The following Lemmas 3.1, 3.2, and 3.3 are shown in [10].

Lemma 3.1. *The following equalities hold:*

$$\begin{aligned}
W_{n+1,0} &= \frac{b_{n+1}}{h_{n+1}} + \alpha_n \frac{b_n}{h_n} + \omega_n \frac{b_{n-1}}{h_{n-1}}, \\
W_{n+1,1} &= \frac{h_1}{h_{n+1}} r_{n+1} + \frac{h_1}{h_{n+1}} \sum_{k=1}^n b_{n+1-k} r_k - \frac{b_n}{h_n} \\
&\quad + \alpha_n \frac{h_1}{h_n} \sum_{k=1}^n b_{n-k} r_k + \omega_n \frac{h_1}{h_{n-1}} \sum_{k=1}^{n-1} b_{n-1-k} r_k, \\
W_{n+1,m} &= \frac{h_m}{h_{n+1}} B_{m,n+1} - \frac{h_{m-1}}{h_n} B_{m-1,n} + \alpha_n \frac{h_m}{h_n} B_{m,n} + \omega_n \frac{h_m}{h_{n-1}} B_{m,n-1}, \\
W_{n+1,n-1} &= \frac{h_{n-1}}{2h_{n+1}} (2b_2 + 2(n-1)(b_1 r_2 + r_3) + (n-1)(n-2)r_2^2) \\
&\quad - \frac{h_{n-2}}{2h_n} (2b_2 + 2(n-2)(b_1 r_2 + r_3) + (n-2)(n-3)r_2^2) \\
&\quad + \frac{h_{n-1}}{h_n} (b_1 + (n-1)r_2) \alpha_n + \omega_n, \\
W_{n+1,n} &= \frac{h_n}{h_{n+1}} (b_1 + nr_2) - \frac{h_{n-1}}{h_n} (b_1 + (n-1)r_2) + \alpha_n.
\end{aligned}$$

Moreover, μ is MRM-applicable for $h(x)$ if and only if $W_{n,m} = 0$ for all $n-1 \geq m \geq 0$ and $\omega_n > 0$ for all $n \geq 0$.

Lemma 3.2. *For given $\{\alpha_n, \omega_n\}$ and $\{b_1, r_2, r_3\}$, the recursion formulas*

$$\begin{aligned}
h_{n+1} &= \frac{h_n^2 (b_1 + nr_2)}{h_{n-1} (b_1 + (n-1)r_2) - h_n \alpha_n}, \\
b_{n+1} &= -\frac{h_{n+1}}{h_n} \alpha_n b_n - \frac{h_{n+1}}{h_{n-1}} \omega_n b_{n-1}, \\
r_{n+1} &= \frac{b_n h_{n+1}}{h_1 h_n} - \sum_{m=1}^n b_{n+1-m} r_m - \frac{h_{n+1}}{h_n} \alpha_n \sum_{m=1}^n b_{n-m} r_m \\
&\quad - \frac{h_{n+1}}{h_{n-1}} \omega_n \sum_{m=1}^{n-1} b_{n-1-m} r_m,
\end{aligned}$$

hold for $n \geq 1$ if $h_{n-1} (b_1 + (n-1)r_2) - h_n \alpha_n \neq 0$ and $h_1 = -\frac{b_1}{\alpha_0}$ if $\alpha_0 \neq 0$.

Lemma 3.3. *For given $h(x)$ and fixed $\{b_1, b_2, r_2, r_3\}$, the Jacobi-Szegő parameters are uniquely determined by*

$$\begin{aligned}
\alpha_n &= -\frac{h_n}{h_{n+1}} (b_1 + nr_2) + \frac{h_{n-1}}{h_n} (b_1 + (n-1)r_2), \quad n \geq 1, \\
\omega_n &= \frac{h_{n-1}}{2h_{n+1}} (2(b_1^2 - b_2 + b_1 r_2) + 2(n-1)(b_1 r_2 - r_3) + (n-1)(n+2)r_2^2) \\
&\quad + \frac{h_{n-2}}{2h_n} (2b_2 + 2(n-2)(b_1 r_2 + r_3) + (n-2)(n-3)r_2^2) \\
&\quad - \left(\frac{h_{n-1}}{h_n} (b_1 + (n-1)r_2) \right)^2, \quad n \geq 2,
\end{aligned}$$

and $\alpha_0 = -\frac{b_1}{h_1}$, $\omega_0 = 1$, $\omega_1 = \frac{b_1^2 - b_2 + b_1 r_2}{h_2} - \frac{b_1^2}{h_1^2}$. Furthermore, the recursion formulas for $\{b_n, r_n\}$ are given by

$$\begin{aligned} b_{n+1} &= -\frac{h_{n+1}}{h_n} \alpha_n b_n - \frac{h_{n+1}}{h_{n-1}} \omega_n b_{n-1}, \quad n \geq 2, \\ r_{n+1} &= \frac{b_n h_{n+1}}{h_1 h_n} - \sum_{m=1}^n b_{n+1-m} r_m - \frac{h_{n+1}}{h_n} \alpha_n \sum_{m=1}^n b_{n-m} r_m \\ &\quad - \frac{h_{n+1}}{h_{n-1}} \omega_n \sum_{m=1}^{n-1} b_{n-1-m} r_m, \quad n \geq 2. \end{aligned}$$

Lemma 3.4. Suppose $A_n = \sum_{j=0}^n \alpha_j \neq 0$ for all $n \geq 0$. Then for given $\{\alpha_n, \omega_n\}$ and $\{b_1, r_2, r_3\}$, we have

$$\frac{h_{n+1}}{h_n} = -\frac{b_1 + nr_2}{\sum_{j=0}^n \alpha_j} \tag{3.7}$$

and

$$\begin{aligned} h_n &= (-1)^n \prod_{k=0}^{n-1} \frac{b_1 + kr_2}{A_k}, \quad n \geq 1, \\ b_{n+1} &= \frac{b_1 + nr_2}{A_n} \alpha_n b_n - \frac{(b_1 + nr_2)(b_1 + (n-1)r_2)}{A_{n-1} A_n} \omega_n b_{n-1}, \quad n \geq 1, \\ r_{n+1} &= \frac{b_1 + nr_2}{h_1 A_n} b_n - \sum_{m=1}^n b_{n+1-m} r_m + \frac{b_1 + nr_2}{A_n} \alpha_n \sum_{m=1}^n b_{n-m} r_m \\ &\quad - \frac{(b_1 + nr_2)(b_1 + (n-1)r_2)}{A_{n-1} A_n} \omega_n \sum_{m=1}^{n-1} b_{n-1-m} r_m, \quad n \geq 2. \end{aligned}$$

Proof. By Lemma 3.2, we have

$$(b_1 + nr_2) \frac{h_n}{h_{n+1}} = (b_1 + (n-1)r_2) \frac{h_{n-1}}{h_n} - \alpha_n.$$

Since $\frac{h_0}{h_1} b_1 = -\alpha_0$, we see that $(b_1 + nr_2) \frac{h_n}{h_{n+1}} = -\sum_{j=0}^n \alpha_j$, which gives Eq. (3.7). But $h_0 = 1$. Hence we have the formula for h_n . By applying Eq. (3.7) to Lemma 3.2, we obtain the formulas for b_{n+1} and r_{n+1} as stated in the lemma. \square

Lemma 3.5. If $\alpha_n = 0$ for all $n \geq 0$, then $r_{2n} = 0$, $b_{2n+1} = 0$ for all $n \geq 0$. For given $\{\alpha_n, \omega_n\}$ and $\{h_1, b_2, r_3\}$, the following recursion formulas hold,

$$\begin{aligned} h_{n+1} &= \frac{(b_2 + (n-1)r_3)h_{n-1}h_n}{(b_2 + (n-2)r_3)h_{n-2} - h_n \omega_n}, \quad n \geq 2 \\ b_{n+1} &= -\frac{h_{n+1}}{h_{n-1}} \omega_n b_{n-1}, \quad n \geq 2 \\ r_{n+1} &= \frac{b_n h_{n+1}}{h_1 h_n} - \sum_{m=1}^n b_{n+1-m} r_m - \frac{h_{n+1}}{h_{n-1}} \omega_n \sum_{m=1}^{n-1} b_{n-1-m} r_m, \quad n \geq 2. \end{aligned}$$

