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SCS 98: Z-Continuity, Z-Hypercompactness and Complete Distributivity

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SEMINAR ON CONTINUITY IN SEMILATTICES

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One of the prominent results in the theory of continuous posets is the fact that *an up-complete poset (DCPO) is continuous iff its lattice of Scott-closed (respectively, Scott-open) sets is completely distributive* (see [4, 12, 16]). Moreover, Hoffmann [13] and Lawson [16] have pointed out *that the continuous posets are precisely the spectra of completely distributive lattices*. Hofmann and Mislove [15] have extended these observations to an equivalence between the category of continuous posets and the category of completely distributive lattices (both endowed with suitable morphisms).

Since the pioneer work of Wright, Wagner and Thatcher [18] on so-called *subset systems* $\mathfrak{Z}$, several fruitful attempts have been made to generalize the theory of continuous and of algebraic posets by replacing directed sets with "$\mathfrak{Z}$-sets". In this vein, a theory of $\mathfrak{Z}$-*continuous* and of $\mathfrak{Z}$-*algebraic* posets has been developed by Novak [17] and, independently, by Bandelt and Erné [2, 3, 5].

However, the translation of the aforementioned categorical equivalence to the general $\mathfrak{Z}$-setting remained fragmentary. Below, we shall establish such an equivalence between categories of $\mathfrak{Z}$-continuous posets and categories of certain completely distributive lattices, the so-called $\mathfrak{Z}$-*supercompactly generated lattices*, but we succeed only under certain additional hypotheses on the subset selections in question. These conditions are met if $\mathfrak{Z}P$ is the collection of all *m*-directed lower sets, or that of all lower sets generated by less than *m* elements (*m* regular). Thus, our general equivalence theorems will include, among other specializations, the equivalence between continuous posets and completely distributive lattices, as well as the equivalence between posets, primely generated semilattices and superalgebraic (i.e. algebraic and completely distributive) lattices.

Let us recall a few relevant definitions. $\mathfrak{Z}$ always denotes a *subset selection* (assigning to each poset P a certain collection $\mathfrak{Z}P$ of subsets). $\mathfrak{Z}^\wedge P$ denotes the collection of all $\mathfrak{Z}$-*ideals*, i.e. lower sets generated by members of $\mathfrak{Z}P$ or by one point (i.e. principal ideals). Furthermore, we shall need the global completion $\mathfrak{Z}^\vee$ where $\mathfrak{Z}^\vee P$ consists of all $\mathfrak{Z}$-*join ideals*, i.e. lower sets Y such that $Z \in \mathfrak{Z}P$, $Z \subseteq Y$ and $x = \bigvee Z$ imply $x \in Y$ (cf. [3, 6, 8]). If $\mathcal{Y} \in \mathfrak{Z}\mathfrak{Z}^\wedge P$ always implies $\bigcup \mathcal{Y} \in \mathfrak{Z}^\wedge P$ then $\mathfrak{Z}$ is said to be *union complete*. Specifically, we shall consider the following *subset systems* in the sense of [18]: $\mathcal{P}_m$ selects all subset of cardinality less than a given cardinal *m*, and $\mathcal{D}_m$ all *m-directed* subsets (cf. [8]). Notice that $\mathcal{D}_m$ is always union complete, whereas $\mathcal{P}_m$ happens to be union complete for regular cardinals *m* only. We write $\mathcal{F}$ for $\mathcal{P}_\omega$ (finite subsets) and $\mathcal{D}$ for $\mathcal{D}_\omega$ (directed subsets). Furthermore, $\mathcal{E}$ selects all singletons, the minimal extension $\mathcal{M} = \mathcal{E}^\wedge$ all principal ideals, and the Alexandroff completion $\mathcal{A} = \mathcal{P}^\wedge$ all lower sets.

Given a class K of isotone (=order-preserving) maps between posets, we say a subset selection $\mathfrak{Z}$ is *K-invariant* provided for each map $\varphi: P \to Q$ in K, we have that $Z \in \mathfrak{Z}^\wedge P$ implies $\downarrow_Q \varphi[Z] \in \mathfrak{Z}^\wedge Q$ (where $\downarrow_Q Y$ or $\downarrow Y$ denotes the lower set generated by Y in Q). Specifically, we write O for the class of all order-preserving maps, E for that of all (order) embeddings, and I for that of all (order) isomorphisms. All subset systems $\mathfrak{Z}$ as well as the associated ideal extensions $\mathfrak{Z}^\wedge$ are O-invariant. The subset selection of all antichains is E-invariant but not O-invariant, whereas none of the I-invariant completions $\mathcal{P}_m^\vee$ or $\mathcal{D}_m^\vee$ except $\mathcal{A} = \mathcal{E}^\vee$ is E-invariant.

