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ON THE EXTENSION OF A BASIC PROPERTY OF CONDITIONAL EXPECTATIONS TO SECOND QUANTIZATION OPERATORS

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Abstract. The aim of this paper is to investigate whether a fundamental property of conditional expectations holds true for second quantization operators. We provide necessary and sufficient conditions for that and we show how they reduce for specific choices of operators. Our proofs are based on two crucial formulas which relates Wick and ordinary products. New proofs for these two formulas are also given.

1. Introduction

Let Y be a random variable defined on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and let $\mathcal{B}$ be a sub-σ-algebra of $\mathcal{F}$. One of the crucial properties of conditional expectations is that if Y is $\mathcal{B}$-measurable then for any bounded X,

$$E[XY|\mathcal{B}] = YE[X|\mathcal{B}], \quad \mathcal{P} - a.e.. \tag{1.1}$$

Now suppose $(\Omega, \mathcal{F}, \mathcal{P})$ to be the classical Wiener space over $[0,T]$ and let $A : \mathcal{L}^2([0,T]) \to \mathcal{L}^2([0,T])$ be a projection operator. It is known (see for instance Janson [5]) that $\Gamma(A)$, the second quantization operator of A, corresponds to the conditional expectation with respect to a certain σ-algebra. In this case the measurability of the random variable Y w.r.t. that σ-algebra can be formulated as

$$\Gamma(A)Y = Y, \tag{1.2}$$

and equality (1.1) becomes

$$\Gamma(A)(X \cdot Y) = Y \cdot \Gamma(A)X, \quad \mathcal{P} - a.e.. \tag{1.3}$$

Now for a general bounded linear operator A is condition (1.2) sufficient to guarantee the validity of (1.3)?

The aim of the present paper is to answer this question showing that in general one needs infinitely many conditions on Y for (1.3) to be true. These conditions can be expressed either in terms of the Hida-Malliavin derivatives of Y or by means of translation operators on the Wiener space.

The first part of this manuscript is devoted to the study of two formulas which relate the Wick and ordinary products of smooth random variables (see formulas

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(3.4) and (3.7) in Section 3 below). We provide new proofs for them and a stronger convergence result for the series there defined. These formulas are then used to prove the first main result (Theorem 4.1) followed by a number of corollaries and remarks. In Theorem 4.9 similar conditions on Y are shown to be equivalent to (1.3).

The paper is organized as follows: Section 2 provides a quick review of the necessary background; more information on the subject can be found for instance in Holden et al. [2], Hu and Yan [4], Janson [5], Kuo [6] or Nualart [9]. Section 3 is devoted to investigate the interplay between Hida-Malliavin derivatives, Wick products and ordinary products in the space $\mathcal{G}$ of smooth random variables. Finally in Section 4 we present the main results together with some observations and examples.

2. Framework

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space which carries a one dimensional Brownian motion $\{B_t\}_{0\leq t\leq T}$. Assume that $\mathcal{F} = \mathcal{F}_T$ where $\{\mathcal{F}_t\}_{0\leq t\leq T}$ denotes the augmented Brownian filtration; as a consequence of this assumption we get that the set

$$\Big\{\mathcal{E}(f) := \exp\Big\{\int_0^T f(s)dB_s - \frac{1}{2}\int_0^T f^2(s)ds\Big\}, f \in \mathcal{L}^2([0,T])\Big\}$$

is total in $\mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$ ($\mathcal{L}^2(\mathcal{P})$ for short); therefore if

$$E[X\mathcal{E}(f)] = E[Y\mathcal{E}(f)] \text{ for all } f \in \mathcal{L}^2([0,T]), \tag{2.1}$$

then $X = Y, \mathcal{P}$-a.e.. According to the Wiener-Itô chaos decomposition theorem any $X \in \mathcal{L}^2(\mathcal{P})$ can be uniquely represented as

$$X = \sum_{n\geq 0} I_n(h_n) \quad (\text{convergence in } \mathcal{L}^2(\mathcal{P})),$$

where $I_0(h_0) := E[X]$ and for $n \geq 1$, $h_n \in \mathcal{L}^2([0,T]^n)$ is a deterministic symmetric function; $I_n(h_n)$ stands for the n-th order multiple Itô integral of h_n w.r.t. the Brownian motion $\{B_t\}_{0\leq t\leq T}$. Moreover one has

$$\begin{aligned}\|X\|^2 &:= E[|X|^2] \\ &= \sum_{n\geq 0} n!|h_n|^2_{\mathcal{L}^2([0,T]^n)}.\end{aligned}$$

It is convenient to introduce the following family of Hilbert spaces of smooth random variables that were defined by Potthoff and Timpel [11] and further studied by Benth and Potthoff [1].

