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Convexities

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SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

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TOPIC Convexities

REFERENCE [COMP] - A COMPENDIUM OF CONTINUOUS LATTICES
[V] FINITE DIMENSIONAL CONVEX STRUCTURES I: GENERAL RESULTS by M. van de Vel (See bibliography for further references)

At the miniworkshop at Tulane in November, '81, I spoke on some open problems in continuous lattices. In this report I would like to develop one of the four areas I mentioned, that of convexity. This past fall M. van de Vel at Vrije Universiteit in Amsterdam sent me a packet of reprints of his work in convexity together with a letter pointing out some very close connections between certain aspects of the theory and the theory of compact semilattices. I am enclosing a extract from the above mentioned reference together with some references to some of the work in this area. Anyone who is interested can write him for preprints and reprints. You can refer to the extract for definitions of terms concerning convexity that I use in the following.

Having been a rather close follower of the work that Hofmann and Lawson did on the spectral theory of continuous distributive lattices, I have sometimes wondered if perhaps there was some generalization to continuous lattices in general. The dissertation of Tiller and the work of Jamison suggested that some notion of convexity might be useful in this regard. But only in the last few days have I hit on an idea of what that convexity structure might be. More about that later.

The first question that one has to deal with is that of choosing those elements which will play the role of the "spectrum." The irreducibles $\neq 1$, IRR L, seem to be a logical candidate for the general case. However one might choose those elements maximal in the complement of a filter (a smaller set) or even those elements maximal in the complement of an open filter (an even smaller set).

West Germany: TH Darmstadt (Gierz, Keimel)
U. Tübingen (Mislove, Visit.)

England: U. Oxford (Scott)

USA: U. California, Riverside (Stralka)
LSU Baton Rouge (Lawson)
Tulane U., New Orleans (Hofmann, Mislove)
U. Tennessee, Knoxville (Carruth, Crawley)

Note that all three of these sets order-generate. Which of the sets turns out to have the nicer properties remains to be seen.

Secondly one needs a topology for this set (and also a name). In my first attempt here I shall choose the set IRR L to work with and equip it with the relative lower topology. I think R.-E. Hoffmann calls this topology the weak topology somewhere, a name I rather like. Note that this all reduces to the hull-kernel topology on PRIME L in the distributive case.

Finally one needs a convexity (or alignment) on IRR L in the sense of van de Vel. And why does one need this? Well, the hope is that one can represent L as the closed convex sets on IRR L. (Except the representation will be an anti-isomorphism since the closed convex sets are used; shades of O. Wyler.) And now comes the candidate for the convexity.

Proposition 1. Let A be a subset of L, a continuous lattice; then Ψ, the trace of of the convexity of all filters (including the empty filter) on L restricted to A is a convexity.

Proof. See [V, Section 1.4].

Note that in the case L is distributive, the convexity simply reduces to the upper or lower sets (depending on your viewpoint) of Spec L, which are recoverable simply from the topology. This is not the case in the more general setting. For a general continuous lattice L, we now let Spec L denote IRR L equipped with the relative lower topology and the filter trace convexity.

Proposition 2. Let L be continuous lattice.

(i) For all $x \in L$, $\uparrow x \cap \text{Spec } L$ is a closed convex set.

(ii) For all $A \subseteq L$, $\bigcap\{\uparrow x \cap \text{Spec } L : x \in A\} = \uparrow \sup A \cap \text{Spec } L$

(iii) For all $x, y \in L$, $\uparrow(x \wedge y) \cap \text{Spec } L$ = convex hull $(\uparrow x \cap \text{Spec } L) \cup (\uparrow y \cap \text{Spec } L)$

= closed convex hull $(\uparrow x \cap \text{Spec } L) \cup (\uparrow y \cap \text{Spec } L)$

Proof. (i) $\uparrow x$ is a filter and closed in the lower topology; thus (i).
(ii) follows as in the distributive case.
(iii) Any filter containing x and y contains $x \wedge y$; (iii) follows.