In due course, we shall need a slightly stronger property than E-invariance, namely the following: an I-invariant subset selection $\mathfrak{Z}$ is *stable* if for all subposets P of posets Q, the condition $P \in \mathfrak{Z}^\wedge P$ is equivalent to $\downarrow_Q P \in \mathfrak{Z}^\wedge Q$. It is easy to see that $\mathfrak{Z}$ is stable iff for any order embedding $\varphi: P \to Q$,

$$Z \in \mathfrak{Z}^\wedge P \iff \downarrow_Q \varphi[Z] \in \mathfrak{Z}^\wedge Q.$$

In particular, any stable subset selection is E-invariant. The converse is not true, as can be seen easily by looking at the union complete subset system of all upper bounded sets. However, all subset selections specified before are stable.

In [9], we have studied so-called $\mathfrak{Z}$-*ary* closure operators and closure systems. A closure system $\mathfrak{X}$ and the associated closure operator $\Gamma: \mathcal{P}P \to \mathcal{P}P$ are called $\mathfrak{Z}$-*ary* provided Γ agrees with its $\mathfrak{Z}$-modification

$$\Gamma^{\mathfrak{Z}}: \mathcal{P}P \to \mathcal{P}P,\ Y \mapsto \bigcup \{\Gamma Z : Z \in \mathfrak{Z}^\wedge P,\ Z \subseteq \downarrow Y\}.$$

Here lower sets and $\mathfrak{Z}$-ideals refer to the *specialization order* given by $x \leq y$ iff x belongs to the closure of {y}. In particular, we shall be interested in the $\mathfrak{Z}$-modification $\Delta_P^{\mathfrak{Z}}$ of the cut operator Δ_P, assigning to each subset Y of P the *cut generated by* Y, i.e. the intersection of all principal ideals containing Y.

A poset P is $\mathfrak{Z}$-*complete* if each member of $\mathfrak{Z}P$ (equivalently, each $\mathfrak{Z}$-ideal) has a join. More generally, a closure space $(P, \mathfrak{X})$ with closure operator Γ is called $\mathfrak{Z}$-*complete* (in [9]: a $\mathfrak{Z}$-*space*) if for each $Z \in \mathfrak{Z}^\wedge P$ there is a unique point y with $\Gamma Z = \Gamma\{y\}$. The $\mathfrak{Z}$-join ideals of a $\mathfrak{Z}$-complete poset P are precisely the fixed points of the operator $\Delta_P^{\mathfrak{Z}}$. But, unfortunately, $\Delta_P^{\mathfrak{Z}}$ is not a closure operator in general, by lack of idempotency. As shown in [9], $\Delta_P^{\mathfrak{Z}}$ is idempotent iff the closure system $\mathfrak{Z}^\Delta P$ is $\mathfrak{Z}$-ary.

In a $\mathfrak{Z}$-complete poset P, the $\mathfrak{Z}$-*below ideal* $\Downarrow^{\mathfrak{Z}} y = \Downarrow_P^{\mathfrak{Z}} y$ of $y \in P$ is the intersection of all $\mathfrak{Z}$-ideals whose join dominates y, and the $\mathfrak{Z}$-*below relation* $\ll_{\mathfrak{Z}}$ is given by $x \ll_{\mathfrak{Z}} y \iff x \in \Downarrow^{\mathfrak{Z}} y$. If each $\mathfrak{Z}$-below ideal $\Downarrow^{\mathfrak{Z}} y$ is in fact a $\mathfrak{Z}$-ideal with join y then P is a $\mathfrak{Z}$-*continuous poset*. If, in addition, the $\mathfrak{Z}$-below relation is idempotent then we speak of a *strongly* $\mathfrak{Z}$-*continuous poset* (cf. [2, 17]).