For $\lambda \geq 1$, let

$$\mathcal{G}_\lambda := \Big\{X = \sum_{n\geq 0} I_n(h_n) \in \mathcal{L}^2(\mathcal{P}) : \sum_{n\geq 0} n!\lambda^{2n}|h_n|^2_{\mathcal{L}^2([0,T]^n)} < +\infty\Big\}.$$

Note that $\mathcal{G}_1 = \mathcal{L}^2(\mathcal{P})$ and for $1 \leq \lambda < \mu$, $\mathcal{G}_\mu \subset \mathcal{G}_\lambda \subset \mathcal{L}^2(\mathcal{P})$. We also denote

$$\mathcal{G} := \bigcap_{\lambda\geq 1} \mathcal{G}_\lambda.$$

If $A : \mathcal{L}^2([0,T]) \to \mathcal{L}^2([0,T])$ is a bounded linear operator then its *second quantization operator* $\Gamma(A) : \mathcal{G} \to \mathcal{G}$ is defined as

$$\begin{aligned}\Gamma(A)X &= \Gamma(A)\sum_{n\geq 0} I_n(h_n) \\ &:= \sum_{n\geq 0} I_n(A^{\otimes n}h_n).\end{aligned}$$

With this notation the space $\mathcal{G}_\lambda$ previously defined can be described as

$$\mathcal{G}_\lambda = \{X \in \mathcal{L}^2(\mathcal{P}) : \|X\|_\lambda := \|\Gamma(\lambda I)X\| < +\infty\},$$

where I stands for the identity operator on $\mathcal{L}^2([0,T])$. In the sequel the operator $\Gamma(\lambda I)$ will be denoted simply by $\Gamma(\lambda)$.

Now let X and Y be elements of $\mathcal{L}^2(\mathcal{P})$ with chaos decompositions,

$$X = \sum_{n\geq 0} I_n(h_n) \text{ and } Y = \sum_{n\geq 0} I_n(g_n);$$

we define a new random variable, named the *Wick product* of X and Y and denoted by $X \diamond Y$, as

$$X \diamond Y := \sum_{n\geq 0} I_n(k_n), \text{ where } k_n := \sum_{j=0}^{n} h_j \hat{\otimes} g_{n-j}, n \geq 0;$$

here the symbol $\hat{\otimes}$ stands for the symmetric tensor product.

A useful property of the Wick product is the following:

$$\Gamma(A)(X \diamond Y) = (\Gamma(A)X) \diamond (\Gamma(A)Y). \tag{2.2}$$

Note that $X \diamond Y$ may not belong to $\mathcal{L}^2(\mathcal{P})$; one can in fact find simple examples where

$$\begin{aligned}\|X \diamond Y\|^2 &= \sum_{n\geq 0} n! |k_n|^2_{\mathcal{L}^2([0,T]^n)} \\ &= +\infty.\end{aligned}$$

A sufficient condition for $X \diamond Y$ to be square integrable is provided by the next theorem which was proved among other things in Kuo, Saito and Stan [7], Theorem 9.

Theorem 2.1. *If $X, Y \in \mathcal{G}_{\sqrt{2}}$, then $X \diamond Y \in \mathcal{L}^2(\mathcal{P})$. More precisely,*

$$\|X \diamond Y\| \leq \|\Gamma(\sqrt{2})X\| \cdot \|\Gamma(\sqrt{2})Y\|, \tag{2.3}$$

or equivalently,

$$\|X \diamond Y\| \leq \|X\|_{\sqrt{2}} \|Y\|_{\sqrt{2}}.$$

Finally let $X = \sum_{n\geq 0} I_n(h_n) \in \mathcal{L}^2(\mathcal{P})$. For $t \in [0,T]$ the random variable:

$$D_t X := \sum_{n\geq 1} n I_{n-1}(h_n(\cdot, t)),$$

where $h_n(\cdot,t)$ is now considered as a function of $n-1$ variables, is called the *Hida-Malliavin derivative* of X at t. By iteration we also define for $k\geq 2$ the k-th order Hida-Malliavin derivative of X at $(t_1,...,t_k)\in[0,T]^k$ as

$$D^k_{t_1,...,t_k}X:=\sum_{n\geq k}n(n-1)\cdots(n-k+1)I_{n-k}(h_n(\cdot,t_1,...,t_k)).$$

It is easy to see that if $X\in\mathcal{G}$ then for any $k\geq 1$ and any $(t_1,...,t_k)\in[0,T]^k$ the random variable $D^k_{t_1,...,t_k}X$ belongs to $\mathcal{L}^2(\mathcal{P})$ and

$$E\Big[\int_{[0,T]^k}|D^k_{t_1,...,t_k}X|^2dt_1...dt_k\Big]<+\infty.$$

For more details we refer to Nualart [9].

3. Formulas for Wick and Ordinary Products

The aim of the present section is to prove the following two formulas that relate Wick and ordinary products via Hida-Malliavin derivatives:

$$X\cdot Y=\sum_{n\geq 0}\frac{1}{n!}\int_{[0,T]^n}D^n_{t_1\cdots t_n}X\diamond D^n_{t_1\cdots t_n}Y dt_1...dt_n, \tag{3.1}$$

and

$$X\diamond Y=\sum_{n\geq 0}\frac{(-1)^n}{n!}\int_{[0,T]^n}D^n_{t_1\cdots t_n}X\cdot D^n_{t_1\cdots t_n}Y dt_1...dt_n. \tag{3.2}$$

These two formulas are already known in the literature: see formulas (3.8) and (3.9) in Nualart and Zakai [10] and formulas (2.3) and (2.4) in Hu and Øksendal [3]. However, none of the above references provides a proof for those formulas.