Proposition 2 tells us that the mapping $x \mapsto \uparrow x \cap \text{Spec } L$ is a lattice homomorphism from L into the complete lattice of all closed convex subsets of Spec L which preserves arbitrary sups. Let us denote this latter lattice by $\Theta(\text{Spec } L)$. (Note that we use inverse inclusion for our order on this lattice.)

In many settings it seems more appropriate to use non-empty closed convex sets. In this case we consider up-complete continuous semilattices. The theory works just as satisfactorily in this setting, and for the most part we adopt this viewpoint in the remainder of our report. Note that one passes back and forth between the two by the deletion or addition of a discrete 1.

2. An Example and a Question

Unfortunately not all continuous lattices can be represented in the previously given scheme.

Example 3. Given two up complete continuous semilattices L and M, one can identify the 0's together and form a new semilattice (where mixed products are 0). If one wants a continuous lattice then a discrete 1 may be attached to this new structure. In particular, let L be a countable product of the two-point lattice 2 and M be 2 and form the semilattice N by pasting at 0. Then the set A = = Prime L is a closed subset of N and is also convex since $F \cap$ IRR N = = A, where F is the filter N M. Thus A is a closed convex set; however, A is not of the form $\uparrow x \cap$ IRR N for any $x \in N$.

One might be tempted to think that perhaps one could find some other alignment which might work. However, the next proposition shows that this is not the case.

Proposition 4. Let S be an up-complete semilattice. Then Ψ, the trace of the filter alignment on IRR S, is the alignment generated by the sets of the form $\uparrow x \cap$ IRR S, $x \in S$.

Proof. If F is a filter, then $F \cap \text{IRR } S = \bigcup\{\uparrow x \cap \text{IRR } S : x \in S\}$, and this union is a directed union since F is a filter. Thus the union is in the alignment generated by all $\uparrow x \cap$ IRR S, since alignments are closed under directed unions. We have already seen that Ψ indeed is an alignment or convexity.

Problem: Find reasonable necessary and sufficient or necessary or sufficient conditions on a continuous semilattice S so that the function of Proposition 2 is onto. This is precisely what one needs to represent S as the closed convex (non-empty if S has no 1) subsets of Spec L. As a shot in the dark, modular might be sufficient. We establish some more sufficient conditions in what follows. Example 3 shows that the representation given in Proposition 2 is not always onto all the closed convex sets.

3. Properties of Convexities

In this section we explore some of the properties of convexities arising from continuous lattices. Throughout this section S will denote an up-complete continuous semilattice.

Proposition 5. Spec S is a topological convex structure, i.e. the polytopes(=convex hulls of finitely many points) are closed.

Proof. Let F be a finite subset of Spec S. Then the convex hull of F, $ch(F) = \uparrow x_1 x_2 \ldots x_n \cap$ Spec S, where $F = \{x_1, \ldots x_n\}$; thus ch(F) is closed.

The property of being a topological convex structure can be alternately characterized by saying the convexity admits a subbase of closed sets. (See [V, Section 1.2].) Another useful property is that of being closure stable, i.e. the closure of a convex set is convex. We turn now to the consideration of this property. Recall from [COMP, V.2] that

(#) Every up-complete continuous semilattice has a smallest closed generating set (which is also order generating), namely IRR S^-

Proposition 6. TAE

(1) Closures of filters in S are again filters. (The closure is taken in the CL-topology.)

(2) Spec S is closure stable, and every non-empty closed convex set is of the form $\uparrow x \cap$ Spec S for some $x \in S$.

Proof. (1) implies (2): Let A be a convex set in Spec S. Then there exists a filter F in S whose intersection with Spec S is A. Since Spec S order generates S and F is an upper set, $F \cap$ Spec S = A order generates F, and thus generates the filter F^-. Applying (#) to F^-, we conclude that A^- contains Spec $S \cap F^-$, which in turn contains $A^- \cap$ Spec S. Thus $A^- \cap$ Spec S = $F^- \cap$ Spec S = $\uparrow \inf F^- \cap$ Spec S since F^- is a closed filter. Thus the closure of A in Spec S is convex.