A map φ between posets P and Q is called $\mathfrak{Z}^\vee$-*continuous (alias* $\mathfrak{Z}$-*continuous)* if inverse images of $\mathfrak{Z}^\vee$-ideals under φ are $\mathfrak{Z}^\vee$-ideals - in other words, if φ is continuous as a map between the closure spaces $(P, \mathfrak{Z}^\vee P)$ and $(Q, \mathfrak{Z}^\vee Q)$. In case of O-invariant subset selections $\mathfrak{Z}$, the latter condition simply means that φ is isotone and preserves $\mathfrak{Z}$-joins [6]. Furthermore, it can be shown that (for arbitrary subset selections $\mathfrak{Z}$) an isotone map φ between strongly $\mathfrak{Z}$-continuous posets is $\mathfrak{Z}^\vee$-continuous iff it is $\mathfrak{Z}$-*below interpolating*, i.e.

$$x \ll_{\mathfrak{Z}} \varphi(z) \text{ implies } x \ll_{\mathfrak{Z}} \varphi(y) \text{ for some } y \ll_{\mathfrak{Z}} z.$$

On the other hand, a straightforward verification shows that φ *is* $\mathfrak{Z}$-*below preserving*, i.e.

$x \ll_{\mathfrak{Z}} y$ implies $\varphi(x) \ll_{\mathfrak{Z}} \varphi(y)$,

iff φ is a *quasiopen* map between the spaces $(P,\mathfrak{Z}^{\vee}P)$ and $(Q,\mathfrak{Z}^{\vee}Q)$, that is, $P \setminus Y \in \mathfrak{Z}^{\vee}P$ entails $Q \setminus \uparrow\varphi[Y] \in \mathfrak{Z}^{\vee}Q$.

The following "topological representation" of strongly $\mathfrak{Z}$-continuous posets as certain *core spaces*, i.e. closure spaces with completely distributive closure systems (see [4, 10]), has been establieshed in [9] - without any restrictions on the subset selection $\mathfrak{Z}$:

THEOREM 1. *Via specialization, one obtains an isomorphism between the category of $\mathfrak{Z}$-ary $\mathfrak{Z}$-complete core spaces with continuous maps and the category of strongly $\mathfrak{Z}$-continuous posets with $\mathfrak{Z}^{\vee}$-continuous (i.e. $\mathfrak{Z}$-below interpolating) maps. Under that isomorphism, the quasiopen continuous maps correspond to the $\mathfrak{Z}$-below preserving and interpolating maps. The converse isomorphism maps a strongly $\mathfrak{Z}$-continuous poset P onto the closure space $(P,\mathfrak{Z}^{\vee}P)$.*

Usually, an element x with $x \ll_{\mathfrak{Z}} x$ is called $\mathfrak{Z}$-*compact*. Now, we say an element x of a complete lattice L is $\mathfrak{Z}$-*hypercompact* if for all $Y \in \mathcal{A}L$ with $x \leq \bigvee Y$, there exists a $Z \in \mathfrak{Z}^{\wedge}L$ with $Z \subseteq Y$ and $x \leq \bigvee Z$. Of course, the smaller $\mathfrak{Z}P$, the stronger is the condition of $\mathfrak{Z}$-hypercompactness. In particular, the $\mathcal{E}$-hypercompact elements are the $\bigvee$-prime (*alias* supercompact) ones, the $\mathcal{F}$-hypercompact elements are the compact elements in the usual sense, and the $\mathcal{P}_{\omega_1}$-hypercompact elements (where ω_1 is the first uncountable cardinal) might be called *"Lindelöf"*, with regard to the corresponding covering property.

Yet another generalization of supercompactness will be relevant for our purposes; namely, an element x of a complete lattice L will be called $\mathfrak{Z}$-*supercompact* if $\Downarrow^{\mathcal{A}}_{L} x \in \mathfrak{Z}^{\wedge}L$ and $x = \bigvee \Downarrow^{\mathcal{A}}_{L} x$.