In a recent paper Hu and Yan [4] prove formula (3.2) under the assumption that X and Y belong to the space $\mathcal{G}_{\sqrt{2}}$ utilizing a limit argument. Here we prove (3.1) and (3.2) under a stronger requirement on the variables X and Y (namely $X,Y\in\mathcal{G}$) but we obtain a better convergence result for the two series involved.

Proposition 3.1. *Let $X,Y\in\mathcal{G}$. Then the series*

$$\sum_{n\geq 0}\frac{1}{n!}\int_{[0,T]^n}D^n_{t_1\cdots t_n}X\diamond D^n_{t_1\cdots t_n}Y dt_1...dt_n \tag{3.3}$$

converges in $\mathcal{G}$. Moreover,

$$X\cdot Y=\sum_{n\geq 0}\frac{1}{n!}\int_{[0,T]^n}D^n_{t_1\cdots t_n}X\diamond D^n_{t_1\cdots t_n}Y dt_1...dt_n. \tag{3.4}$$

Proof. We begin by showing that for any $\lambda\geq 1$,

$$\sum_{n\geq 0}\frac{1}{n!}\Big\|\int_{[0,T]^n}D^n_{t_1\cdots t_n}X\cdot D^n_{t_1\cdots t_n}Y dt_1...dt_n\Big\|_{\lambda}<+\infty;$$

this will imply the convergence in $\mathcal{G}$ of the series (3.3). To ease the notation we write D^n_t for $D^n_{t_1\cdots t_n}$ and dt for $dt_1...dt_n$.

Using the Cauchy-Schwarz inequality and inequality (2.3) we get

$$\begin{aligned}
&\sum_{n\geq 0}\frac{1}{n!}\Big\|\int_{[0,T]^n} D_t^n X\diamond D_t^n Y dt\Big\|_\lambda \\
&\leq \sum_{n\geq 0}\frac{1}{n!}\int_{[0,T]^n}\|D_t^n X\diamond D_t^n Y\|_\lambda dt \\
&= \sum_{n\geq 0}\frac{1}{n!}\int_{[0,T]^n}\|\Gamma(\lambda)(D_t^n X\diamond D_t^n Y)\|dt \\
&= \sum_{n\geq 0}\frac{1}{n!}\int_{[0,T]^n}\|(\Gamma(\lambda)D_t^n X)\diamond(\Gamma(\lambda)D_t^n Y)\|dt \\
&\leq \sum_{n\geq 0}\frac{1}{n!}\int_{[0,T]^n}\|\Gamma(\sqrt{2}\lambda)D_t^n X\|\cdot\|\Gamma(\sqrt{2}\lambda)D_t^n Y\|dt \\
&\leq \sum_{n\geq 0}\frac{1}{n!}\Big(\int_{[0,T]^n}\|\Gamma(\sqrt{2}\lambda)D_t^n X\|^2 dt\Big)^{\frac{1}{2}} \\
&\quad\cdot\Big(\int_{[0,T]^n}\|\Gamma(\sqrt{2}\lambda)D_t^n Y\|^2 dt\Big)^{\frac{1}{2}} \\
&= \sum_{n\geq 0}\Big(\int_{[0,T]^n}\frac{\|\Gamma(\sqrt{2}\lambda)D_t^n X\|^2}{n!}dt\Big)^{\frac{1}{2}} \\
&\quad\cdot\Big(\int_{[0,T]^n}\frac{\|\Gamma(\sqrt{2}\lambda)D_t^n Y\|^2}{n!}dt\Big)^{\frac{1}{2}} \\
&\leq \Big(\sum_{n\geq 0}\int_{[0,T]^n}\frac{\|\Gamma(\sqrt{2}\lambda)D_t^n X\|^2}{n!}dt\Big)^{\frac{1}{2}} \\
&\quad\cdot\Big(\sum_{n\geq 0}\int_{[0,T]^n}\frac{\|\Gamma(\sqrt{2}\lambda)D_t^n Y\|^2}{n!}dt\Big)^{\frac{1}{2}}.
\end{aligned}$$

Now observe that if $X=\sum_{k\geq 0}I_k(h_k)$, then

$$\begin{aligned}
&\sum_{n\geq 0}\int_{[0,T]^n}\frac{\|\Gamma(\sqrt{2}\lambda)D_t^n X\|^2}{n!}dt \\
&= \sum_{n\geq 0}\frac{1}{n!}\sum_{k\geq n}k(k-1)\cdots(k-n+1)k!(2\lambda^2)^{k-n}|h_k|^2 \\
&= \sum_{n\geq 0}\sum_{k\geq n}\binom{k}{n}k!(2\lambda^2)^{k-n}|h_k|^2 \\
&= \sum_{k\geq 0}k!|h_k|^2(2\lambda^2+1)^k \\
&= \|X\|^2_{\sqrt{2\lambda^2+1}}.
\end{aligned}$$

Therefore, since the same holds for the random variable Y, we conclude that

$$\sum_{n\geq 0}\frac{1}{n!}\Big\|\int_{[0,T]^n} D_t^n X\diamond D_t^n Y dt\Big\|_{\lambda}\leq \|X\|_{\sqrt{2\lambda^2+1}}\|Y\|_{\sqrt{2\lambda^2+1}}, \tag{3.5}$$

which implies the convergence of the series (3.3).