Now let A be a closed convex set in Spec S (closed relative to Spec S). Then there exists a filter F such that $F \cap$ Spec S = A. Then A^-(taken in S) generates the filter F^-, and thus by (#) contains Spec $S \cap F^-$. Thus A = $A^- \cap$ Spec S $\supseteq F^- \cap$ Spec S = $\uparrow \inf F^- \cap$ Spec S $\supseteq$ A, and so A = $\uparrow \inf F^- \cap$ Spec S, completing the implication.

(2) implies (1): Let F be a filter in S whose intersection with Spec S is A. By hypothesis $A^- \cap$ Spec S is a closed convex subset of Spec S, and hence of the form $\uparrow x \cap$ Spec S for some x. Now A order generates F and hence generates F^-; since A is contained in the closed filter $\uparrow x$, $F^- \subseteq \uparrow x$. Conversely since F^- is a compact semilattice

containing $\uparrow x \cap$ Spec S, ~~XXXXXXXXXXXXXXXX(XXXXSpecXS)XXXX~~ a generating set for $\uparrow x$, it contains $\uparrow x$. Thus the two are equal, and so F^- is a filter. This completes the proof.

1. PRELIMINARIES

1.1. SET THEORETIC CONVEXITIES.

A *convexity* (or: an *alignment*) on a set X is a collection $\mathcal{C}$ of subsets of X such that

(1) $\emptyset, X \in \mathcal{C}$;

(2) $\mathcal{C}$ is closed under intersection;

(3) $\mathcal{C}$ is closed under chain union.

The members of $\mathcal{C}$ are called *convex sets.* Usually, the term "convexity" refers to a family of sets with the properties (1) and (2) only. As the axiom (3) is both essential and standard in our treatment of the theory we prefer to "assume that all convexities in consideration are alignments". The pair $(X,\mathcal{C})$, consisting of a set equipped with a convexity, will henceforth be called a *convex structure.* If no confusion can arise, we will write X instead of $(X,\mathcal{C})$.

A convex set with a convex complement will be called a *half-space.* The (*convex*) *hull* of a set A is defined to be the set

$$h(A) = \cap \{ C : A \subset C, C \text{ convex}\}.$$

The hull of a finite set is also called a *polytope*, and the hull of a two-point set is called an *interval* or *segment* between these points.

The axiom (3) is equivalent with the important *domain finiteness condition* (also known as the *finitary property*), which states that a set is convex iff it includes the hull of each of its *finite* subsets. Equivalently, the union of an *upward filtered* family of convex sets is convex again. See [J_1,p.6].

A collection $\mathcal{B} \subset \mathcal{C}$ is called a *base for* the convexity $\mathcal{C}$ if each member of $\mathcal{C}$ can be obtained as the union of an upward filtered subfamily of $\mathcal{B}$.

Equivalently, $\mathcal{B}$ contains all $\mathcal{C}$-polytopes. A collection $\mathcal{S} \subset \mathcal{C}$ is a *subbase for* $\mathcal{C}$ if the intersections of members of $\mathcal{S}$ constitute a base for $\mathcal{C}$. We then say that $\mathcal{C}$ is *generated by* $\mathcal{S}$.

1.2. TOPOLOGICAL CONVEXITIES.

If a set X is equipped with both a topology and a convexity such that all polytopes are closed, then X is called a *topological convex structure*. An equivalent condition is that the convexity on X admits a subbase of *closed* sets. The convexity on X will then be called a *topological convexity*. It will be assumed throughout that *singletons are convex* (making the underlying space into a T_1 space), and that a topological convexity is *closure stable*, that is: the closure of each convex set is convex again. See [J_1], [V_1].

For conveniency, we will sometimes denote the collection of all *nonempty closed convex* sets of a topological convexity $\mathcal{C}$ by $\mathcal{C}^*$.

1.3. C.P. MAPS AND SEPARATION PROPERTIES.

Let $(X,\mathcal{C})$ and $(X',\mathcal{C}')$ be convex structures, and let $f : X \to X'$ be a function. We say that f is *convexity preserving relative to* $\mathcal{C}$ *and* $\mathcal{C}'$ (briefly, f is *C.P.*) if $f^{-1}(C') \in \mathcal{C}$ for each $C' \in \mathcal{C}'$. Equivalently, ($h = h_{\mathcal{C}}$; $h' = h_{\mathcal{C}'}$)

$$fh(A) \subset h'f(A),\ A \subset X \text{ finite}.$$

See [vMV_2], [V_1].