Furthermore, let $\Omega_n = \sum_{j=1}^n \omega_j$. Then for $n \geq 1$, we have

$$\frac{h_{n+1}}{h_{n-1}} = -\frac{b_2 + (n-1)r_3}{\Omega_n}, \tag{3.8}$$

$$\begin{aligned}
 h_{2n} &= (-1)^n \prod_{k=1}^n \frac{(b_2 + 2(k-1)r_3)}{\Omega_{2k-1}}, \\
 h_{2n+1} &= (-1)^n h_1 \prod_{k=1}^n \frac{(b_2 + (2k-1)r_3)}{\Omega_{2k}},
 \end{aligned}
 \tag{3.9}$$

$$b_{2n} = (-1)^n h_{2n} \prod_{k=1}^n \omega_{2k-1} = \prod_{k=1}^n \frac{\omega_{2k-1}(b_2 + 2(k-1)r_3)}{\Omega_{2k-1}},
 \tag{3.10}$$

$$\begin{aligned}
 r_{2n+1} &= \prod_{k=1}^n \frac{\omega_{2k-1}(b_2 + (2k-1)r_3)}{\Omega_{2k}} - \sum_{m=1}^n b_{2(n+1-m)} r_{2m-1} \\
 &\quad + \frac{\omega_{2n}(b_2 + (2n-1)r_3)}{\Omega_{2n}} \sum_{m=1}^n b_{2(n-m)} r_{2m-1}.
 \end{aligned}
 \tag{3.11}$$

Proof. Suppose $\alpha_n = 0$ for all $n \geq 0$. Since $W_{1,0} = \frac{b_1}{h_1} = 0$, we have $b_1 = 0$. But $W_{2,0} = \frac{b_2+h_2\omega_1}{h_2} = 0$ and $W_{2,1} = \frac{h_1r_2}{h_2} = 0$. Hence $r_2 = 0$ and $h_2 = -\frac{b_2}{\omega_1}$. Solving $W_{n+1,n-1} = 0$ for h_{n+1} , we obtain the first recursion formula in the lemma for h_{n+1} . Then we can rewrite it as

$$\frac{h_{n-1}}{h_{n+1}}(b_2 + (n-1)r_3) = \frac{h_{n-2}}{h_n}(b_2 + (n-2)r_3) - \omega_n,$$

which yields the equality

$$\frac{h_{n-1}}{h_{n+1}}(b_2 + (n-1)r_3) = - \sum_{j=1}^n \omega_j.$$

Therefore we have Eq. (3.8). The other equalities are obtained from Lemma 3.2. Since $b_1 = 0$ and $b_{n+1} = -\frac{h_{n+1}}{h_{n-1}}\omega_n b_{n-1}$, we see that $b_{2n+1} = 0$ for all $n \geq 0$. Assume that $r_{2j} = 0$ for $n \geq j \geq 0$, then

$$r_{2n+2} = \frac{b_{2n+1}h_{2n+2}}{h_1h_{2n+1}} - \sum_{m=1}^{2n+1} b_{2n+2-m}r_m - \frac{h_{2n+2}}{h_{2n}}\omega_{2n-1} \sum_{m=1}^{2n} b_{2n-m}r_m = 0.$$

Thus by induction we have $r_{2n} = 0$ for all $n \geq 0$. □

Remark 3.6. (i) We have normalized the constants so that

$$h_0 = 1, \quad b_0 = 1, \quad r_1 = 1.
 \tag{3.9}$$

This does not impose essential restrictions. Since $\theta(0) = h(0)$, $B(0)h(0) = b_0h_0 = 1$ holds. Suppose that $(t, x) = B(t)h(\rho(t)x)$ is a generating function for orthogonal polynomials. If $h_0 \neq 1$, put $\widehat{h}(x) = \frac{h(x)}{h(0)}$ and $\widehat{B}(t) = h(0)B(t)$. Then $\widehat{(t, x)} = \widehat{B}(t)\widehat{h}(\rho(t)x) = (t, x)$ is a generating function satisfying Eq. (3.9). On the other hand, if $r_1 \neq 1$, then put $\alpha = \frac{1}{r_1}$, $\widehat{\rho}(t) = \rho(\alpha t)$ and $\widehat{b}(t) = B(\alpha t)$. Then $\widehat{(t, x)} = \widehat{B}(t)h(\widehat{\rho}(t)x)$ is a generating function satisfying Eq. (3.9).

(ii) Suppose $(t, x) = B(t)h(\rho(t)x)$ is a generating function of $\{P_n\}$. For given $\tilde{h}_1, \tilde{r}_1 \neq 0$, the scaling transforms

$$\tilde{h}(x) = h\left(\frac{\tilde{h}_1}{h_1}x\right), \quad \tilde{\rho}(t) = \frac{h_1}{\tilde{h}_1}\rho\left(\frac{\tilde{h}_1\tilde{r}_1}{h_1r_1}t\right), \quad \tilde{B}(t) = B\left(\frac{\tilde{h}_1\tilde{r}_1}{h_1r_1}t\right), \tag{3.10}$$

give a modified generating function

$$\tilde{\sim}(t, x) = \tilde{B}(t)\tilde{h}(\tilde{\rho}(t)x) = \left(\frac{\tilde{h}_1\tilde{r}_1}{h_1r_1}t, x\right)$$

satisfying $\tilde{h}'(0) = \tilde{h}_1$ and $\tilde{r}'(0) = \tilde{r}_1$.

Now let us apply the above lemmas to the case $h(x) = {}_0F_1(-; \kappa; x)$, namely, $h_n = \frac{1}{n!(\kappa)_n}$ to obtain the Jacobi-Szegö parameters $\{\alpha_n, \omega_n\}$.

Lemma 3.7. For $h_n = \frac{1}{n!(\kappa)_n}$ and given b_1, b_2, r_2, r_3 , the Jacobi-Szegö parameters $\{\alpha_n, \omega_n\}$ are given by

$$\begin{aligned} \alpha_n &= -(3r_2n^2 + (2b_1 + (2\kappa - 1)r_2)n + \kappa b_1), \quad n \geq 0, \\ \omega_n &= n(\kappa + n - 1)\left((8r_2^2 - 5r_3)n^2 + (2b_1^2 - 4b_2 + 6b_1r_2 + 2(2\kappa - 7)r_2^2 - 3(\kappa - 3)r_3)\right. \\ &\quad \left. n + \kappa b_1^2 - 2(\kappa - 1)b_2 + 2(\kappa - 2)b_1r_2 - 2(2\kappa - 3)r_2^2 + (3\kappa - 4)r_3\right), \quad n \geq 2, \end{aligned}$$

and $\omega_0 = 1, \omega_1 = \kappa(-2(\kappa + 1)b_2 + (\kappa + 2)b_1^2 + 2(\kappa + 1)b_1r_2)$. Furthermore, $\tilde{W}_{n,m} \equiv \frac{h_n}{h_m}n(n + \kappa - 1)W_{n,m}$ are given by

$$\begin{aligned} \tilde{W}_{n,m} &= (n(n - 1) + \kappa n)B_{m,n} - m(m + \kappa - 1)B_{m-1,n-1} \\ &\quad - (3r_2(n - 1)(n - 2) + 2(b_1 + (\kappa + 1)r_2)(n - 1) + \kappa b_1)B_{m,n-1} \\ &\quad + \left((8r_2^2 - 5r_3)(n - 2)(n - 3) + (2b_1^2 - 4b_2 + 6b_1r_2 + 2(2\kappa + 5)r_2^2 - 3(\kappa + 2)r_3)(n - 2)\right. \\ &\quad \left. (\kappa + 2)b_1^2 - 2(\kappa + 1)(b_2 - b_1r_2)\right)B_{m,n-2}. \end{aligned}$$

4. Classification by Using Differential Equations

Put $B_m(t) = B(t)\rho(t)^m$, then by Eq. (3.6),

$$B_m(t) = \sum_{n=m}^{\infty} B_{m,n}t^n. \tag{4.1}$$

It is easily seen that

$$\sum_{n=m}^{\infty} n^p B_{m,n-q}t^n = t^q \left(t \frac{d}{dt} + q\right)^p B_m(t)$$

and

$$\sum_{n=m}^{\infty} (n - q)(n - q - 1) \cdots (n + 1 - q - p) B_{m,n-q}t^n = t^q \left(\frac{d}{dt}\right)^p B_m(t) \tag{4.2}$$

for non-negative integers p and q .