We now prove formula (3.4). Fix $f\in\mathcal{L}^2([0,T])$; then

$$\begin{aligned} E[XY\mathcal{E}(f)] &= E_{\mathcal{Q}(f)}[XY] \\ &= \sum_{n\geq 0} n!\langle h_n^f, g_n^f\rangle_{\mathcal{L}^2([0,T]^n)}, \end{aligned}$$

where $\{h_n^f\}_{n\geq 0}$ and $\{g_n^f\}_{n\geq 0}$ denote the kernels in the Wiener-Itô chaos decomposition of respectively X and Y w.r.t. the probability measure $d\mathcal{Q}(f):=\mathcal{E}(f)d\mathcal{P}$ and the Brownian motion $\{B_t^f:=B_t-\int_0^t f(s)ds\}_{0\leq t\leq T}$. Since the space $\mathcal{G}$ is invariant under deterministic translations of the underlying white noise measure (see Lanconelli [8]) we have that X and Y are also elements of $\mathcal{G}$ under the new white noise measure $\mathcal{Q}(f)$. Therefore by the Stroock-Taylor formula (see Nualart [9]) we can deduce that:

$$h_n^f(t)=\frac{1}{n!}E_{\mathcal{Q}(f)}[(D^f)_t^n X],$$

where D^f denotes the Hida-Malliavin derivative w.r.t. the Brownian motion $\{B_t^f\}_{0\leq t\leq T}$. Moreover since f is deterministic the derivative operator D^f coincides with D; hence

$$h_n^f(t)=\frac{1}{n!}E[D_t^n X\mathcal{E}(f)].$$

The same reasoning can be carried for Y and this yields:

$$\begin{aligned} E[XY\mathcal{E}(f)] &= \sum_{n\geq 0} n!\langle h_n^f, g_n^f\rangle_{\mathcal{L}^2([0,T]^n)} \\ &= \sum_{n\geq 0} n!\langle \frac{1}{n!}E[D_t^n X\mathcal{E}(f)], \frac{1}{n!}E[D_t^n Y\mathcal{E}(f)]\rangle_{\mathcal{L}^2([0,T]^n)} \\ &= \sum_{n\geq 0}\frac{1}{n!}\langle E[D_t^n X\mathcal{E}(f)], E[D_t^n Y\mathcal{E}(f)]\rangle_{\mathcal{L}^2([0,T]^n)} \\ &= \sum_{n\geq 0}\frac{1}{n!}\int_{[0,T]^n} E[D_t^n X\mathcal{E}(f)]E[D_t^n Y\mathcal{E}(f)]dt \\ &= \sum_{n\geq 0}\frac{1}{n!}\int_{[0,T]^n} E[(D_t^n X\diamond D_t^n Y)\mathcal{E}(f)]dt \\ &= E\Big[\Big(\sum_{n\geq 0}\frac{1}{n!}\int_{[0,T]^n} D_t^n X\diamond D_t^n Y dt\Big)\mathcal{E}(f)\Big]. \end{aligned}$$

Since $f\in\mathcal{L}^2([0,T])$ is arbitrary the proof of formula (3.4) is complete. □

Proposition 3.2. *Let $X, Y \in \mathcal{G}$. Then the series*

$$\sum_{n\geq 0} \frac{1}{n!} \int_{[0,T]^n} D^n_{t_1\cdots t_n} X \cdot D^n_{t_1\cdots t_n} Y dt_1...dt_n \tag{3.6}$$

converges in $\mathcal{G}$. Moreover,

$$X \diamond Y = \sum_{n\geq 0} \frac{(-1)^n}{n!} \int_{[0,T]^n} D^n_{t_1\cdots t_n} X \cdot D^n_{t_1\cdots t_n} Y dt_1...dt_n. \tag{3.7}$$

Proof. We keep the notation of the previous proof. Exploiting Proposition 3.1 and the estimate (3.5) we get:

$$\begin{aligned}
&\sum_{n\geq 0} \frac{1}{n!} \Big\| \int_{[0,T]^n} D^n_t X \cdot D^n_t Y dt \Big\|_\lambda \\
&\leq \sum_{n\geq 0} \frac{1}{n!} \int_{[0,T]^n} \| D^n_t X \cdot D^n_t Y \|_\lambda dt \\
&\leq \sum_{n\geq 0} \frac{1}{n!} \int_{[0,T]^n} \| D^n_t X \|_{\lambda_1} \| D^n_t Y \|_{\lambda_1} dt \\
&\leq \sum_{n\geq 0} \frac{1}{n!} \Big(\int_{[0,T]^n} \| D^n_t X \|^2_{\lambda_1} dt \Big)^{\frac{1}{2}} \Big(\int_{[0,T]^n} \| D^n_t Y \|^2_{\lambda_1} dt \Big)^{\frac{1}{2}} \\
&\leq \Big(\sum_{n\geq 0} \frac{1}{n!} \int_{[0,T]^n} \| D^n_t X \|^2_{\lambda_1} dt \Big)^{\frac{1}{2}} \Big(\sum_{n\geq 0} \frac{1}{n!} \int_{[0,T]^n} \| D^n_t Y \|^2_{\lambda_1} dt \Big)^{\frac{1}{2}} \\
&= \| X \|_{\lambda_2} \| Y \|_{\lambda_2},
\end{aligned}$$

where $\lambda_1 := \sqrt{2\lambda^2 + 1}$ and $\lambda_2 := \sqrt{\lambda_1^2 + 1}$. This proves the convergence of the series (3.6).

