Seminar on Continuity in Semilattices

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In the following, L always denotes a complete lattice together with a compact (Hausdorff) topology such that the translations $t_a : L \to L \,;\, x \to a \wedge x$ are continuous for all $a \in L$. Our aim is to show, the the operation $\wedge$ is jointly continuous.

We start with 2 technical notations:

1. A point $l \in L$ is called a λ-point, if there is a sequence of neighborhoods $(U_n)_n$ of l such that $\overline{U_{n+1}} \subseteq U_n$ and such that l is a minimal element of $\bigcap_{n \in N} U_n$.
2. If $A \subseteq L$ is any subset and if $x \in L$ is any point, then the star with center x and radius A is defined by $st_x(A) := \bigcup\{[a,x] : a \in A\}$

<u>3. Lemma</u>. (i) For all $a \in L$ the sets $\downarrow a$ and $\uparrow a$ are closed.

(ii) updirected and downdirected sets converge to their infimum and supremum.

(iii) every point is a supremum of λ-points.

Proof. (i) $\uparrow a = t_a^{-1}(a)$ and singletons are closed. Hence $\uparrow a$ is closed by the continuity of t_a. Next, assume that $a_i \in \downarrow a$ is a converging net. Then $a \wedge \lim a_i = t_a(\lim a_i) = \lim (t_a(a_i)) = \lim a \wedge a_i = \lim a_i$, i.e. $\lim a_i \leq a$. Hence $\downarrow a$ is closed.

(ii) Let D be any updirected set and let M be the set of upper bounds of M. Then for every $d \in D$, the net D is eventually in $\uparrow d \cap \downarrow m$, $m \in M$. Hence every cluster point of D is contained in $\bigcap_{d \in D} \uparrow d \cap \bigcap_{m \in M} \downarrow m = \{\sup D\}$, i.e. D has only one cluster point and this cluster point is equal to sup D. Therefore D converges to sup D.

(iii) We show: in every neighborhood of $a \in L$ there is a λ-point $l \in L$ which is less or equal to a. Indeed, let U be a neighborhood of a. By compactness, we may choose a sequence of neighborhoods (U_n) such that (i) all U_n are open and contained in U and (ii) for all n we have $\overline{U_{n+1}} \subseteq U_n$

Then $a \in \bigcap U_n = \bigcap \overline{U}_n$ and the latter set is closed. Hence by Zorn's lemma and (ii) we can pick a minimal element $1 \in \bigcap U_n$ which is smaller than a. Clearly, 1 is a λ-point.

4. Lemma. If 1 L is a λ-point, then 1 has a countable base in $\downarrow 1$.

Proof. Clear.

5. Lemma. If 1 L is a λ-point and if $(a_n)_n$ is a sequence contained in $\downarrow 1$ such that $1 = \lim a_n$, then there exists a subsequence $(b_n)_n$ such that $1 = \liminf b_n$.

Proof. Let $(U_n)_n$ be a sequence of open sets such that $\overline{U}_{n+1} \subseteq U_n$ and such that $\{1\} = \bigcap U_n \cap \downarrow 1$. Choose n_1 such that for all $n \geq n_1$ we have $a_n \in U_1$. Define $b_1 := a_{n_1}$ and $V_1 := U_1$. If b_m and $V_m \ni 1, b_m$ are already defined, define $V_{m+1} := U_{m+1} \cap t_{b_m}^{-1}(V_m)$. Clearly, V_{m+1} is a open neighborhood of 1 as translations are continuous. Choose a number n_{m+1} such that for all $n' \geq n_{m+1}$ have $a_{n'} \in V_{m+1}$ and such that $n_{m+1} \geq n_1, \ldots, n_m$. Define $b_{m+1} := a_{n_{m+1}}$. Then an easy calculation shows $b_m \wedge \ldots \wedge b_n \in V_n \subseteq U_n$ for all $m \geq n$, hence $\bigwedge_{m \geq n} b_m \in \overline{U}_n$. But this implies that for all $n \in N$ the set $\downarrow \liminf b_n \cap \overline{U}_n \neq \emptyset$. Compactness yields that $\downarrow \liminf b_n \cap \bigcap \overline{U}_n \neq \emptyset$. Clearly, $\liminf b_n \leq 1$. As 1 is minimal in $\bigcap U_n$, this yields $1 = \liminf b_n$.