The set of all $\mathfrak{Z}$-compact elements of a complete lattice L, sometimes referred to as the $\mathfrak{Z}$-*core* or the $\mathfrak{Z}$-*spectrum* of L, is denoted by $\mathcal{C}_{\mathfrak{Z}}L$, that of all $\mathfrak{Z}$-hypercompact elements by $\mathcal{H}_{\mathfrak{Z}}L$, and that of all $\mathfrak{Z}$-supercompact elements by $\mathcal{S}_{\mathfrak{Z}}L$. The basic relationships between the various types of compactness are collected together in

LEMMA 2. *Let $\mathfrak{Z}$ be any subset selection and L a complete lattice.*
(1) *Each $\mathfrak{Z}$-supercompact element of L is $\mathfrak{Z}$-hypercompact.*
(2) *Each $\mathfrak{Z}$-hypercompact element of L is $\mathfrak{Z}^{\vee}$-compact.*
(3) *$\mathcal{S}_{\mathfrak{Z}}L = \mathcal{H}_{\mathfrak{Z}}L$ if L is completely distributive.*
(4) *$\mathcal{H}_{\mathfrak{Z}}L = \mathcal{C}_{\mathfrak{Z}^{\vee}}L$ if the operator $\Delta_L^{\mathfrak{Z}}$ is idempotent (i.e. $\mathfrak{Z}^{\vee}L$ is $\mathfrak{Z}$-ary).*

PROOF. (1) If x is $\mathfrak{Z}$-supercompact and $x \leq \bigvee Y$ then $\Downarrow^{\mathcal{A}}_{L} x \subseteq \downarrow Y$, $\Downarrow^{\mathcal{A}}_{L} x \in \mathfrak{Z}^{\wedge}L$ and $x = \bigvee \Downarrow^{\mathcal{A}}_{L} x$.
(2) If x is $\mathfrak{Z}$-hypercompact and $x \leq \bigvee Y$ for some $Y \in \mathfrak{Z}^{\vee}L$ then we find a $Z \in \mathfrak{Z}^{\wedge}L$ with $x \leq \bigvee Z$ and $Z \subseteq \downarrow Y = Y$, whence $\bigvee Z \in Y$ and $x \in Y$.
(3) If x is a $\mathfrak{Z}$-hypercompact element of the completely distributive lattice L then $x = \bigvee \Downarrow^{\mathcal{A}} x$ and consequently $x = \bigvee Z$ for some $Z \in \mathfrak{Z}^{\wedge}L$ with $Z \subseteq \Downarrow^{\mathcal{A}}_{L} x$. But this can happen only if $\Downarrow^{\mathcal{A}}_{L} x = \downarrow Z \in \mathfrak{Z}^{\wedge}L$.
(4) If x is $\mathfrak{Z}^{\vee}$-compact in L and $\Delta_L^{\mathfrak{Z}}$ is idempotent then for each lower set $Y \in \mathcal{A}L$, $x \leq \bigvee Y = \bigvee \Delta_L^{\mathfrak{Z}} Y$ implies $x \in \Delta_L^{\mathfrak{Z}} Y$ since this is a $\mathfrak{Z}^{\vee}$-ideal. Thus, by definition of $\Delta_L^{\mathfrak{Z}}$, $x \leq \bigvee Z$ for some $Z \in \mathfrak{Z}^{\wedge}L$ with $Z \subseteq Y$. □

EXAMPLES.

(1) $\mathcal{E}^{\vee}$*-compact* = $\mathcal{E}$*-hypercompact* = $\mathcal{E}$*-supercompact* = *supercompact* = $\bigvee$*-prime*.

(2) $\mathcal{F}^{\vee}$*-compact* = $\mathcal{F}$*-hypercompact* = *compact*,
$\mathcal{F}$*-supercompact* = *join of finitely many* $\bigvee$*-primes*.

(3) Of course, every $\mathcal{D}^{\vee}$-compact element, *a fortiori* every $\mathcal{D}$-hypercompact element, is $\vee$-prime, i.e. $\mathcal{F}$-compact, since $\mathcal{F}^{\wedge}P \subseteq \mathcal{D}^{\vee}P$. The converse implication is not valid in all distributive complete lattices (for example, in $L = \mathcal{F}\omega \cup \{\omega\}$, the greatest element ω is $\vee$-prime but not $\mathcal{D}^{\vee}$-compact since $\omega = \bigcup \mathcal{Y}$ where $\mathcal{Y} = \mathcal{P}_2\omega \in \mathcal{D}^{\vee}L$).
The fact that a $\vee$-prime element x of a completely distributive lattice L is always $\mathcal{D}$-hypercompact is not trivial and requires the Boolean Ultrafilter Theorem (cf. *The Lemma* in [11, V-1]): if $x \leq \bigvee Y$ for some $Y \subseteq L$ then the set $\mathcal{J} = \{Z \subseteq Y : x \not\leq \bigvee Z\}$ is a proper ideal in $\mathcal{P}Y$ because x is $\vee$-prime; hence there exists an ultrafilter $\mathcal{U}$ on Y with $\mathcal{J} \cap \mathcal{U} = \emptyset$, and it follows from complete distributivity that