Using Proposition 3.1 we get

$$\begin{aligned}
&\sum_{n\geq 0} \frac{(-1)^n}{n!} \int_{[0,T]^n} D^n_t X \cdot D^n_t Y dt \\
&= \sum_{n\geq 0} \frac{(-1)^n}{n!} \int_{[0,T]^n} \Big(\sum_{k\geq 0} \frac{1}{k!} \int_{[0,T]^n} D^k_s D^n_t X \diamond D^k_s D^n_t Y ds \Big) dt \\
&= \sum_{n\geq 0} \sum_{k\geq 0} \frac{(-1)^n}{n!k!} \int_{[0,T]^{n+k}} D^{n+k}_r X \diamond D^{n+k}_r Y dr \\
&= \sum_{n\geq 0} \sum_{k\geq 0} \frac{(-1)^n}{(n+k)!} \frac{(n+k)!}{n!k!} \int_{[0,T]^{n+k}} D^{n+k}_r X \diamond D^{n+k}_r Y dr \\
&= \sum_{n\geq 0} \sum_{j\geq n} \frac{(-1)^n}{j!} \frac{j!}{n!(j-n)!} \int_{[0,T]^j} D^j_r X \diamond D^j_r Y dr \\
&= \sum_{j\geq 0} \frac{1}{j!} \int_{[0,T]^j} D^j_r X \diamond D^j_r Y dr \cdot \sum_{n=0}^{j} (-1)^n \frac{j!}{n!(j-n)!}
\end{aligned}$$

$$= \sum_{j\geq 0}\frac{1}{j!}\int_{[0,T]^j} D_r^j X \diamond D_r^j Y\, dr \cdot \delta_{0j}$$
$$= X \diamond Y.$$

□

4. Main Results

We are now going to state and prove the first main result of the present paper. It provides a necessary and sufficient condition for a basic property of the conditional expectation to be true also for general second quantization operators.

If A is a linear operator acting on $\mathcal{L}^2([0,T])$ we denote by $Ah(t), t\in[0,T]$ the function obtained by applying A on the function h. The adjoint operator of A will be denoted by A^* and the symbol $(dt)^n$ will represent the n-th dimensional Lebesgue measure on $[0,T]^n$.

To ease the notation further we write

$$\langle D^k X \diamond D^k Y\rangle_{\mathcal{L}^2([0,T]^k)}$$

for

$$\int_{[0,T]^k} D^k_{t_1,...,t_k} X \diamond D^k_{t_1,...,t_k} Y\, dt_1...dt_k.$$

Theorem 4.1. *Let $A:\mathcal{L}^2([0,T])\to\mathcal{L}^2([0,T])$ be a bounded linear operator and let $Y\in\mathcal{G}$. Then the following statements are equivalent:*

(1) *For all $X\in\mathcal{G}$, $\Gamma(A)(X\cdot Y)=Y\cdot\Gamma(A)X$, $\mathcal{P}$-a.e..*
(2) *For all $n\geq 0$, $\Gamma(A)D^nY=(A^*)^{\otimes n}D^nY$, $\mathcal{P}\otimes(dt)^n$-a.e..*

Proof. (1) ⇒ (2): Assume that for all $X\in\mathcal{G}$, $\Gamma(A)(X\cdot Y)=Y\cdot\Gamma(A)X$. The proof of this implication will be carried by induction on n.

For $X=1$ the previous equation becomes $\Gamma(A)Y=Y$ which corresponds to our assertion for $n=0$. Now assume that the assertion is true for any $k\leq n-1$ and let us prove it for $k=n$.

Let $X=I_n(h_n)$ for some symmetric $h_n\in\mathcal{L}^2([0,T]^n)$. Then by formula (3.4) (to ease the notation we write $\langle\cdot,\cdot\rangle_k$ for $\langle\cdot,\cdot\rangle_{\mathcal{L}^2([0,T]^k)}$, $0\leq k\leq n$),