On the other hand, multiply $n(n + \kappa - 1)t^n$ to $\widetilde{W}_{n,m}$ in Lemma 3.7 and take the summation. Since $B_{m,n} = 0$ for $m > n$, we get

$$\begin{aligned}
 \text{(i)} \quad & \sum_{n=1}^{\infty} (n(n-1) + \kappa n) B_{m,n} t^n = t^2 B_m''(t) + \kappa t B_m'(t), \\
 \text{(ii)} \quad & \sum_{n=1}^{\infty} m(m + \kappa - 1) B_{m-1,n-1} t^n = m(m + \kappa - 1) t B_{m-1}(t), \\
 \text{(iii)} \quad & \sum_{n=1}^{\infty} (3r_2(n-1)(n-2) + 2(b_1 + (\kappa + 1)r_2)(n-1) + \kappa b_1) B_{m,n-1} t^n \\
 & = 3r_2 t^3 B_m''(t) + 2(b_1 + (\kappa + 1)r_2) t^2 B_m'(t) + \kappa b_1 t B_m(t), \\
 \text{(iv)} \quad & \sum_{n=2}^{\infty} \left((8r_2^2 - 5r_3)(n-2)(n-3) + (2b_1^2 - 4b_2 + 6b_1 r_2 + 2(2\kappa + 5)r_2^2 \right. \\
 & \quad \left. - 3(\kappa + 2)r_3)(n-2) + (\kappa + 2)b_1^2 - 2(\kappa + 1)(b_2 - b_1 r_2) \right) B_{m,n-2} t^n \\
 & = (8r_2^2 - 5r_3) t^4 B_m''(t) + (2b_1^2 - 4b_2 + 6b_1 r_2 + 2(2\kappa + 5)r_2^2 \\
 & \quad - 3(\kappa + 2)r_3) t^3 B_m'(t) + ((\kappa + 2)b_1^2 - 2(\kappa + 1)(b_2 - b_1 r_2)) t^2 B_m(t).
 \end{aligned}$$

Since $\widetilde{W}_{n,m}$ must vanish, we have a differential equation

$$\begin{aligned}
 & -m(m + \kappa - 1) B_{m-1}(t) + (-\kappa b_1 + ((\kappa + 2)b_1^2 - 2(\kappa + 1)(b_2 - b_1 r_2))t) B_m(t) \\
 & + (\kappa - 2(b_1 + (\kappa + 1)r_2)t + (2b_1^2 - 4b_2 + 6b_1 r_2 + 2(2\kappa + 5)r_2^2 \\
 & - 3(\kappa + 2)r_3)t^2) B_m'(t) + t(1 - 3r_2 t + (8r_2^2 - 5r_3)t^2) B_m''(t) = 0
 \end{aligned}$$

by Lemma 3.3 and Eq. (4.2). Thus we have proved the next lemma.

Lemma 4.1. *Let $c_m = -m(m + \kappa - 1)$ and*

$$\begin{aligned}
 p_0(t) &= -\kappa b_1 + ((\kappa + 2)b_1^2 - 2(\kappa + 1)(b_2 - b_1 r_2))t, \\
 p_1(t) &= (\kappa - 2(b_1 + (\kappa + 1)r_2)t + (2b_1^2 - 4b_2 + 6b_1 r_2 + 2(2\kappa + 5)r_2^2 \\
 & \quad - 3(\kappa + 2)r_3)t^2), \\
 p_2(t) &= t(1 - 3r_2 t + (8r_2^2 - 5r_3)t^2).
 \end{aligned}$$

Then the differential equation

$$p_0(t) B_m(t) + p_1(t) B_m'(t) + p_2(t) B_m''(t) + c_m B_{m-1}(t) = 0 \tag{4.3}$$

holds for all $m \geq 0$ if and only if $\widetilde{W}_{n,m} = 0$ holds for all $n \geq m \geq 0$.

Now, we will first obtain conditions on parameters b_1, b_2, r_2 and r_3 under which all $\widetilde{W}_{n,m}$ vanish. From Eq. (4.3), we have for $m = 0$,

$$D_0 = p_0(t) B(t) + p_1(t) B'(t) + p_2(t) B''(t) = 0 \tag{4.4}$$

and for $m = 1$,

$$p_0(t) B(t) \rho(t) + p_1(t) (B(t) \rho(t))' + p_2(t) (B(t) \rho(t))'' + c_1 B(t) = 0.$$

Then we use Eq. (4.4) to show that

$$\begin{aligned}
 & p_0(t) B(t) \rho(t) + p_1(t) (B(t) \rho(t))' + p_2(t) (B(t) \rho(t))'' \\
 & = (p_0(t) B(t) + p_1(t) B'(t) + p_2(t) B''(t)) \rho(t)
 \end{aligned}$$

$$\begin{aligned}
 &+ (p_1(t)B(t) + 2p_2(t)B'(t))\rho'(t) + p_2(t)B(t)\rho''(t) \\
 &= (p_1(t)B(t) + 2p_2(t)B'(t))\rho'(t) + p_2(t)B(t)\rho''(t) + c_1B(t),
 \end{aligned}$$

which implies that

$$D_1 = (p_2(t)\rho''(t) + p_1(t)\rho'(t) - \kappa)B(t) + 2p_2(t)\rho'(t)B'(t) = 0. \tag{4.5}$$

For $m = 2$ in Eq. (4.3), we use Eqs. (4.4) and (4.5) and the fact that $c_2 + 2\kappa = -2$ to show that

$$\begin{aligned}
 &p_0(t)B(t)\rho^2(t) + p_1(t)(B(t)\rho^2(t))' + p_2(t)(B(t)\rho^2(t))'' + c_2B(t)\rho(t) \\
 &= (p_0(t)B(t) + p_1(t)B'(t) + p_2(t)B''(t))\rho^2(t) \\
 &\quad + 2((p_2(t)\rho''(t) + p_1(t)\rho'(t) - \kappa)B(t) + 2p_2(t)\rho'(t)B'(t))\rho(t) \\
 &\quad + 2p_2(t)\rho'(t)^2B(t) + (c_2 + 2\kappa)\rho(t)B(t) \\
 &= 2(p_2(t)\rho'(t)^2 - \rho(t))B(t).
 \end{aligned}$$

Therefore

$$D_2 = p_2(t)\rho'(t)^2 - \rho(t) = 0. \tag{4.6}$$

Since $\rho(0) = 0$, the solution $\rho(t)$ of Eq. (4.6) is given by

$$\rho(t) = \left(\frac{1}{2}E(p_2; t)\right)^2, \quad E(p_2; t) = \int_0^t \frac{du}{\sqrt{p_2(u)}}, \tag{4.7}$$

where $E(p_2; t)$ can be represented by using an elliptic integral. Note that $\rho(t)$ is analytic around $t = 0$ in view of Eq. (2.10). From (4.6), we see that $\frac{1}{p_2(t)\rho'(t)} = \frac{\rho'(t)}{\rho(t)}$. Hence by Eq. (4.5),

$$\frac{B'(t)}{B(t)} = -\frac{1}{2}\frac{p_1(t)}{p_2(t)} - \frac{1}{2}\frac{\rho''(t)}{\rho'(t)} + \frac{\kappa}{2}\frac{\rho'(t)}{\rho(t)}.$$

Integrate both sides of this equation to obtain

$$\begin{aligned}
 B(t) &= \frac{1}{\sqrt{\rho'(t)}}\rho(t)^{\kappa/2} \exp\left[-\frac{1}{2}\int \frac{p_1(t)}{p_2(t)} dt\right] \\
 &= 2^{-\kappa+1/2}p_2(t)^{1/4}E(p_2; t)^{\kappa-1/2} \exp\left[-\frac{1}{2}\int \frac{p_1(t)}{p_2(t)} dt\right].
 \end{aligned} \tag{4.8}$$

Here we have used Eq. (4.7) with a suitable integral constant satisfying $B(+0) = 1$. Thus $\rho(t)$ and $B(t)$ are determined explicitly. However, it is not guaranteed that $B(t)$ satisfies (4.4). We will discuss the conditions on parameters b_1, b_2, r_2 and r_3 later. Now we prove that the differential equations in Eq. (4.3) hold for all $k \geq 0$ if the three equations $D_0 = 0, D_1 = 0, D_2 = 0$ are satisfied.

Theorem 4.2. *Assume that the following differential equations hold,*

$$\begin{aligned}
 D_0 &= p_0(t)B(t) + p_1(t)B'(t) + p_2(t)B''(t) = 0, \\
 D_1 &= (p_2(t)\rho''(t) + p_1(t)\rho'(t) - \kappa)B(t) + 2p_2(t)\rho'(t)B'(t) = 0, \\
 D_2 &= p_2(t)\rho'(t)^2 - \rho(t) = 0.
 \end{aligned}$$

Then

$$\tilde{D}_k = p_0(t)B_k(t) + p_1(t)B'_k(t) + p_2(t)B''_k(t) + c_k B_{k-1}(t) = 0$$

holds for all $k \geq 0$.