$$x \leq \bigwedge\{\bigvee U : U \in \mathcal{U}\} = \bigvee\{\bigwedge U : U \in \mathcal{U}\}.$$

As $\{\bigwedge U : U \in \mathcal{U}\}$ is a directed subset of $\downarrow Y$, this proves $\mathcal{D}$-hypercompactness of x.
Hence, in completely distributive lattices, we have:

$\mathcal{D}^{\vee}$*-compact* = $\mathcal{D}$*-hypercompact* = $\mathcal{D}$*-supercompact* = $\mathcal{F}$*-compact* = $\vee$*-prime* = *coprime*.

(4) Let m be an irregular cardinal number. Then $P = \mathcal{P}_m m$, partially ordered by inclusion, is not $\mathcal{P}_m$-complete (whence $\mathcal{P}_m$ fails to be union complete). The power set $L = \mathcal{P}m$ is a completely distributive lattice, but it is not $\mathcal{P}_m$-continuous because

$$W := \downarrow_L^{\mathcal{P}_m} m = \mathcal{P}_2 m$$

is not in $\mathcal{P}_m L$. The greatest element m of L is $\mathcal{P}_m^{\vee}$-compact: indeed, any $\mathcal{P}_m^{\vee}$-ideal Y of L with $m = \bigcup Y$ contains all singletons and therefore all members of P. By irregularity of m, there exists a $Z \in \mathcal{P}_m P = \mathcal{P}_m L \cap \mathcal{P}P$ with $m = \bigcup Z$, whence $m \in Y$. However, m is *not* $\mathcal{P}_m$-hypercompact since $m = \bigcup W$ but $m \neq \bigcup X$ for all $X \in \mathcal{P}_m L$ with $X \subseteq W$. This example shows that idempotency of the operator $\Delta_L^{\mathfrak{Z}}$ is essential in Lemma 2 (4).

COROLLARY 3. *For a cardinal number* m, *the following conditions are equivalent:*
(a) m *is regular.*
(b) $\mathcal{P}_m$ *is union complete.*
(c) *For any poset* P, *the operator* $\Delta_P^{\mathcal{P}_m}$ *is idempotent (hence a closure operator).*
(d) *Every* $\mathcal{P}_m^{\vee}$*-compact element is* $\mathcal{P}_m$*-hypercompact (and conversely).*
(e) *Every* $\mathcal{P}_m^{\vee}$*-compact element of a completely distributive lattice is* $\mathcal{P}_m$*-supercompact.*

A $\mathfrak{Z}$-complete poset P is said to be $\mathfrak{Z}$*-compactly generated* if each element of P is a join of $\mathfrak{Z}$-compact elements. Similarly, a complete lattice L is called $\mathfrak{Z}$*-hypercompactly* *(*$\mathfrak{Z}$*-supercompactly) generated* if each element of L is a join of $\mathfrak{Z}$-hypercompact ($\mathfrak{Z}$-supercompact) elements. From Lemma 2 and the fact that a complete lattice L is completely distributive iff $y = \bigvee \downarrow_L^{\mathcal{A}} y$ for all $y \in L$, we infer:

PROPOSITION 4. (1) *A complete lattice* L *is* $\mathfrak{Z}$*-supercompactly generated iff* L *is* $\mathfrak{Z}$*-hypercompactly generated and completely distributive.*

(2) *Every* $\mathfrak{Z}$*-hypercompactly generated complete lattice* L *is* $\mathfrak{Z}^\vee$*-compactly generated. The converse holds if the operator* $\Delta_L^{\mathfrak{Z}}$ *is idempotent.*

(3) *Every* $\mathfrak{Z}^\vee$*-compactly generated complete lattice is* $\mathfrak{Z}^\vee$*-continuous.*

EXAMPLES.