$$\begin{aligned}
&\Gamma(A)(I_n(h_n)Y)\\
&= \Gamma(A)\Big(\sum_{k=0}^{n-1}\frac{1}{k!}\langle D^kI_n(h_n)\diamond D^kY\rangle_k+\frac{1}{n!}\langle n!h_n, D^nY\rangle_n\Big)\\
&= \sum_{k=0}^{n-1}\frac{1}{k!}\langle\Gamma(A)(D^kI_n(h_n))\diamond\Gamma(A)(D^kY)\rangle_k+\langle h_n,\Gamma(A)D^nY\rangle_n\\
&= \sum_{k=0}^{n-1}\frac{1}{k!}\langle\Gamma(A)(D^kI_n(h_n))\diamond(A^*)^{\otimes k}D^kY\rangle_k+\langle h_n,\Gamma(A)D^nY\rangle_n\\
&= \sum_{k=0}^{n-1}\frac{1}{k!}\langle A^{\otimes k}\Gamma(A)(D^kI_n(h_n))\diamond D^kY\rangle_k+\langle h_n,\Gamma(A)D^nY\rangle_n
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{n-1} \frac{1}{k!} \langle D^k \Gamma(A) I_n(h_n) \diamond D^k Y \rangle_k + \langle h_n, \Gamma(A) D^n Y \rangle_n \\
&= \Gamma(A) I_n(h_n) \cdot Y - \frac{1}{n!} \langle D^n \Gamma(A) I_n(h_n) \diamond D^n Y \rangle_n + \langle h_n, \Gamma(A) D^n Y \rangle_n \\
&= \Gamma(A) I_n(h_n) \cdot Y - \langle A^{\otimes n} h_n, D^n Y \rangle_n + \langle h_n, \Gamma(A) D^n Y \rangle_n \\
&= \Gamma(A) I_n(h_n) \cdot Y - \langle h_n, (A^*)^{\otimes n} D^n Y - \Gamma(A) D^n Y \rangle_n .
\end{aligned}
$$

Comparing the first and the last terms of the above chain of equalities with our hypothesis we deduce that

$$
\langle h_n, (A^*)^{\otimes n} D^n Y - \Gamma(A) D^n Y \rangle_{\mathcal{L}^2([0,T]^n)} = 0, \mathcal{P}\text{-a.e..}
$$

Moreover since the identity above holds for any $h_n \in \mathcal{L}^2([0,T]^n)$ we conclude that

$$
(A^*)^{\otimes n} D^n Y = \Gamma(A) D^n Y, \mathcal{P} \otimes (dt)^n\text{-a.e..}
$$

(2) $\Rightarrow$ (1): Fix $f \in \mathcal{L}^2([0,T])$; then

$$
\begin{aligned}
E[\Gamma(A)(XY)\mathcal{E}(f)] &= E[XY\mathcal{E}(A^* f)] \\
&= \sum_{n\geq 0} n! \langle \frac{E[D^n X \mathcal{E}(A^* f)]}{n!}, \frac{E[D^n Y \mathcal{E}(A^* f)]}{n!} \rangle_{\mathcal{L}^2([0,T]^n)} \\
&= \sum_{n\geq 0} \frac{1}{n!} \langle E[\Gamma(A) D^n X \mathcal{E}(f)], E[\Gamma(A) D^n Y \mathcal{E}(f)] \rangle_{\mathcal{L}^2([0,T]^n)} \\
&= \sum_{n\geq 0} \frac{1}{n!} \langle E[\Gamma(A) D^n X \mathcal{E}(f)], E[(A^*)^{\otimes n} D^n Y \mathcal{E}(f)] \rangle_{\mathcal{L}^2([0,T]^n)} \\
&= \sum_{n\geq 0} \frac{1}{n!} \langle E[A^{\otimes n} \Gamma(A) D^n X \mathcal{E}(f)], E[D^n Y \mathcal{E}(f)] \rangle_{\mathcal{L}^2([0,T]^n)} \\
&= \sum_{n\geq 0} \frac{1}{n!} \langle E[D^n \Gamma(A) X \mathcal{E}(f)], E[D^n Y \mathcal{E}(f)] \rangle_{\mathcal{L}^2([0,T]^n)} \\
&= \sum_{n\geq 0} \frac{1}{n!} \int_{[0,T]^n} E[D_t^n \Gamma(A) X \mathcal{E}(f)] E[D_t^n Y \mathcal{E}(f)] dt \\
&= \sum_{n\geq 0} \frac{1}{n!} \int_{[0,T]^n} E[(D_t^n \Gamma(A) X \diamond D_t^n Y) \mathcal{E}(f)] dt \\
&= E\Big[\Big(\sum_{n\geq 0} \frac{1}{n!} \int_{[0,T]^n} D_t^n(\Gamma(A)X) \diamond D_t^n Y \, dt\Big) \mathcal{E}(f)\Big] \\
&= E[(\Gamma(A) X \cdot Y) \mathcal{E}(f)].
\end{aligned}
$$

The proof is complete. □

Remark 4.2. If $A : \mathcal{L}^2([0,T]) \to \mathcal{L}^2([0,T])$ is an orthogonal operator, i.e. $AA^* = I$, then the condition

$$
\Gamma(A) Y = Y \text{ implies } \Gamma(A) D^n Y = (A^*)^{\otimes n} D^n Y \text{ for all } n \geq 1.
$$

In fact,

$$\begin{aligned} \Gamma(A)Y = Y \quad &\Rightarrow \quad D^n\Gamma(A)Y = D^nY \\ &\Rightarrow \quad A^{\otimes n}\Gamma(A)D^nY = D^nY \\ &\Rightarrow \quad (A^*A)^{\otimes n}\Gamma(A)D^nY = (A^*)^{\otimes n}D^nY \\ &\Rightarrow \quad \Gamma(A)D^nY = (A^*)^{\otimes n}D^nY. \end{aligned}$$

Corollary 4.3. *Let $A : \mathcal{L}^2([0,T]) \to \mathcal{L}^2([0,T])$ be an orthogonal operator and let $Y \in \mathcal{G}$. Then the following statements are equivalent:*

(1) *For all $X \in \mathcal{G}$, $\Gamma(A)(X \cdot Y) = Y \cdot \Gamma(A)X$, $\mathcal{P}$-a.e..*
(2) $\Gamma(A)Y = Y$, *$\mathcal{P}$-a.e..*

Remark 4.4. It is known (see for instance Janson [5]) that if A is an orthogonal operator then

$$\Gamma(A)(X \cdot Y) = (\Gamma(A)X) \cdot (\Gamma(A)Y);$$

therefore if $\Gamma(A)Y = Y$ we get that

$$\Gamma(A)(X \cdot Y) = Y \cdot (\Gamma(A)X),$$

which is the statement of Corollary 4.3.