Proof. Observe the equalities

$$\begin{aligned}
& \tilde{D}_k - \rho^k(t)D_0 - k\rho^{k-1}(t)D_1 \\
&= p_0(t)\rho^k(t)B(t) + p_1(t)(\rho^k(t)B(t))' + p_2(t)(\rho^k(t)B(t))'' + c_k\rho^{k-1}(t)B(t) \\
&\quad - (p_0(t)\rho^k(t)B(t) + p_1(t)\rho^k(t)B'(t) + p_2(t)\rho^k(t)B''(t)) - k\rho^{k-1}(t)D_1 \\
&= \left(kp_1(t)\rho^{k-1}(t)\rho'(t)B(t) + 2kp_2(t)\rho^{k-1}(t)\rho'(t)B'(t) \right. \\
&\quad \left. + k(k-1)p_2(t)\rho^{k-2}(t)\rho'(t)^2B(t) + kp_2(t)\rho^{k-1}(t)\rho''(t)B(t) \right) \\
&\quad - \left(kp_2(t)\rho^{k-1}(t)\rho''(t)B(t) + kp_1(t)\rho^{k-1}(t)\rho'(t)B(t) \right. \\
&\quad \left. - k\rho^k(t)B(t) + 2kp_2(t)\rho^{k-1}(t)\rho'(t)B'(t) \right) + c_k\rho^{k-1}(t)B(t) \\
&= k(k-1)\rho^{k-1}(t)B(t)(p_2(t)\rho'(t)^2 - \rho(t)) \\
&= k(k-1)\rho^{k-1}(t)B(t)D_2 \\
&= 0.
\end{aligned}$$

Hence we see the assertion. \square

Note that

$$\begin{aligned}
B'(t) &= \left(\frac{p_2'(t)}{4p_2(t)} - \frac{p_1(t)}{2p_2(t)} + \frac{2\kappa - 1}{2\sqrt{p_2(t)}E(p_2; t)} \right) B(t), \\
B''(t) &= \left(\frac{p_2'(t)}{4p_2(t)} - \frac{p_1(t)}{2p_2(t)} + \frac{2\kappa - 1}{2\sqrt{p_2(t)}E(p_2; t)} \right)^2 B(t) \\
&\quad + \frac{p_2''(t)p_2(t) - p_2'(t)^2}{4p_2(t)^2} - \frac{p_1'(t)p_2(t) - p_1(t)p_2'(t)}{2p_2(t)^2} \\
&\quad - \left(\frac{(2\kappa - 1)p_2'(t)}{4(\sqrt{p_2(t)})^3 E(p_2; t)} - \frac{(2\kappa - 1)}{2p_2(t)E(p_2; t)^2} \right) B(t),
\end{aligned}$$

Hence from Eqs. (4.4) and (4.8), we have

$$\begin{aligned}
D_0 &= \frac{1}{16p_2(t)E(p_2; t)^2} \left(4(2\kappa - 1)(2\kappa - 3)p_2(t) + \left(-4p_1^2(t) + 16p_0(t)p_2(t) \right. \right. \\
&\quad \left. \left. - 8p_2(t)p_1'(t) + 8p_1(t)p_2'(t) - 3p_2'(t)^2 + 4p_2(t)p_2''(t) \right) E(p_2; t)^2 \right).
\end{aligned}$$

Since $p_0(t)$, $p_1(t)$, $p_2(t)$ are polynomials in t , the condition $D_0 = 0$ is equivalent to that both of $(2\kappa - 1)(2\kappa - 3) = 0$ and

$$-4p_1^2(t) + 16p_0(t)p_2(t) - 8p_2(t)p_1'(t) + p_1(t)p_2'(t) - 3p_2'(t)^2 + 4p_2(t)p_2''(t) = 0$$

hold if $E(p_2; t)^2$ is not a polynomial in t . On the other hand, $E(p_2; t)^2$ is a polynomial if and only if $p_2(t) = t$. But the condition $p_2(t) = t(1 + 3r_2t + (8r_2^2 - 5r_3)t^2) = t$ is equivalent to $r_2 = r_3 = 0$. Therefore, we have the following classification:

(I) $r_2 = r_3 = 0$.

Then $D_0 = 0$ implies that $(b_1^2 - 2b_2)((b_1^2 - 2b_2)t - 2b_1) = 0$, which is equivalent to $b_2 = \frac{1}{2}b_1^2$.

(II) $\kappa = \frac{1}{2}$.Then $D_0 = 0$ implies that

$$\begin{aligned} 4b_1^3 - 8b_1b_2 + 3b_1^2r_2 + 6b_2r_2 - 16b_1r_2^2 + 10b_1r_3 &= 0, \\ 2b_1^4 - 8b_1^2b_2 + 8b_2^2 + 12b_1^3r_2 - 24b_1b_2r_2 - 14r_1^2r_2^2 + 32b_2r_2^2 - 24b_1r_2^3 \\ + (20b_1^2 - 20b_2 + 15b_1r_2)r_3 &= 0. \end{aligned}$$

If $b_1 \neq 0$, then

$$\begin{aligned} r_3 &= \frac{1}{10b_1}(-4b_1^3 + 8b_1b_2 - 3b_1^2r_2 - 6b_2r_2 + 16b_1r_2^2), \\ (3b_1^2 - 2b_2)(2b_1 - 3r_2)(2b_1^2 - 4b_2 + 3b_1r_2) &= 0. \end{aligned}$$

Hence the case $\kappa = \frac{1}{2}$ is classified as:(1) $b_1 = 0$.Then $D_0 = 0$ implies that $b_2(-3r_2 + 2(2b_2 + 8r_2^2 - 5r_3)t) = 0$. Hence either $b_2 = 0$ or $r_2 = 0$, $r_3 = \frac{2}{5}(b_2 + 4r_2^2)$ must hold. Therefore,

- (i) $b_1 = b_2 = 0$,
- (ii) $b_1 = 0$, $r_2 = 0$, $r_3 = \frac{2}{5}(b_2 + 4r_2^2)$.

(2) $r_3 = \frac{1}{10b_1}(-4b_1^3 + 8b_1b_2 - 3b_1^2r_2 - 6b_2r_2 + 16b_1r_2^2)$.Then $D_0 = 0$ implies that $(3b_1^2 - 2b_2)(2b_1 - 3r_2)(2b_1^2 - 4b_2 + 3b_1r_2) = 0$.

Hence this case is classified as:

- (i) $b_2 = \frac{3b_1^2}{2}$,
- (ii) $r_2 = \frac{2b_1}{3}$,
- (iii) $r_2 = -\frac{2(b_1^2 - 2b_2)}{3b_1}$.

(III) $\kappa = \frac{3}{2}$.For this case, $D_0 = 0$ implies that

$$\begin{aligned} -4b_1^3 + 8b_1b_2 - b_1^2r_2 - 10b_2r_2 + 24b_1r_2^2 - 12r_2^3 - 14b_1r_3 + 10r_2r_3 &= 0, \\ 2b_1^4 - 8b_1^2b_2 + 8b_2^2 + 12b_1^3r_2 - 24b_1b_2r_2 - 22b_1^2r_2^2 + 48b_2r_2^2 - 32b_1r_2^3 \\ + 24r_2^4 + 24b_1^2r_3 - 28b_2r_3 + 17b_1r_2r_3 - 34r_2^2r_3 + 12r_3^2 &= 0. \end{aligned}$$

If $7b_1 - 5r_2 \neq 0$, then from the first equation we have

$$r_3 = \frac{1}{2(7b_1 - 5r_2)}(-4b_1^3 + 8b_1b_2 - b_1^2r_2 - 10b_2r_2 + 24b_1r_2^2 - 12r_2^3).$$

Thus this case is classified as:

(1) $r_2 = \frac{7}{5}b_1$.Then $D_0 = 0$ implies that either

$$b_2 = \frac{363}{250}b_1^2, (11271b_1^2 - 500r_3)(338b_1^2 - 125r_3) = 0$$

or

$$b_1 = 0, (b_2 - 3r_3)(2b_2 - r_3) = 0.$$

Therefore, this case is classified as:

- (i) $b_2 = \frac{363}{250}b_1^2$, $r_3 = \frac{1127}{500}b_1^2$,
- (ii) $b_2 = \frac{363}{250}b_1^2$, $r_3 = \frac{338}{125}b_1^2$,
- (iii) $b_1 = 0$, $r_3 = 2b_2$,

- (iv) $b_1 = 0, r_3 = \frac{1}{3}b_2.$
- (2) $r_3 = \frac{3}{7b_1 - 5r_2}(-4b_1^3 + 8ab_1b_2 - b_1^2r_2 - 10b_2r_2 + 24b_1r_2^2 - 12r_2^3).$
Then $D_0 = 0$ implies that

$$(b_1 - 2r_2)(2b_1 - r_2)(10b_1^2 - 20b_2 + 15b_1r_2 - r_2^2) \times (19b_1^2 - 10b_2 - 20b_1r_2 + 12r_2^2) = 0.$$

Hence this case is classified as:

- (i) $r_2 = \frac{b_1}{2},$
- (ii) $r_2 = 2b_1,$
- (iii) $b_2 = \frac{1}{20}(10b_1^2 + 15b_1r_2 - r_2^2),$
- (iv) $b_2 = \frac{1}{10}(19b_1^2 - 20b_1r_2 + 12r_2^2).$

For each case, we can obtain Jacobi-Szegő parameters $\{\alpha_n, \omega_n\}$ and $\rho(t), B(t)$ by Lemma 3.3 and Eqs. (4.7) and (4.8). Applying Favard’s Theorem, we can conclude the following theorems.

