Seminar on Continuity in Semilattices

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Odds and Ends

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ODDS AND ENDS (JDL)

① Chains and $\mathscr{CL}$

A. A canonical way to pick strict subchains

Let $S \in \mathscr{CL}$, T a closed chain. (Sup closed is all that seems to be of interest here) The idea of this construction is to retain as many as possible of $\{\sup J\}$ where J is a maximal interval in T with $a, b \in J$, $a < b \Rightarrow a \not\ll b$.

We first throw out the open intervals that appear in each J. We also throw out $\inf J$ if it is not $\sup J'$ for some other interval J'. Let T' be what is left.

Now define a relation ρ on T' by $x \rho y$ if $x, y \in J$ for one of the maximal intervals. Let $\rho^{\#}$ be the transitive closure of ρ. If A is a $\rho^{\#}$ equivalence class, there are three cases to consider

Case (i). A has a largest element m. Then $A = \{x_i : i = 0, 1, \ldots m\}$ if A is finite or $A = \{x_i : i \in \omega\}$ if A is infinite where $x_0 = m$ and $x_i \rho x_{i+1}$ for all i, but $x_{i+j} \ll x_i$ for $j > 1$.

For this case we keep all elements whose index is an even integer.

Case (ii) A has a least element p, but no greatest element. This time number the elements from the bottom up and retain the even ones.

Case (iii) A has no least or greatest element. One has 2 choices in this case, ~~which~~ and the one chosen is arbitrary. Pick $p \in A$. Number up and down from p with all the integers, and again retain the even ones!

ODDS AND ENDS (JDL)

① Chains in $\mathcal{CL}$

A. Picking strict subchains canonically

Let $S \in \mathcal{CL}$, T a closed chain.
We wish to pick a subchain of T which is a maximal strict subchain M of T and for which the inclusion mapping from M to S preserves arbitrary sups.

Step 1. Define a relation ρ on $T \times T$ by $x \rho y$ if $x \not\ll y$ and $y \not\ll x$, or $x = y$. Every interval of T having all of its elements ρ related is contained in a maximal one, say J. Furthermore $\sup J \in J$. The maximal such J cover T (many may be singletons), but they need not be pairwise disjoint.

Step 2. Let J be a maximal ρ-interval. J may or may not have an inf which is again in J. If $\inf J \in J$, delete all elements of J except $\inf J$ and $\sup J$. If $\inf J \notin J$, then we all free to delete all elements of J except one. If we wish to be the least arbitrary, we can keep only $\sup J$, since $\sup J \in J$. Clearly if the final result is strict, we can keep at most one element of each J. After these deletions for each maximal J, let M_1 be the chain that is left. Each element x of M_1 has at most one element w above it and at most one y below it $\ni x \rho y$ and $x \rho w$.

Step 3. Consider the reflexive, symmetric relation ρ restricted to M_1. Let $\rho^{\#}$ be its transitive closure. Let Q be a $\rho^{\#}$ equivalence class. There are three cases to consider:

Case (i) Q has a least element q.
Then $Q = \{x_i : 0 \le i \le n\}$ or $Q = \{x_i : 0 \le i\}$ (depending on whether Q is finite or infinite)

At the end of this process one has a strict maximal chain. One obtains furthermore a chain for which the inclusion mapping preserves arbitrary infs.