A function $f : X \to [0,1]$ is said to *separate the subsets* A,B of X if $f(A) \subset \{0\}$, $f(B) \subset \{1\}$. A continuous function will be called a *map*. In the sequel, [0,1] will always be equipped with the *linear convexity*, i.e. the (topological) convexity generated by the sets of type

$$[0,t],\ [t,1],\ t \in [0,1].$$

A topological convex structure $(X,\mathcal{C})$ (or: its convexity $\mathcal{C}$) is said to be:

semi-regular, if each $C \in \mathcal{C}^*$ can be separated from each $x \in X \setminus C$ by a C.P. map $X \to [0,1]$;

regular, if each $C \in \mathcal{C}^*$ can be separated from each polytope $D \subset X \setminus C$ by a C.P. map $X \to [0,1]$;

normal, if every two disjoint sets $C,D \in \mathcal{C}^*$ can be separated by a C.P. map $X \to [0,1]$.

See [V_1, 1.5] for an equivalent description in terms of *screening* (the latter notion will be explained in section 2 below). A nice motivation for the above definitions can be found in [vMW] or [V_3].

It was shown in [V_1, 2.4] that a regular convexity on a *compact* space is normal. On *non-compact spaces*, normality of a convexity seems to be a rather unrealistic assumption. E.g. a locally convex linear space equipped with its linear (topological) convexity is not normal unless its algebraic dimension is at most 1. They are, however, all regular, and it appears that the latter is the best possible separation property in general. Most of our results below can already be obtained using semi-regularity.

1.4. TRACE OF A CONVEXITY.

If $(X,\mathcal{C})$ is a convex structure, and if $Y \subset X$, then

$$\mathcal{C} \upharpoonright Y = \{C \cap Y \mid C \in \mathcal{C}\}$$

is again a convexity (cf. [J, p. 22]) which will be called the *trace of* $\mathcal{C}$ *on* Y. We will usually consider the case where Y is a *convex* subset. Then $\mathcal{C} \upharpoonright Y$ is a topological convexity (including: closure stability) if $\mathcal{C}$ is. In this case, Y inherits both semi-regularity and regularity from $(X,\mathcal{C})$.

In the sequel a convex set in a topological convexity will *always* be equipped with the trace convexity.

2.

The closure of a set A will henceforth be denoted either by $\overline{A}$ or by Cl(A). If O is an *open* set, then its boundary $\overline{O} \setminus O$ will also be denoted by $\dot{O}$.

A *hyperplane* of a topological convexity $(X, \mathcal{C})$ is a set of type $\dot{O}$, where $O \subset X$ is an open half-space. Note that a hyperplane is a convex closed set (by closure stability), and that sets of type

$$f^{-1}[0,t) \text{ or } f^{-1}(t,1], \quad t \in [0,1],$$

are open half-spaces of X if $f : X \to [0,1]$ is a C.P. map.

2.2. SCREENING AND SEPARATORS.

Let A,A',B,B' be subsets of X. We say that (A',B') *screens* (A,B) if

$$A \subset A' \setminus B', \quad B \subset B' \setminus A', \quad A' \cup B' = X.$$

A pair (A',B') is called a *screening pair* if it screens some pair of nonempty sets.

A closed subset C of a space X is called a *separator of* X if $X \setminus C$ is disconnected. If O_1, O_2 are nonempty disjoint open sets of X with

$$X \setminus C = O_1 \cup O_2,$$

and if moreover $A_1 \subset O_1$, $A_2 \subset O_2$, then C is also said to *separate* A_1 *from* A_2 (or: to *separate between* A_1, A_2). As is well-known, the sets $C_1 = C \cup O_1$, $C_2 = C \cup O_2$, are closed, and (C_1, C_2) is a screening pair. Conversely, if (C_1, C_2) is a screening pair of closed sets, then $C_1 \cap C_2$ is a separator of X.