Theorem 4.3. *For the case (I) $r_2 = r_3 = 0$, only the followings are MRM-applicable, namely, we must have $b_2 = b_1^2 \neq 0$ and*

$$\alpha_n = -b_1(\kappa + 2n), \quad \omega_n = b_1^2n(\kappa - 1 + n),$$

$$\rho(t) = t, \quad B(t) = e^{b_1t}, \quad (t, x) = e^{b_1t} {}_0F_1(-; \kappa, tx).$$

Moreover, the corresponding measure is a deformation of the gamma distribution (2.1) by the dilation τ_{1/b_1} and the corresponding orthogonal polynomials are given by $\{L_n^{\kappa-1}(x/b_1)\}.$

Proof. Since $p_2(t) = t$ by $r_2 = r_3 = 0$, we have

$$E(p_2; t) = \int_0^t \frac{du}{\sqrt{p_2(u)}} = \int_0^t \frac{du}{\sqrt{u}} = 2\sqrt{t}.$$

Hence $\rho(t) = \frac{1}{4}E(p_2; t)^2 = t.$ Since $p_1(t) = \kappa - 2b_1t$ and $\int \frac{p_1(t)}{p_2(t)} dt = -2b_1 + \kappa \log t,$ we have $B(t) = 2^{-\kappa+1/2}t^{1/4}(2\sqrt{t})^{\kappa-1/2}t^{-\kappa/2}e^{b_1t} = e^{b_1t}.$ □

We have introduced the notation $E(a, b; t)$ in (2.10) as

$$E(a, b; t) = \int_0^t \frac{du}{\sqrt{u(1 + bu + au^2)}} = E(q; t), \quad q(t) = t(1 + bt + at^2),$$

where $E(q; t)$ is defined by Eq. (4.7). Let us remark that

$${}_0F_1(-; \frac{1}{2}; x) = \cosh(2\sqrt{x}), \quad {}_0F_1(-; \frac{3}{2}; x) = \frac{1}{2\sqrt{x}} \cosh(2\sqrt{x}).$$

For the case $\kappa = \frac{1}{2}$, Eq. (4.8) can be rewritten as

$$B(t) = \exp \left[\frac{1}{2} \int_0^t \left(\frac{p_2'(u)}{2p_2(u)} - \frac{p_1(u)}{p_2(u)} \right) du \right]. \tag{4.9}$$

Theorem 4.4. *For the case (II) $\kappa = \frac{1}{2}$, only the followings are MRM-applicable:*

(1) $b_1 = 0$. *In this case, we must have $b_2 < 0$, $r_2 = 0$, $r_3 = \frac{2}{5}b_2$, and*

$$\begin{aligned} \alpha_n &= 0, \\ \omega_n &= -\frac{b_2}{2}n^2(2n-1)(2n+1), \\ \rho(t) &= \frac{1}{4}E(-2b_2, 0; t)^2, \\ B(t) &= \frac{1}{\sqrt{1-2b_2t^2}}, \\ (t, x) &= \frac{1}{\sqrt{1-2b_2t^2}} \cosh(E(-2b_2, 0; t)\sqrt{x}). \end{aligned}$$

(2) $b_1 \neq 0$, $r_3 = \frac{1}{10b_1}(-4b_1^3 + 8b_1b_2 - 3b_1^2r_2 - 6b_2r_2 + 16b_1r_2^2)$. *In this case, we have the following subcases:*

(i) $b_2 = \frac{3b_1^2}{2}$, $r_3 = \frac{2}{5}(2b_1^2 - 3b_1r_2 + 4r_2^2)$ *with $b_1(2b_1 - 3r_3) < 0$ and*

$$\begin{aligned} \alpha_n &= -\frac{1}{2}(6r_2n^2 + 4b_1n + b_1), \\ \omega_n &= -\frac{1}{2}b_1(2b_1 - 3r_2)n^2(2n-1)^2, \\ \rho(t) &= \frac{1}{2}E(-4b_1^2 + 6b_1r_2, -3r_2; t)^2, \\ B(t) &= \frac{1}{\sqrt{1-2b_1t}}, \\ (t, x) &= \frac{\cosh(E(-4b_1^2 + 6b_1r_2, -3r_2; t)\sqrt{x})}{\sqrt{1-2b_1t}}. \end{aligned}$$

(ii) $r_2 = \frac{2}{3}b_1$, $r_3 = \frac{1}{45}(5b_1^2 + 18b_2)$ *with $3b_1^2 - 2b_2 > 0$ and*

$$\begin{aligned} \alpha_n &= -\frac{b_1}{2}(2n+1)^2, \\ \omega_n &= \frac{1}{4}(3b_1^2 - 2b_2)n^2(2n-1)(2n+1), \\ \rho(t) &= \frac{1}{4}E(3b_1^2 - 2b_2, -2b_1; t)^2, \\ B(t) &= \frac{1}{\sqrt{1-2b_1t + (3b_1^2 - 2b_2)t^2}}, \\ (t, x) &= \frac{\cosh(E(3b_1 - b_2, -2b_1; t)\sqrt{x})}{\sqrt{1-2b_1t + (3b_1^2 - 2b_2)t^2}}. \end{aligned}$$

(iii) $r_2 = -\frac{2(b_1^2-2b_2)}{3b_1}$, $r_3 = \frac{23(b_1^2-2b_2)^2}{45b_1^2}$ *with $\frac{b_1^2}{6} > b_2 > 0$ and*

$$\begin{aligned} \alpha_n &= \frac{1}{2b_1}(4(b_1^2 - 2b_2)n^2 - 4b_1^2n - b_1^2), \\ \omega_n &= \frac{1}{4b_1^2}n(2n-1)(2(b_1^2 - 2b_2)n - 3b_1^2 + 2b_2) \\ &\quad \times ((b_1^2 - 2b_2)n - 2b_1^2 + 2b_2), \\ \rho(t) &= \frac{b_1}{b_1^2 - 2b_2} \left(\tanh^{-1} \left(\sqrt{b_1 - 2b_2/b_1} \sqrt{t} \right) \right)^2, \end{aligned}$$

$$\begin{aligned}
B(t) &= \left(1 + \frac{b_1^2 - 2b_2}{b_1} t\right)^{\frac{b_1^2}{b_1^2 - 2b_2}}. \\
(t, x) &= \left(1 + \frac{b_1^2 - 2b_2}{b_1} t\right)^{\frac{b_1^2}{b_1^2 - 2b_2}} \\
&\quad \times \cosh \left(2\sqrt{\frac{b_1}{b_1^2 - 2b_2}} x \tanh^{-1} \left(\sqrt{b_1 - 2b_2/b_1} \sqrt{t}\right)\right).
\end{aligned}$$

Proof. The proof is similar to that of Theorem 4.3. \square

For the case $\kappa = \frac{3}{2}$, Eq. (4.8) becomes

$$B(t) = \frac{E(p_2; t)}{2\sqrt{t}} \exp \left[\frac{1}{2} \int_0^t \left(\frac{p_2'(u)}{2p_2(u)} - \frac{p_1(u)}{p_2(u)} + \frac{1}{u} \right) du \right]. \quad (4.10)$$

Theorem 4.5. For the case (III) $\kappa = \frac{3}{2}$, only the followings are MRM-applicable:

(1) The case $r_2 = \frac{7}{5}b_1$. In this case, we have the following subcases:

