Seminar on Continuity in Semilattices

Volume 1 | Issue 1

March 2023

Odds and Ends

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Recommended Citation
Lawson, Jimmie D. (2023) "Odds and Ends," Seminar on Continuity in Semilattices: Vol. 1: Iss. 1, Article 105.
Available at: https://repository.lsu.edu/scs/vol1/iss1/105
1. Chains and CL

A canonical way to pick strict subchains

Let $S \subseteq \mathbb{N}$, $T$ a closed chain. (Sup closed is all that seems to be of interest here)

The idea of this construction is to retain as many as possible of $\sup J$ where $J$ is a maximal interval in $\mathbb{N}$ with $a, b \in J, a < b \Rightarrow a \neq b$.

We first throw out the open intervals that appear in each $J$. We also throw out inf $J$ if it is not sup $J'$ for some other interval $J'$.

Let $T'$ be what is left.

Now define a relation $\preceq$ by $x \preceq y$ if $x,y \in J$ for one of the maximal intervals. Let $\equiv$ be the transitive closure of $\preceq$. If $A$ is a $\equiv$ equivalence class, there are three cases to consider:

Case (i). $A$ has a largest element $m$. Then $A = \{x_i : i \geq 0, i \leq m\}$ if $A$ is finite or $A = \{x_i : i \in \mathbb{N}\}$ if $A$ is infinite, where $x_0 = m$ and $x_i \preceq x_{i+1}$ for all $i$, but $x_{i+1} \preceq x_i$ for $j > 1$.

For this case we keep all elements whose index is an even integer.

Case (ii) A has a least element $p$, but no greatest element.

This time number the elements from the bottom up and retain the even ones.

Case (iii) A has no least or greatest element.

One has 2 choices in this case, but both are arbitrary. Pick $p \in A$. Number up and down from $p$ with all the integers, and again retain the even ones.
Chains in $E_L$

A. Picking strict subchains canonically

Let $S \subseteq E_L$, $T$ a closed chain.

We wish to pick a subchain of $T$ which is a maximal strict subchain of $T$ and for which the inclusion mapping from $M$ to $S$ preserves arbitrary sups.

Step 1. Define a relation $\mathcal{P}$ on $T \times T$ by $x \mathcal{P} y$ if $x \leq y$ and $y \leq x$, or $x = y$. Every interval of $T$ having all of its elements $\mathcal{P}$ related is contained in a maximal one, say $J$. Furthermore, $\text{sup} J \in J$. The maximal such $J$ cover $T$ (many may be singletons), but they need not be pairwise disjoint.

Step 2. Let $J$ be a maximal $\mathcal{P}$-interval. $J$ may or may not have an inf which is again in $J$. If $\inf J \in J$, delete all elements of $J$ except $\inf J$ and $\sup J$. If $\inf J \notin J$, then we are free to delete all elements of $J$ except one. If we wish to be the least arbitrary, we can keep only $\sup J$, since $\sup J \in J$. Clearly, if the final result is strict, we can keep at most one element $\theta$ of each $J$. After these deletions, for each maximal $J$, let $M_j$ be the chain that is left. Each element $x$ of $M_j$ has at most one element above it and at most one below it, so $x \leq y$ and $y \leq x$.

Step 3. Consider the reflexive, symmetric relation $\mathcal{P}$ restricted to $M$. Let $\mathcal{P}^*$ be its transitive closure. Let $Q$ be a $\mathcal{P}^*$ equivalence class. There are three cases to consider:

Case (i) $Q$ has a least element $q$.

Then $Q = \{x_i : 0 \leq i < n\}$ or $Q = \{x_i : 0 \leq i < 3\}$ (depending on whether $Q$ is finite or infinite).
At the end of this process one has a strict maximal chain. One obtains furthermore a chain for which the inclusion mapping preserves arbitrary infs.