Remark 4.5. If $A : \mathcal{L}^2([0,T]) \to \mathcal{L}^2([0,T])$ is a projection, i.e. $A^2 = A$ and $A^* = A$, then the condition

$$\Gamma(A)Y = Y \text{ implies } \Gamma(A)D^nY = (A^*)^{\otimes n}D^nY \text{ for all } n \geq 1.$$

In fact, if $Y = \sum_{k\geq 0} I_k(h_k)$, the condition

$$\Gamma(A)D^nY = (A^*)^{\otimes n}D^nY,$$

corresponds to the conditions

$$A^{\otimes n}h_k = A^{\otimes k-n}h_k, \text{ for any } k \geq n. \tag{4.1}$$

Since by assumption

$$A^{\otimes k}h_k = h_k, \text{ for any } k \geq 0,$$

conditions (4.1) are trivially true because for any $j \leq k$,

$$A^{\otimes j}h_k = h_k.$$

Corollary 4.6. *Let $A : \mathcal{L}^2([0,T]) \to \mathcal{L}^2([0,T])$ be a projection and let $Y \in \mathcal{G}$. Then the following statements are equivalent:*

(1) *For all $X \in \mathcal{G}$, $\Gamma(A)(X \cdot Y) = Y \cdot \Gamma(A)X$, $\mathcal{P}$-a.e..*
(2) $\Gamma(A)Y = Y$, *$\mathcal{P}$-a.e..*

Remark 4.7. In Janson [5] it is shown that if A is an orthogonal projection then $\Gamma(A)$ is a conditional expectation w.r.t. a certain sigma-algebra. For conditional expectations it is known that

$$E[XY|\mathcal{B}] = YE[X|\mathcal{B}] \quad \Longleftrightarrow \quad E[Y|\mathcal{B}] = Y,$$

where $\mathcal{B}$ is a sub-σ-algebra of $\mathcal{F}$. Corollary 4.6 shows that the previous equivalence follows as a particular case of our main theorem.

The next example shows that for some A and Y the conditon

$$\Gamma(A)D^nY = (A^*)^{\otimes n}D^nY, \tag{4.2}$$

fails to be true only for some $n \in \mathbb{N}$.

Example 4.8. Let $h, g \in \mathcal{L}^2([0,T])$ be such that

$$|h|_{\mathcal{L}^2([0,T])} = |g|_{\mathcal{L}^2([0,T])} = 1 \text{ and } \langle h, g\rangle_{\mathcal{L}^2([0,T])} = 0.$$

Define the operator $A : \mathcal{L}^2([0,T]) \to \mathcal{L}^2([0,T])$ as follows

$$(Af)(t) := \lambda\langle f, h\rangle_{\mathcal{L}^2([0,T])}h(t) + \frac{1}{\lambda}\langle f, g\rangle_{\mathcal{L}^2([0,T])}g(t), \quad t \in [0,T],$$

for some $\lambda \neq 0$ and choose $Y = I_1(h) \diamond I_1(g)$. Then

$$\begin{aligned} \Gamma(A)(I_1(h) \diamond I_1(g)) &= I_1(Ah) \diamond I_1(Ag) \\ &= (\lambda I_1(h)) \diamond \Big(\frac{1}{\lambda}I_1(g)\Big) \\ &= I_1(h) \diamond I_1(g). \end{aligned}$$

However,

$$\begin{aligned} \Gamma(A)D_t(I_1(h) \diamond I_1(g)) &= \Gamma(A)(h(t)I_1(g) + g(t)I_1(h)) \\ &= h(t)I_1(Ag) + g(t)I_1(Ah) \\ &= \frac{1}{\lambda}h(t)I_1(g) + \lambda g(t)I_1(h), \end{aligned}$$

while

$$\begin{aligned} AD_t(I_1(h) \diamond I_1(g)) &= A(h(t)I_1(g) + g(t)I_1(h)) \\ &= Ah(t)I_1(g) + Ag(t)I_1(h) \\ &= \lambda h(t)I_1(g) + \frac{1}{\lambda}g(t)I_1(h). \end{aligned}$$

Hence

$$\Gamma(A)D_t(I_1(h) \diamond I_1(g)) \neq AD_t(I_1(h) \diamond I_1(g)).$$

Moreover, since

$$D^2_{t_1t_2}Y = h(t_1)g(t_2) + h(t_2)g(t_1),$$

one has

$$\Gamma(A)D^2Y = Y = (A^*)^{\otimes 2}D^2Y.$$

Higher order derivatives are identically zero and hence the condition (4.2) is trivially true for $n \geq 3$. We have therefore proved that with A and Y as above condition (4.2) fails only for $n = 2$.