(i) $b_2 = \frac{363}{250}b_1^2$, $r_2 = \frac{7}{5}b_1$, $r_3 = \frac{1127}{500}b_1^2$ with $b_1 \neq 0$, and

$$\begin{aligned}
\alpha_n &= -\frac{3}{10}b_1(14n^2 + 16n + 5), \\
\omega_n &= \frac{9}{400}b_1^2n(2n+1)(7n+1)(14n-5), \\
\rho(t) &= \frac{10}{21b_1} \tan^{-1} \left(\sqrt{\frac{21b_1}{10}} \sqrt{t} \right)^2, \\
B(t) &= \frac{10^{9/14}}{\sqrt{21b_1}\sqrt{t}(10-21b_1t)^{1/7}} \tan^{-1} \left(\sqrt{\frac{21b_1}{10}} \sqrt{t} \right), \\
(t, x) &= \frac{1}{2\sqrt{10}\sqrt{t}\sqrt{x}(10-21b_1t)^{1/7}} \\
&\quad \times \sinh \left(\sqrt{\frac{10}{21b_1}} \sqrt{x} \tan^{-1} \left(\sqrt{\frac{21b_1}{10}} \sqrt{t} \right) \right).
\end{aligned}$$

(ii) $b_2 = \frac{363}{250}b_1^2$, $r_2 = \frac{7}{5}b_1$, $r_3 = \frac{338}{125}b_1^2$ with $b_1 \neq 0$, and

$$\begin{aligned}
\alpha_n &= -\frac{3}{10}b_1(14n^2 + 16n + 5), \\
\omega_n &= \frac{27}{50}b_1^2n^2(2n+1)^2, \\
\rho(t) &= \frac{1}{4}E \left(\frac{54}{25}b_1^2, -\frac{21}{5}b_1; t \right)^2, \\
B(t) &= \frac{\sqrt{5}}{2\sqrt{t(5-3b_1t)}} E \left(\frac{54}{25}b_1^2, -\frac{21}{5}b_1; t \right), \\
(t, x) &= \frac{\sqrt{5}}{2\sqrt{t(5-3b_1t)}\sqrt{x}} \\
&\quad \times \sinh \left(\sqrt{x} E \left(\frac{54}{25}b_1^2, -\frac{21}{5}b_1; t \right) \right).
\end{aligned}$$

(iii) $b_1 = 0$, $r_2 = \frac{7}{5}b_1$, $r_3 = 2b_2$ with $b_2 < 0$, and

$$\begin{aligned}\alpha_n &= 0, \\ \omega_n &= -\frac{5b_2}{2}n^2(2n+1)(2n-1), \\ \rho(t) &= \frac{1}{4}E(-10b_2, 0; t)^2, \\ B(t) &= \frac{1}{2\sqrt{t}}E(-10b_2, 0; t), \\ (t, x) &= \frac{1}{2\sqrt{tx}}\sinh(E(-10b_2, 0; t)\sqrt{x}).\end{aligned}$$

(iv) $b_1 = 0$, $r_2 = \frac{7}{5}b_1$, $r_3 = \frac{1}{3}b_2$ with $b_2 < 0$, and

$$\begin{aligned}\alpha_n &= 0, \\ \omega_n &= -\frac{5b_2}{12}n(n+1)(2n+1)^2, \\ \rho(t) &= \frac{1}{4}E\left(-\frac{5}{12}, 0; t\right)^2, \\ B(t) &= \frac{1}{2\sqrt{t}}E\left(-\frac{5}{12}, 0; t\right), \\ (t, x) &= \frac{\sqrt{3}}{2\sqrt{3-5b_2t^2}\sqrt{tx}}\sinh(E(-10b_2, 0; t)\sqrt{x}).\end{aligned}$$

(2) The case $r_3 = \frac{-1}{2(7b_1-5r_2)}(4b_1^3 - 8b_1b_2 + b_1^2r_2 + 10b_2r_2 - 24b_1r_2^2 + 12r_2^3)$. In this case, we have the following subcases:

(i) $r_2 = \frac{b_1}{2}$, $r_3 = \frac{b_2}{3}$ with $6b_1^2 - 5b_2 > 0$, and

$$\begin{aligned}\alpha_n &= -\frac{3}{2}b_1(n+1)^2, \\ \omega_n &= \frac{1}{12}(6b_1^2 - 5b_2)n(n+1)(2n+1)^2, \\ \rho(t) &= \frac{1}{4}E\left(\frac{12b_1^2 - 5b_2}{6}, -\frac{3b_1}{2}; t\right)^2, \\ B(t) &= \frac{\sqrt{3}}{\sqrt{2}\sqrt{t(6-9b_1t+(12b_1^2-5b_2)t^2)}} \\ &\quad \times E\left(\frac{12b_1^2-5b_2}{6}, -\frac{3b_1}{2}; t\right), \\ (t, x) &= \frac{\sqrt{3}}{\sqrt{2}\sqrt{x}\sqrt{t(6-9b_1t+(12b_1^2-10b_2)t^2)}} \\ &\quad \times \sinh E\left(\frac{12b_1^2-5b_2}{6}, -\frac{3b_1}{2}; t\right).\end{aligned}$$

(ii) $r_2 = 2b_1$, $r_3 = b_1^2 + 2b_2$ with $27b_1^2 - 10b_2 > 0$, and

$$\begin{aligned}\alpha_n &= -\frac{3}{2}b_1(2n+1)^2, \\ \omega_n &= \frac{1}{4}(27b_1^2 - 10b_2)n^2(2n-1)(2n+1), \\ \rho(t) &= \frac{1}{4}E(27b_1^2 - 10b_2, -6b_1; t)^2,\end{aligned}$$

$$B(t) = \frac{1}{2\sqrt{t}} E(27b_1^2 - 10b_2, -6b_1; t),$$

$$(t, x) = \frac{1}{2\sqrt{t}\sqrt{x}} \sinh(\sqrt{x} E(27b_1^2 - 10b_2, -6b_1; t)).$$

$$(iii) \quad b_2 = \frac{1}{20}(10b_1^2 + 15b_1r_2 - r_2^2), \quad r_3 = \frac{23}{20}r_2^2 \quad \text{with } r_2(b_1 + r_2) > 0, \text{ and}$$

$$\alpha_n = -\frac{1}{2}(6r_2n^2 + 4(b_1 + r_2)n + 3b_1),$$

$$\omega_n = \frac{1}{16}n(2n+1)(3r_2n + 2b_1 - r_2)(6r_2n + 4b_1 - 5r_2),$$

$$\rho(t) = \frac{1}{4}E\left(\frac{9}{4}b_1^2, -3r_2; t\right)^2,$$

$$B(t) = \frac{1}{\sqrt{3r_2}\sqrt{t}(2-3r_2t)^{\frac{2b_1-r_2}{3r_2}}} E\left(\frac{9}{4}b_1^2, -3r_2; t\right),$$

$$(t, x) = \frac{(2-3r_2t)^{\frac{-2b_1+r_2}{3r_2}}}{4\sqrt{t}\sqrt{x}} \sinh \sqrt{x} E\left(\frac{9}{4}b_1^2, -3r_2; t\right).$$

$$(iv) \quad b_2 = \frac{1}{10}(19b_1^2 - 20b_1r_2 + 12r_2^2), \quad r_3 = \frac{2}{5}(2b_1^2 - 5b_1r_2 + 6r_2^2) \quad \text{with } (b_1 - 2r_2)(2b_1 - r_2) < 0, \text{ and}$$

$$\alpha_n = -\frac{1}{2}(6r_2n^2 + 4(b_1 + r_2)n + 3b_1),$$

$$\omega_n = -\frac{1}{2}(b_1 - 2r_2)(2b_1 - r_2)n^2(2n+1)^2,$$

$$\rho(t) = \frac{1}{4}E(-(4b_1^2 - 10b_1r_2 + 4r_2^2), -3r_2; t)^2,$$

$$B(t) = \frac{1}{2\sqrt{t}\sqrt{1-(2b_1+r_2)t}} \times E(-(4b_1^2 - 10b_1r_2 + 4r_2^2), -3r_2; t),$$

$$(t, x) = \frac{1}{2\sqrt{t}\sqrt{1-2(2b_1-r_2)t\sqrt{x}}} \times \sinh(\sqrt{x} E(-(4b_1^2 - 10b_1r_2 + 4r_2^2), -3r_2; t)).$$

Proof. The proof is similar to that of Theorem 4.3. □

5. Wilson Polynomials and MRM-applicability

As seen from §4, the Jacobi-Szegö parameters of MRM-applicable orthogonal polynomials satisfy the condition that α_n 's are at most second degree and ω_n 's are at most fourth degree in n . It is known that Jacobi-Szegö parameters of Wilson polynomials are rational expressions in n . We are interested in MRM-applicable Wilson polynomials for $h(x) = {}_0F_1(-; \kappa; x)$.