(1) $\mathcal{E}$*-hypercompactly generated* = $\mathcal{E}$*-supercompactly generated* = $\mathcal{F}$*-supercompactly generated* = *superalgebraic* = *algebraic and completely distributive.*

(2) $\mathcal{F}$*-hypercompactly generated* = $\mathcal{D}$*-compactly generated* = *algebraic and complete* = *compactly generated and complete.*

(3) $\mathcal{P}$*-hypercompactly generated* = *complete,* $\mathcal{P}$*-supercompactly generated* = *completely distributive* = *supercontinuous.*

(4) The lattice $L = \mathcal{P}m$ is $\mathcal{P}_m$-supercompactly generated, the $\mathcal{P}_m$-supercompact elements being the members of $P = \mathcal{P}_m m$. However, L is not $\mathcal{P}_m$-continuous for $m > 1$, and P is not $\mathcal{P}_m$-continuous if m is irregular.

PROPOSITION 5. *If* $\mathfrak{Z}$ *is an E-invariant subset selection then for any* $\mathfrak{Z}$*-ary closure system* $\mathfrak{X}$, *all point closures are* $\mathfrak{Z}$*-hypercompact, and consequently* $\mathfrak{X}$ *is a* $\mathfrak{Z}$*-hypercompactly generated lattice. Conversely, if* $\mathfrak{X}$ *is a closure system in which every point closure is* $\mathfrak{Z}$*-hypercompact then* $\mathfrak{X}$ *is* $\mathfrak{Z}$*-ary, provided* $\mathfrak{Z} = \mathcal{P}_m$ *or* $\mathfrak{Z}$ *is O-invariant and the underlying set* $P = \bigcup \mathfrak{X}$ *is a complete lattice with respect to specialization.*

PROOF. Let $\Gamma : \mathcal{P}P \to \mathcal{P}P$ denote the $\mathfrak{Z}$-ary closure operator of $\mathfrak{X}$ and suppose that $\downarrow y \subseteq \bigvee_{\mathfrak{X}} \mathcal{Y} = \Gamma(\bigcup \mathcal{Y})$ for some $\mathcal{Y} \subseteq \mathfrak{X}$. Then we find a $Z \in \mathfrak{Z}^\wedge P$ with $Z \subseteq \bigcup \mathcal{Y}$ and $y \in \Gamma Z$. For the principal ideal embedding $\eta : P \to \mathfrak{X}$, $y \mapsto \downarrow y = \Gamma\{y\}$, we obtain $\mathcal{V} = \downarrow_{\mathfrak{X}} \eta[Z] \in \mathfrak{Z}^\wedge \mathfrak{X}$, $\mathcal{V} \subseteq \downarrow_{\mathfrak{X}} \mathcal{Y}$ and $\downarrow y \subseteq \Gamma Z = \bigvee_{\mathfrak{X}} \mathcal{V}$. Hence $\downarrow y$ is $\mathfrak{Z}$-hypercompact.

Now assume that every point closure of $\mathfrak{X}$ is $\mathfrak{Z}$-hypercompact. If $y \in \Gamma Y$ then $\downarrow y \subseteq \Gamma Y = \bigvee_{\mathfrak{X}} \eta[Y]$, so there exists a $\mathcal{W} \in \mathfrak{Z}^\wedge \mathfrak{X}$ with $\mathcal{W} \subseteq \downarrow_{\mathfrak{X}} \eta[Y]$ and $\downarrow y \subseteq \bigvee_{\mathfrak{X}} \mathcal{W}$. If $\mathfrak{Z} = \mathcal{P}_m$ then we find a $Z \in \mathfrak{Z}^\wedge P$ with $Z \subseteq \downarrow Y$ and $\mathcal{W} \subseteq \downarrow_{\mathfrak{X}} \eta[Z]$, whence $y \in \Gamma(\bigcup \mathcal{W}) \subseteq \Gamma(\bigcup \eta[Z]) = \Gamma Z$. If $\mathfrak{Z}$ is O-invariant and P is complete then we may take $Z = \downarrow\{\bigvee W : W \in \mathcal{W}\}$. ▫