6. Lemma. If 1 is a λ-point and if U is an open neighborhood of 1, then there is an open neighborhood V of 1 such that $st_1(V) \subseteq U$.

Proof. We define a set $W \subseteq \downarrow 1$ by $W := \{a : [a,1] \subseteq U\}$. Clearly, it is enough to show that W is a neighborhood of 1 in $\downarrow 1$. If W were not a neighborhood of 1 in $\downarrow 1$, then there would be a net $(a_i)_{i \in I} \subseteq \downarrow 1 \setminus$

such that $\lim a_i = 1$. As J has countable base in $\Downarrow 1$, we may assume that (a_i) is a sequence and by (5) we may assume that $1 = \lim\inf$. But then we have $\{1\} = \bigcap_n [\bigwedge_{m\geq n} a_m, 1] \subseteq U$. As intervals are closed and as U is open and as the intersection is down-directed, compactness yields an $n \in N$ such that $[\bigwedge_{m\geq n} a_m, 1] \subseteq U$, i.e. $[a_n, 1] \subseteq U$, i.e. $a_n \in W$, a contradiction.

7. Lemma. If $U \subseteq L$ is open, then $\uparrow U$ is open.

Proof. Let $a \in \uparrow U$. Then there is an $u \in U$ such that $a \geq u$. Now $t_u^{-1}(U)$ is an open set contained in $\uparrow U$ which contains a.

8. Lemma. A point $1 \in L$ is a λ-point if and only if there is a sequence of open upper sets $(U_n)_{n\ N}$ such that (i) $\overline{U}_{n+1} \subseteq U_n$ and (ii) $\bigcap U_n = \uparrow 1$ as well as (iii) $t_1(U_{n+1}) \subseteq U_n$.

Proof. By the previous results and by the continuity of t_1 we can find a sequence of open neighborhoods of 1, call it (V_n), such that (i) 1 is minimal in $\bigcap V_n$, (ii) $\overline{V}_{n+1} \subseteq V_n$, (iii) $st_1(V_{n+1}) \subseteq V_n$ and (iv) $t_1(V_{n+1}) \subseteq V_n$. Define a sequence (U_n) of open upper sets by $U_n = V_{3n}$. Then the sequence U_n has the desired properties:
Let $a \in \overline{U}_n$. Then there is a net $(a_i)_i$ with $a_i \in U_n$ and $\lim a_i = a$ For every i there is a $b_i \in V_{3n}$ such that $b_i \leq a_i$. But then $1 \wedge b_i \in V_{3n-1}$ and $1 \wedge b_i \leq 1 \wedge a_i \leq 1$, hence $1 \wedge a_i \in St_1(V_{3n-1}) \subseteq V_{3n-2}$. But this implies $1 \wedge a = 1 \wedge \lim a_i = \lim 1 \wedge a_i \in \overline{V}_{3n-2} \subseteq V_{3n-3}$, i.e. $a \in \uparrow V_{3n-3} = U_{n-1}$. This proves $\overline{U}_{n+1} \subseteq U_n$. Next, let $x \in U_{n+1}$. Then there is a $y \in V_{3n+3}$ such that $x \geq y$. Using the arguments given above we can conclude that $1 \wedge x \in V_{3n+1} \subseteq V_{3n} \subseteq \uparrow U_n$, i.e. $t_1(U_{n+1}) \subseteq U_n$. Finally, let $x \in U_n$ for all $n \in N$. As we have just seen this implies $1 \wedge x \in V_{3n}$ for all n, hence $1 \wedge x \in \bigcap V_{3n} = \bigcap V_n$. As 1 is minimal in this latter set and as

$1 \wedge x \leq 1$, we have $1 \leq x$. This proves (ii).

Conversely, if we have a sequence of open upper set (U_n) which satisfies (i) and (ii), then 1 is clearly a λ-point.