Wilson polynomials $\{W_n(x; a, b, c, d)\}$ (see [20], §3.8 of [1]) are defined by

$$W_n(x^2; a, b, c, d) = (a+b)_n(a+c)_n(a+d)_n$$

$$\times {}_4F_3(-n, n+a+b+c+d-1, a+ix+a-ix; a+b, a+c, a+d; 1). \quad (5.1)$$

Their normalizing orthogonal polynomials are given by

$$P_n(x) = P_n(x; a, b, c, d) = (-1)^n \frac{W_n(x; a, b, c, d; x)}{(n + a + b + c + d - 1)_n}. \tag{5.2}$$

The density of the corresponding probability measure is

$$f(x; a, b, c, d) = \frac{\Gamma(a + b + c + d)}{4\pi\sqrt{x}\Gamma(a + b)\Gamma(a + c)\Gamma(a + d)\Gamma(b + c)\Gamma(b + d)\Gamma(c + d)} \times \left| \frac{\Gamma(a + i\sqrt{x})\Gamma(b + i\sqrt{x})\Gamma(c + i\sqrt{x})\Gamma(d + i\sqrt{x})}{\Gamma(2i\sqrt{x})} \right|^2. \tag{5.3}$$

The Jacobi-Szegö parameters are given by

$$\alpha_n = B_n + C_n - a^2, \quad \omega_n = B_{n-1}C_n, \tag{5.4}$$

where

$$B_n = \frac{(n + a + b + c + d - 1)(n + a + b)(n + a + c)(n + a + d)}{(2n + a + b + c + d - 1)(2n + a + b + c + d)},$$

$$C_n = \frac{n(n + b + c - 1)(n + b + d - 1)(n + c + d - 1)}{(2n + a + b + c + d - 2)(2n + a + b + c + d - 1)}.$$

The following is the list of parameters such that α_n 's and ω_n 's are polynomials of degrees 2 and 4, respectively, in n .

Proposition 5.1. *The Jacobi-Szegö parameters of Wilson polynomials are polynomials in n only for the following cases:*

(1) *The set $\{a, b, c, d\}$ is of the form $\{p, p - \frac{1}{2}, q, q - \frac{1}{2}\}$. Then*

$$\alpha_n = \frac{1}{4}(2n^2 + 4(p + q - 1)n + 4pq - p - q),$$

$$\omega_n = \frac{1}{64}n(2n + 4p - 3)(2n + 4q - 3)(n + 2p + 2q - 3).$$

(2) *The set $\{a, b, c, d\}$ is of the form $\{p, p + \frac{1}{2}, q, q - \frac{1}{2}\}$. Then*

$$\alpha_n = \frac{1}{8}(4n^2 + 4(2p + 2q - 1)n + 8pq - 2p + 2q - 1),$$

$$\omega_n = \frac{1}{64}n(2n + 4p - 1)(2n + 4q - 3)(n + 2p + 2q - 2).$$

(3) *The set $\{a, b, c, d\}$ is of the form $\{p, p + \frac{1}{2}, q, q + \frac{1}{2}\}$. Then*

$$\alpha_n = \frac{1}{4}(2n^2 + 4(p + q)n + 4pq + p + q),$$

$$\omega_n = \frac{1}{64}n(n + 2p + 2q)(2n + 4q - 1)(n + 2p + 2q - 1).$$

(4) *The set $\{a, b, c, d\}$ is of the form $\{p, \frac{1}{2} - p, q, \frac{1}{2} - q\}$. Then*

$$\alpha_n = \frac{1}{4}(2n^2 - 2p^2 - 2q^2 + p + q),$$

$$\omega_n = \frac{1}{64}(2n + 2p - 2q - 1)(2n - 2p + 2q - 1)(n - p - q)(n + p + q - 1).$$

(5) The set $\{a, b, c, d\}$ is of the form $\{p, \frac{1}{2} - p, q, \frac{3}{2} - q\}$. Then

$$\begin{aligned}\alpha_n &= \frac{1}{8}(4n^2 + 4n - 4p^2 - 4q^2 + 2p + 6q - 1), \\ \omega_n &= \frac{1}{64}(2n + 2p - 2q + 1)(2n - 2p + 2q - 1)(n + p + q - 1) \\ &\quad \times (n - p - q + 1).\end{aligned}$$

(6) The set $\{a, b, c, d\}$ is of the form $\{p, \frac{3}{2} - p, q, \frac{3}{2} - q\}$. Then

$$\begin{aligned}\alpha_n &= \frac{1}{4}(2n^2 + 4n - 2p^2 - 2q^2 + 3p + 3q), \\ \omega_n &= \frac{1}{64}(2n + 2p - 2q + 1)(2n - 2p + 2q + 1)(n - p - q + 2) \\ &\quad \times (n + p + q - 1).\end{aligned}$$

Proof. From Eq. (5.4), the remainder in deviding the numerator of α_n by its denominator is given by

$$(a + b - c - d)(a - b + c - d)(a - b - c + d)(-2 + a + b + c + d).$$

Hence $d = a + b - c$, $d = a - b + c$, $d = -a + b + c$ or $d = 2 - a - b - c$ must hold. If $d = a + b - c$, then

$$\omega_n = \frac{n(n + 2a + 2b - 2)(n + 2a + b - c - 1)(n + a + 2b - c - 1) \times (n + a + c - 1)(n + b + c - 1)}{4(2n + 2a + 2b - 3)(2n + 2a + 2b - 1)}$$

with the remainder given by

$$-(4(a - c)^2 - 1)(4(b - c)^2 - 1)(2(a + b) - 3)(2(a + b) - 1).$$

Thus $a - c = \pm \frac{1}{2}$, $b - c = \pm \frac{1}{2}$, $a + b = \frac{1}{2}$, or $a + b = \frac{3}{2}$. Hence possible sets of parameters are

$$\begin{aligned}&\left(a, b, a - \frac{1}{2}, b + \frac{1}{2}\right), \quad \left(a, b, a + \frac{1}{2}, b - \frac{1}{2}\right), \quad \left(a, b, b - \frac{1}{2}, a + \frac{1}{2}\right), \\ &\left(a, b, b + \frac{1}{2}, a - \frac{1}{2}\right), \quad \left(a, \frac{1}{2} - a, c, \frac{1}{2} - c\right), \quad \left(a, \frac{3}{2} - a, c, \frac{3}{2} - c\right).\end{aligned}$$

Other cases can be treated similarly. \square

We have seen in §2 all possible cases on the MRM-applicability for the normalized factor $h(x) = {}_0F_1(-; \kappa; x)$. By comparing the function forms in n of Jacobi-Szegő parameters in Theorems 4.3, 4.4, and 4.5 with Proposition 5.1, we see special Wilson polynomials, which can be obtained by the form in Eq. (2.9). The next theorem provides a complete list of MRM-applicable Wilson polynomials for $h(x) = {}_0F_1(-; \kappa; x)$.

For convenience, we denote by $(a, b, c, d) \sim (a', b', c', d')$ that (a, b, c, d) is a permutation of (a', b', c', d') .

Theorem 5.2. *The normalized Wilson polynomials $\{P_n(x; a, b, c, d)\}$ defined by Eq. (5.1) are MRM-applicable if and only if either $(a, b, c, d) \sim (p, p + \frac{1}{2}, \frac{1}{2}, 0)$ with $p > 0$ or $(a, b, c, d) \sim (p, p + \frac{1}{2}, \frac{1}{2}, 1)$ with $p \geq 0$.*

(1) The case $(a, b, c, d) \sim (p, p + \frac{1}{2}, \frac{1}{2}, 0)$. In this case, we have

$$\begin{aligned}\alpha_n &= \frac{1}{4}(2n^2 + 4pn + p), \\ \omega_n &= \frac{1}{64}n(2n-1)(2n+4p-1)(n+2p-1), \\ P_n(x; a, b, c, d) &= (-16)^{-n}(4p)_{2n} \\ &\quad \times {}_4F_3(-n, n+2p, p-i\sqrt{x}, p+i\sqrt{x}; p, p+\frac{1}{2}, 2p+\frac{1}{2}; 1), \\ f(a, b, c, d; x) &= \frac{2^{4p}|\Gamma(2(p+i\sqrt{x}))|^2}{\pi\Gamma(4p)\sqrt{x}}.\end{aligned}$$