We know that for any strongly $\mathfrak{Z}$-continuous poset P, the $\mathfrak{Z}^\vee$-ideal completion $L = \mathfrak{Z}^\vee P$ is completely distributive [3, 17]. In order to show that L is even $\mathfrak{Z}$-supercompactly generated, it will suffice to verify the equation $\mathcal{S}_{\mathfrak{Z}} L = \mathcal{U} P$ (since every $\mathfrak{Z}^\vee$-ideal is a union of principal ideals). For this, it will be convenient to have an explicit description of the $\mathcal{A}$-below relation in arbitrary completely distributive closure systems $\mathfrak{X}$. Recall from [4] that the restricted closure operator $\Gamma : \mathcal{A}P \to \mathcal{A}P$ of such a closure system has a lower adjoint $L : \mathcal{A}P \to \mathcal{A}P$, where $LY = \{x \in P : \exists y \in Y \ \forall Z \subseteq P \ (y \in \Gamma Z \Rightarrow x \in \downarrow Z)\}$ $(Y \subseteq P)$.

LEMMA 6. *Let* $(P, \mathfrak{X})$ *be a core space with closure operator* Γ. *Then for all* $Y \subseteq P$,

$$\downarrow_{\mathfrak{X}}^{\mathcal{A}} \Gamma Y = \{X \in \mathfrak{X} : X \subseteq \downarrow x \text{ for some } x \in LY\}.$$

PROOF. By known facts about core spaces, the closure operator Γ of $\mathfrak{X}$ induces an isomorphism between the kernel system $\mathfrak{X}' = \{LX : X \in \mathfrak{X}\}$ and the closure system $\mathfrak{X}$.

Furthermore, $LY = \bigcup\{L\{x\}: x \in Y\}$ and therefore $\Gamma Y = \Gamma LY = \bigvee_{\mathfrak{X}}\{\downarrow x : x \in LY\}$, whence $\downarrow^{\mathcal{A}}_{\mathfrak{X}} \Gamma Y \subseteq \downarrow_{\mathfrak{X}}\{\downarrow x : x \in LY\}$. For the converse inclusion, consider any $\mathcal{Y} \in \mathcal{A}\mathfrak{X}$ with $\Gamma Y \subseteq \bigvee_{\mathfrak{X}} \mathcal{Y}$. Then $LY \subseteq \bigvee_{\mathfrak{X}} L[\mathcal{Y}] = \bigcup L[\mathcal{Y}] \subseteq \bigcup \mathcal{Y}$, and it follows that $\downarrow_{\mathfrak{X}}\{\downarrow x : x \in LY\} \subseteq \mathcal{Y}$. □

COROLLARY 7. *If $\mathfrak{Z}$ is a stable subset selection then for each subset Y of a core space $(P,\mathfrak{X})$ with closure operator Γ,*

$$LY \in \mathfrak{Z}^{\wedge}P \iff \Gamma Y \in \mathcal{S}_{\mathfrak{Z}}\mathfrak{X} = \mathcal{H}_{\mathfrak{Z}}\mathfrak{X}.$$

PROOF. Since $\mathfrak{X}$ is completely distributive, we have $\mathcal{S}_{\mathfrak{Z}}\mathfrak{X} = \mathcal{H}_{\mathfrak{Z}}\mathfrak{X}$ and $\Gamma Y = \bigvee \downarrow^{\mathcal{A}}_{\mathfrak{X}} \Gamma Y$. Using the principal ideal embedding $\eta : P \to \mathfrak{X}, y \mapsto \Gamma\{y\}$, we infer from Lemma 6:

$$LY \in \mathfrak{Z}^{\wedge}P \iff \downarrow^{\mathcal{A}}_{\mathfrak{X}} \Gamma Y = \downarrow_{\mathfrak{X}} \eta[LY] \in \mathfrak{Z}^{\wedge}\mathfrak{X} \iff \Gamma Y \in \mathcal{S}_{\mathfrak{Z}}\mathfrak{X}. \;\square$$

The next result provides us with various characterizations of $\mathfrak{Z}$-ary core spaces:

THEOREM 8. *Let $\mathfrak{Z}$ be a stable subset selection. Then the following statements on a core space $(P,\mathfrak{X})$ are equivalent:*

(a) *$\mathfrak{X}$ is $\mathfrak{Z}$-ary.*

(b) *$L\downarrow y \in \mathfrak{Z}^{\wedge}P$ for all $y \in P$.*

(c) *Each point closure is $\mathfrak{Z}$-hypercompact in $\mathfrak{X}$.*

(d) *Each member of $\mathfrak{X}$ is a union of $\mathfrak{Z}$-hypercompact elements of $\mathfrak{X}$.*