In the course of proving a result in $[V_1, 5.4]$, we used an argument concerning convex separators, but the partial result obtained from it was not mentioned explicitly. This result - essential for our present work - is the following one:

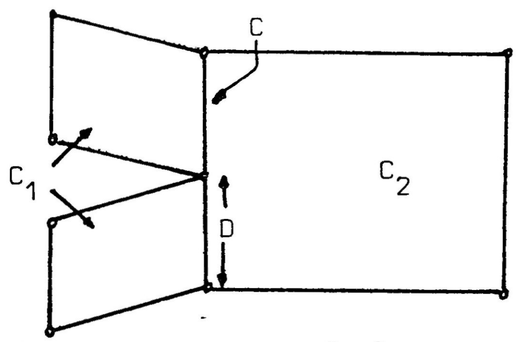
2.3. THEOREM.

Let X *be a semi-regular convex structure with connected convex sets, and let* (C_1, C_2) *be a screening pair of convex closed sets. Then*

(1) there is a minimal screening pair (D_1, D_2) *of convex closed sets such that* $D_1 \subset C_1$ and $D_2 \subset C_2$.

(2) If $C = C_1 \cap C_2$ *is a convex closed separator corresponding to a minimal screening pair* (C_1, C_2), *then for each dense convex set* $B \subset X$ *the set* $B \cap C$ *is dense in* C.

The reader should be warned that a separator which corresponds to a minimal screening pair need not be minimal as a separator. The picture below presents such a separator C, together with a strictly smaller separator D:



For this reason the terminology of $[V_1]$ - referring to C above as a "minimal separator" - is somewhat unfortunate, but we have no better terminology available except for a full description of the property. We will therefore maintain the terminology of $[V_1]$, but within quotes " ".

If $(X,\mathcal{C})$ is a topological convexity, $\mathcal{C}^\star$ will always denote the collection of *nonempty convex closed* sets of $(X,\mathcal{C})$. Note that $\mathcal{C}^\star \subset H(X)$. The *convex closure* $\overline{h}(A)$ of $A \subset X$ is defined to be the set $\mathrm{Cl}\, h(A)$. This gives rise to an operator

$$\overline{h} : H(X) \to H(X).$$

$(X,\mathcal{C})$ is called *continuous* if $\overline{h}$ is continuous (cf. [J, p. 45]), in which case $(X,\mathcal{C})$ is closure stable (cf. [V_1, thm 2.6]). For compact X, continuity of $(X,\mathcal{C})$ is equivalent to the following statement: $\mathcal{C}^\star$ is closed in H(X), and if $C \in \mathcal{C}^\star$ is contained in an open set $O \subset X$, then there exists a convex closed $D \subset X$ with

$$C \subset \mathrm{int}\, D \subset D \subset O \quad \text{(cf. [J, p. 46])}.$$

Continuity of $\overline{h}$ is a "reasonable" requirement on compact spaces only. See [V_1, thm 2.6] for weaker conditions on general spaces, and see [vMV_1] for other equivalent conditions on compact spaces.

4. COMPATIBLE UNIFORM STRUCTURES.

If $\mathcal{U}$ is a cover of X, and if $A \subset X$, then the *$\mathcal{U}$-star of* A is defined to be the set

$$\mathrm{st}\,(A,\mathcal{U}) = \cup\, \{U \mid U \in \mathcal{U},\ A \cap U \neq \emptyset\}.$$

4.1. DEFINITIONS.

Let $(X,\mathcal{C})$ be a topological convexity structure, and let μ be a (covering) uniformity structure for X. We say that μ is *compatible with* $\mathcal{C}$ if for each $\mathcal{U} \in \mu$ there is a $\mathcal{V} \in \mu$ with the following property: if $C \in \mathcal{C}$, then

$$h(\mathrm{st}\,(C,\mathcal{V})) \subset \mathrm{st}\,(C,\mathcal{U}).$$

If there is a uniformity structure for X compatible with $\mathcal{C}$, then $(X,\mathcal{C})$ will be called a *uniformizable convexity*. If the compatible uniformity is understood in the symbol X (as is the case with the topological structure), then $(X,\mathcal{C})$ will be called a *uniform convexity structure* (compare with: topological convexity structure).

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