Moreover, we have the corresponding MRM-factor $h(x)$, ρ -, B - and φ -functions,

$$\begin{aligned}h(x) &= {}_0F_1(-; \frac{1}{2}; x) = \cosh 2\sqrt{x}, \\ \rho(t) &= 4\left(\tan^{-1} \frac{\sqrt{t}}{2}\right)^2, \\ B(t) &= \frac{4^{2p}}{(4+t)^{2p}}, \\ \varphi(t, x) &= \frac{4^{2p}}{(4+t)^{2p}} \cosh\left(4\sqrt{x} \tan^{-1} \frac{\sqrt{t}}{2}\right).\end{aligned}$$

(2) The case $(a, b, c, d) \sim (p, p + \frac{1}{2}, \frac{1}{2}, 1)$. In this case, we have

$$\begin{aligned}\alpha_n &= \frac{1}{8}(4n^2 + 4(2p+1)n + 6p+1), \\ \omega_n &= \frac{1}{64}n(2n+1)(2n+4p-1)(n+2p), \\ P_n(x; a, b, c, d) &= (-16)^{-n}(4p+1)_{2n} \\ &\quad \times {}_4F_3(-n, n+2p+1, p-i\sqrt{x}, p+i\sqrt{x}; p+\frac{1}{2}, p+1, 2p+\frac{1}{2}; 1), \\ f(a, b, c, d; x) &= \frac{2^{4(p+1)}\sqrt{x}|\Gamma(2(p+i\sqrt{x}))|^2}{\pi\Gamma(4p+1)}.\end{aligned}$$

Moreover, we have the corresponding MRM-factor $h(x)$, ρ -, B - and φ -functions,

$$\begin{aligned}h(x) &= {}_0F_1(-; \frac{3}{2}; x) = \frac{1}{2\sqrt{x}} \sinh 2\sqrt{x}, \\ \rho(t) &= 4\left(\tan^{-1} \frac{\sqrt{t}}{2}\right)^2, \\ B(t) &= \frac{2 \cdot 4^{2p}}{(4+t)^{2p}\sqrt{t}} \tan^{-1} \frac{\sqrt{t}}{2}, \\ \varphi(t, x) &= \frac{4^{2p}}{2(4+t)^{2p}\sqrt{tx}} \sinh\left(4\sqrt{x} \tan^{-1} \frac{\sqrt{t}}{2}\right).\end{aligned}$$

Finally, let us point out a special case of Theorem 5.2. For $p = \frac{1}{2}$ in Case (1), we have $(a, b, c, d) \sim (0, \frac{1}{2}, \frac{1}{2}, 1)$. On the other hand, for $p = 0$ in Case (2), we also have $(a, b, c, d) \sim (0, \frac{1}{2}, \frac{1}{2}, 1)$. Then

$$\begin{aligned}\alpha_n &= \frac{1}{8}(2n+1)^2, \\ \omega_n &= \frac{1}{64}n^2(2n-1)(2n+1), \\ P_n(x; a, b, c, d) &= (-1)^n \frac{(2n)!}{2^{4n}} {}_4F_3(-n, n+1, -i\sqrt{x}, i\sqrt{x}; \frac{1}{2}, \frac{1}{2}, 1; 1), \\ f(a, b, c, d; x) &= \frac{2^4 \sqrt{x}}{\pi} |\Gamma(2i\sqrt{x})|^2.\end{aligned}$$

This case is MRM-applicable for both functions ${}_0F(-; \frac{1}{2}; x)$ and ${}_0F(-; \frac{3}{2}; x)$. The corresponding ρ -, B - and ω -functions are stated in Examples 2.2 and 2.3.

6. MRM-factors for Given Measures

In §2, we saw that the gamma distribution μ_κ is MRM-applicable for several MRM-factors $h(x)$. It is natural to ask the question whether there are other MRM-factors. The answer is given by the next theorem.

Theorem 6.1. *All MRM-factors of the gamma distribution μ_κ are given by*

$${}_0F_1(-; \kappa; x) \quad \text{and} \quad {}_1F_1(c; \kappa; x) \quad (c \neq 0, -1, -2, -3, \dots) \quad (6.1)$$

up to a scaling.

Proof. The orthogonal polynomials are Laguerre polynomials with Jacobi-Szegő parameters

$$\alpha_n = -(\kappa + 2n), \quad n \geq 0; \quad \omega_n = n(\kappa + n - 1), \quad n \geq 1, \quad \omega_0 = 1,$$

(see [3].) Therefore,

$$\sum_{j=0}^{n-1} \alpha_j = -\sum_{j=0}^{n-1} (\kappa + 2j) = -n(\kappa + n - 1) = -\omega_n.$$

Applying Lemma 3.4, we get

$$h_n = \prod_{k=0}^{n-1} \frac{b_1 + kr_2}{(k+1)(\kappa+k)} = \begin{cases} \frac{b_1^n}{(\kappa)_n n!}, & \text{if } r_2 = 0, \\ r_2^n \frac{\left(\frac{b_1}{r_2}\right)_n}{(\kappa)_n n!}, & \text{if } r_2 \neq 0, \end{cases}$$

which yields that

$$h(x) = \sum_{n=0}^{\infty} h_n x^n = \begin{cases} {}_0F_1(-; \kappa; b_1 x), & \text{if } r_2 = 0, \\ {}_1F_1\left(\frac{b_1}{r_2}; \kappa; r_2 x\right), & \text{if } r_2 \neq 0. \end{cases}$$

By the condition $h_n \neq 0$, we obtain the condition on c . □

We have obtained all MRM-applicable measures for $h(x) = e^x$. This leads to a new problem of finding all possible MRM-factors of these measures. The following answer will be proved in a forthcoming paper using Lemma 3.5. First, we mention that $\rho(t)$ is an odd function and $B(t)$ is an even function, if μ is symmetric or equivalently if $\alpha_n = 0$ for all $n \geq 0$. Put

$$h_e(x) = \frac{1}{2}(h(x) + h(-x)), \quad h_o(x) = \frac{1}{2}(h(x) - h(-x))$$

and let $h^c(x) = h_e(x) + ch_o(x)$ for $c \neq 0$. Then the triple $((h^c(x), \rho(t), B(t)))$ also gives a generating function. As mentioned already, we have a freedom of scaling. In the followings, we will state the assertion “up to a trivial deformation.”

Theorem 6.2. *All MRM-factors of the standard Gaussian measure $N(0, 1)$ are given by*

$$h(x) = e^x, \\ \tilde{h}(x) = {}_1F_1\left(\frac{c}{2}; \frac{1}{2}; -x^2\right) + {}_1F_1\left(\frac{c}{2} + \frac{1}{2}; \frac{3}{2}; -x^2\right)x, \quad c \neq 0, -1, -2, -3, \dots$$

up to a trivial deformation and the corresponding ρ and B -functions are given respectively by

$$\rho(t) = t, \quad B(t) = e^{-\frac{1}{2}t^2} \quad \text{and} \quad \tilde{\rho}(t) = \frac{t}{\sqrt{1+2t^2}}, \quad \tilde{B}(t) = (1+2t^2)^{-\frac{1}{2}\gamma}.$$

Theorem 6.3. *Let M_κ be the Meixner distribution, whose characteristic function is $(\cosh t)^{-\kappa}$ and let $M_{\kappa,\beta}$ be the β -shift of M_κ . Then we have the following assertions:*

(i) *All MRM-factors of M_κ are unique up to a trivial deformation for $\kappa \neq 2$ and*

$$h(x) = e^x, \quad \rho(t) = \tan^{-1} t, \quad B(t) = (1+t^2)^{-\frac{\kappa}{2}}.$$

(ii) *There are only two MRM-factors of M_2 up to a trivial deformation and*

$$h(x) = e^x, \quad \rho(t) = \tan^{-1} t, \quad B(t) = (1+t^2)^{-1}, \\ \tilde{h}(x) = \frac{1}{x}(e^x - 1), \quad \tilde{\rho}(t) = \tan^{-1} t, \quad \tilde{B}(t) = \frac{1}{t} \tan^{-1} t.$$

Remark 6.4. The MRM-factors $h(x)$ in the above theorems can be represented by hypergeometric functions as follows:

$$\frac{e^x}{x} = {}_1F_1(c; c; x), \\ \frac{e^x - 1}{x} = {}_1F_1(1; 2; x).$$

Our frame work can also be applied to probability measures having atoms with infinite supports. We can show that MRM-factor for the other probability measures with the MRM-factor $h(x) = e^x$ is unique. In the forthcoming paper [11], improved versions of above theorems will be shown together with other cases.

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