The next theorem provides another necessary and sufficient condition on the random variable Y for the property (1.3) to be true. In the sequel we denote by T_f the so called translation operator which is given by

$$T_fY(\omega) := Y\Big(\omega + \int f\Big).$$

We refer the reader to Kuo [6] for the details on this operator.

Theorem 4.9. *Let* $Y \in \mathcal{G}$ *and* $A : \mathcal{L}^2([0,T]) \to \mathcal{L}^2([0,T])$ *be a bounded linear operator.*

(1) *If for all* $X \in \mathcal{G}$, $\Gamma(A)(X \cdot Y) = Y \cdot \Gamma(A)X$, *then*

$$\Gamma(A)T_f Y = T_{Af}Y, \text{ for all } f \in \mathcal{L}^2([0,T]).$$

(2) *If* $\Gamma(A^*)T_f Y = T_{A^*f}Y$, *for all* $f \in \mathcal{L}^2([0,T])$, *then*

$$\Gamma(A)(X \cdot Y) = Y \cdot \Gamma(A)X, \text{ for all } X \in \mathcal{G}.$$

(3) *If A is self-adjoint the following two statements are equivalent:*
- $\Gamma(A)(X \cdot Y) = Y \cdot \Gamma(A)X$, *for all* $X \in \mathcal{G}$.
- $\Gamma(A)T_f Y = T_{Af}Y$, *for all* $f \in \mathcal{L}^2([0,T])$.

Proof. (1): Let us assume that for all $X \in \mathcal{G}$, $\Gamma(A)(X \cdot Y) = Y \cdot \Gamma(A)X$. Choosing $X = \mathcal{E}(f)$ for some $f \in \mathcal{L}^2([0,T])$ the previous equality reads:

$$\Gamma(A)(\mathcal{E}(f) \cdot Y) = Y \cdot \mathcal{E}(Af). \tag{4.3}$$

By means of the Gjessing's formula (see Holden et al. [2]) the left hand side of (4.3) can be rewritten as:

$$\begin{aligned} \Gamma(A)(\mathcal{E}(f) \cdot Y) &= \Gamma(A)(\mathcal{E}(f) \diamond T_f Y) \\ &= \mathcal{E}(Af) \diamond \Gamma(A)T_f Y \\ &= \mathcal{E}(Af) \cdot T_{-Af}\Gamma(A)T_f Y. \end{aligned}$$

Therefore

$$T_{-Af}\Gamma(A)T_f Y = Y,$$

which is equivalent to our assertion.

(2): Now suppose that $T_{-A^*f}\Gamma(A^*)T_f Y = Y$, for all $f \in \mathcal{L}^2([0,T])$. We use the Gjessing's formula again to obtain:

$$\begin{aligned} E[(\Gamma(A)X \cdot Y)\mathcal{E}(f)] &= E[X\Gamma(A^*)(Y\mathcal{E}(f))] \\ &= E[X\Gamma(A^*)(T_f Y \diamond \mathcal{E}(f))] \\ &= E[X(\Gamma(A^*)T_f Y \diamond \mathcal{E}(A^*f))] \\ &= E[X(T_{-A^*f}\Gamma(A^*)T_f Y \cdot \mathcal{E}(A^*f))] \\ &= E[X \cdot Y \cdot \mathcal{E}(A^*f))] \\ &= E[\Gamma(A)(X \cdot Y) \cdot \mathcal{E}(f))]. \end{aligned}$$

The proof is complete.

(3): It follows immediately from (1) and (2). □

Remark 4.10. If we assume that $\|A\| \leq 1$, then we can relax the assumptions on X and Y. It is in fact known (see Janson [5]) that for $\|A\| \leq 1$ the operator $\Gamma(A)$ can be extended to a bounded linear operator from $\mathcal{L}^q(\mathcal{P})$ into itself for all $q \geq 1$. Therefore Theorem 4.9 remains true for all X and Y such that $XY \in \mathcal{L}^q(\mathcal{P})$ for some $q > 1$.

Remark 4.11. It was proved in Potthoff and Timpel [11] that for $Y \in \mathcal{G}$ one has

$$T_f Y = \sum_{n\geq 0} \frac{1}{n!} \langle D^n Y, f^{\otimes n} \rangle_{\mathcal{L}^2([0,T])},$$

where the series converges in $\mathcal{G}$. By means of this result the condition

$$\Gamma(A) T_f Y = T_{Af} Y, \text{ for all } f \in \mathcal{L}^2([0,T]),$$

is equivalent to

$$\sum_{n\geq 0} \frac{1}{n!} \langle \Gamma(A) D^n Y, f^{\otimes n} \rangle_{\mathcal{L}^2([0,T])} = \sum_{n\geq 0} \frac{1}{n!} \langle D^n Y, (Af)^{\otimes n} \rangle_{\mathcal{L}^2([0,T])},$$

which implies $\Gamma(A) D^n Y = (A^*)^{\otimes n} D^n Y$. This is precisely the condition of Theorem 4.1.

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