9. Corollary. The supremum of two λ-points is again a λ-point.

Proof. Let $1_1, 1_2$ be two λ-points. Then there are two sequences of open upper sets (U_n) and (V_n) fulfilling (i) and (ii) of (8). Define $W_n := U_n \cap V_n$. Then (W_n) is also a sequence of open upper sets such that $\overline{W}_{n+1} \subseteq W_n$ and such that $\bigcap W_n = \bigcap U_n \cap \bigcap V_n = \uparrow 1_1 \cap \uparrow 1_2 = \uparrow 1_1 \wedge 1_2$.

10. Corollary. Every Scott open set is open and conversely, every open upper set is Scott open.

Proof. Let U be any Scott open set. Let $a \in U$ be any point. As the set of all λ-point below a is up-directed and has supremum a, we can find a λ-point $1 \leq a$ such that $1 \in U$. Choose a sequence of open upper sets (U_n) fulfilling (i) - (iii) of (8). We show: there is a $n \in N$ such that $U_n \subseteq U$. Assume not. Then for every $n \in N$ we can find an $a_n \in U_n \setminus U$. By (iii), we have $a_n \wedge 1 \in U_{n-1} \setminus U$. As the sets $\downarrow 1 \cap U_n$ form a countable base for 1 in $\downarrow 1$, we have $\lim(a_n \wedge 1) = 1$. Therefore we can select a subsequence (b_n) such that $1 = \liminf b_n$. Clearly, as each b_n is of the form $a_m \wedge 1$, we have $b_n \notin U$. But this is a contradiction, because the liminf is a limit point in the Scott topology. - The other direction is easy, as up-directed set converge to their supremum.

11. Corollary. The graph of the odering is closed in LxL.

Proof. Let (a_i, b_i) be a net in LxL such that $a_i \leq b_i$ and such that both (a_i) and (b_i) converge. Let $a = \lim a_i$ and $b = \lim b_i$. Further, let 1 be any λ-point below a. Select a sequence (U_n) of open upper set such that $\overline{(U_{n+1})} \subseteq U_n$ and such that $\bigcap U_n = \uparrow 1$. As $1 \leq a$, we have $a \in U_n$ for each $n \in N$. Therefore the net a_i is eventually in U_n. As U_n is an upper set, the net b_i is eventually in U_n and hence $b = \lim b_i \in \overline{U}_n \subseteq U_{n-1}$. As this holds for every n, we can conclude that $b \in \bigcap U_n = \uparrow 1$, i.e. $b \geq 1$. As this is true for every λ-point below a and as a is the supremum of λ-points, we finally have $b \geq a$.

<u>12. Theorem</u> (J.D.Lawson). The mapping $\wedge : LxL \to L$ is jointly continuous.

<u>Proof</u>. First, notice that LxL in the product topology has the same properties than L. Hence every Scott open set of LxL is an open upper set. Now let U be any Scott open set of LxL and let $x \in U$ be a point. By the above remark we can find open sets V_1 and V_2 of L such that $x \in V_1 x V_2 \subseteq U$. If we let $U_1 = \uparrow V_1$ and $U_2 = \uparrow V_2$, we have $x \in U_1 x U_2 \subseteq U$. This means that the Scott topology of LxL is the product of the Scott topology on L. Hence $\wedge : LxL \to L$ is Scott continuous. So if U is any open upper set in L, the set $\{(x,y) : x \wedge y \in U\}$ is open in LxL. Finally, let V be any open lower set and let $a \wedge b \in V$. This implies $\downarrow a \cap \downarrow b \subseteq V$. Now by a result of Nachbin, we have $\downarrow a = \bigcap \{\overline{V} : a \in V \text{ and } \downarrow V = V\}$ and $\downarrow b = \bigcap \{\overline{V} : b \in V \text{ and } V = \downarrow V\}$. An compactness argument new yield open lower sets $V_1 \ni a$ and $V_2 \ni b$ such that $V_1 \cap V_2 \subseteq V$. Because $V_1 \cap V_2 = V_1 \wedge V_2$, we have that $\{(x,y) : x \wedge y \in V\}$ is open in LxL for every open lower set V. Again by a result of Nachbin, we can conclude that $\wedge : LxL \to L$ is continuous, because the open upper sets together with the open lower sets form a subbase for the topology in every compact partially ordered space with closed graph of $\leq$.