

Seminar on Continuity in Semilattices

Volume 1 | Issue 1

Article 102

9-14-1981

Meet-Continuous Lattices in which Meet is not Continuous

Marcel Erné

Leibniz University Hannover, 30167, Hannover, Germany, erne@math.uni-hannover.de

Hartmut Gatzke

Leibniz Universität Hannover D-30167 Hannover Germany

Follow this and additional works at: <https://repository.lsu.edu/scs>



Part of the [Mathematics Commons](#)

Recommended Citation

Erné, Marcel and Gatzke, Hartmut (1981) "Meet-Continuous Lattices in which Meet is not Continuous," *Seminar on Continuity in Semilattices*: Vol. 1: Iss. 1, Article 102.

Available at: <https://repository.lsu.edu/scs/vol1/iss1/102>

NAME: Marcel Ern  and
Hartmut Gatzke

Date	M	D	Y
	9	14	81

TOPIC: Meet-continuous lattices in which meet is not continuous

- REFERENCES: [Bi] G.Birkhoff, Lattice Theory
- [C] A Compendium of CL
- [E 1] M.Ern , Order-topological lattices. Glasgow Math.J. 21 (1980) 57 - 68
- [E 2] M.Ern , Topologies on products of partially ordered sets III: Order convergence and order topology. Algebra Universalis 13 (1981) 1 - 23
- [Fla] J.Flachsmeyer, Einige topologische Fragen in der Theorie der Booleschen Algebren. Arch.Math. 16 (1965) 25 - 33
- [Flo] E.E.Floyd, Boolean algebras with pathological order topologies. Pacific J.Math. 5 (1955) 687 - 689.

At the Bremen Workshop 1982 on Continuous Lattices, the problem arose to find examples of meet-continuous lattices in which the binary meet-operation is not (jointly) continuous with respect to the Scott topology. As complete Boolean lattices are always join- and meet-continuous, it appears reasonable to look for examples within this class. It turns out that a complete Boolean lattice whose meet-operation is continuous in the Scott topology (or order topology) must be Hausdorff in its order topology (see the Theorem below). Papers of Floyd [Flo] and Flachsmeyer [Fla] provide us with enough examples of Boolean lattices whose order topology fails to be Hausdorff, e.g. the lattice of regular open subsets of \mathbb{R} . Such lattices cannot be topological (semi)lattices with respect to the order topology $\theta(B)$ or the Scott topology $\sigma(B)$; in particular,

$$\theta(B \times B) \neq \theta(B) \times \theta(B) \quad \text{and} \quad \sigma(B \times B) \neq \sigma(B) \times \sigma(B) .$$

THEOREM. Consider the following statements on a complete Boolean lattice B :

- (a) B is atomic (i.e. isomorphic to a power set).
- (b) B is continuous.
- (c) $\mathcal{O}(B \times B) = \mathcal{O}(B) \times \mathcal{O}(B)$.
- (d) $\sigma(B \times B) = \sigma(B) \times \sigma(B)$.
- (e) $(B, \mathcal{O}(B))$ is a topological lattice (\vee -semilattice, \wedge -semilattice).
- (f) $(B, \sigma(B))$ is a topological lattice.
- (g) $(B, \sigma(B))$ is a topological \wedge -semilattice.
- (h) $(B, \sigma(B))$ is a topological \vee -semilattice.
- (i) The Bi-Scott topology $\sigma(B) \vee \sigma(B^{op})$ is Hausdorff.
- (j) The order topology $\mathcal{O}(B)$ is Hausdorff.
- (k) The Scott topology $\sigma(B)$ is sober.

The following implications are always true:

$$\begin{array}{c}
 (a) \Leftrightarrow (b) \Rightarrow (c) \Rightarrow (d) \\
 \qquad \qquad \qquad \downarrow \qquad \downarrow \\
 \qquad \qquad \qquad (e) \Rightarrow (f) \Rightarrow (g) \Rightarrow (i) \Rightarrow (j) \\
 \qquad \qquad \qquad \qquad \qquad \downarrow \\
 \qquad \qquad \qquad (h) \Rightarrow (k) .
 \end{array}$$

All implications except $(g) \Rightarrow (i)$ are easily verified or well-known from [C], [E 1] and [E 2]. The implication $(g) \Rightarrow (i)$ may be strengthened as follows. If $(B, \sigma(B))$ is a topological \wedge -semilattice then for $x \not\leq y$ in B there exist $U \in \sigma(B)$ and $V \in \sigma(B^{op})$ with $x \in U$, $y \in V$ and $U \cap V = \emptyset$. In other words, the relation \leq is closed with respect to the order topology, and in particular, $\mathcal{O}(B)$ is Hausdorff.

PROOF. $x \not\leq y$ implies $x \in B \setminus \downarrow y \in \sigma(B)$. As $x = x \wedge x$, there exists $U \in \sigma(B)$ with $x \in U$ and $U \wedge U \subseteq B \setminus \downarrow y$, i.e. $u \wedge v \not\leq y$ for all $u, v \in U$. Define $V := \{v \in B : u \wedge v \leq y \text{ for some } u \in U\}$. Then $y \in V$, $U \cap V = \emptyset$, and $V \in \sigma(B^{op})$. Indeed, if F is a filter in B with $\bigwedge F \in V$ then $u \wedge \bigwedge F \leq y$ for some $u \in U$, whence $u \leq y \vee (\bigwedge F)' = \bigvee \{y \vee z' : z \in F\}$. This is a directed sup, and since $u \in U \in \sigma(B)$, it follows that $y \vee z' \in U$ for some $z \in F$, i.e. $w \wedge z \leq y$ for some $w \in U$, and so $z \in F \cap V$.

Certainly it would be interesting to investigate which of the implications in the Theorem may be inverted.