If $\mathfrak{Z}$ is union complete then these conditions are also equivalent to the following two:

(e) *$LZ \in \mathfrak{Z}^{\wedge}P$ for all $Z \in \mathfrak{Z}^{\wedge}P$.*

(f) *For all $Z \in \mathfrak{Z}^{\wedge}P$, the closure of Z is $\mathfrak{Z}$-hypercompact.*

Each of the above six conditions implies that $\mathfrak{X}$ is $\mathfrak{Z}$-hypercompactly generated. Moreover, the prefix "hyper" may be replaced with "super" by complete distributivity of $\mathfrak{X}$.

PROOF. (a) $\iff$ (b): See [9].

(b) $\iff$ (c): Apply Corollary 7.

(c) $\iff$ (d): Clear.

Of course, (e) implies (b), while (f) implies (c), and (d) implies that $\mathfrak{X}$ is $\mathfrak{Z}$-hypercompactly generated.

Now assume that $\mathfrak{Z}$ is union complete.

(b) $\Rightarrow$ (e): Using the embedding $\iota : P \to \mathfrak{Z}^{\wedge}P$, $y \mapsto L\downarrow y$, we see that $Z \in \mathfrak{Z}^{\wedge}P$ implies $\downarrow_{\mathfrak{Z}^{\wedge}P}\iota[Z] \in \mathfrak{Z}^{\wedge}\mathfrak{Z}^{\wedge}P$, whence $LZ = \bigcup \iota[Z] = \bigcup \downarrow_{\mathfrak{Z}^{\wedge}P}\iota[Z] \in \mathfrak{Z}^{\wedge}P$.

(e) $\iff$ (f): See Corollary 7 again. □

For $\mathfrak{Z} = \mathcal{P}_m$, the equivalence of the statements (a),(c) and (d) remains valid in arbitrary closure spaces (cf. Proposition 5).

Now to the first part of the announced equivalence theorem for $\mathfrak{Z}$-continuous posets.

PROPOSITION 9. *Let $\mathfrak{Z}$ be a union complete stable subset selection. Then for each $\mathfrak{Z}$-continuous poset P, the $\mathfrak{Z}^{\vee}$-ideal completion $\mathfrak{Z}^{\vee}P$ is a $\mathfrak{Z}$-supercompactly generated lattice $\mathfrak{X}$ with $P \simeq \mathcal{A}\mathcal{L}P = \mathcal{H}_{\mathfrak{Z}}\mathfrak{X} = \mathcal{S}_{\mathfrak{Z}}\mathfrak{X}$.*

PROOF. We know from Theorem 1 that $(P, \mathcal{Z}^\vee P)$ is a $\mathcal{Z}$-ary core space. Hence, by Theorem 8, $\mathcal{X} = \mathcal{Z}^\vee P$ is a $\mathcal{Z}$-supercompactly generated lattice with $\mathcal{M}P \subseteq \mathcal{S}_{\mathcal{Z}}\mathcal{X}$. Conversely, if $(P,\mathcal{X})$ is any $\mathcal{Z}$-ary core space then for each $\mathcal{Z}$-supercompact member Y of $\mathcal{X}$, we have $LY \in \mathcal{Z}^\wedge P$ (by Corollary 7) and $\bigvee LY = \bigvee\{\bigvee L{\downarrow}y : y \in Y\} = \bigvee Y$. If $\mathcal{X} = \mathcal{Z}^\vee P$ then $LY \subseteq Y \in \mathcal{X}$ and $LY \in \mathcal{Z}^\wedge P$ imply that $Y = {\downarrow}\bigvee LY = {\downarrow}\bigvee Y$ is a principal ideal. □

In [9], we have called a T_0 closure space $\mathcal{Z}$-*sober* if the $\mathcal{Z}$-compact closed sets are just the point closures. Now, we say a T_0 closure space $(P,\mathcal{X})$ or its closure system $\mathcal{X}$ is $\mathcal{Z}$-*hypersober* ($\mathcal{Z}$-*supersober*) if the point closures are precisely the $\mathcal{Z}$-hypercompact ($\mathcal{Z}$-supercompact) elements of $\mathcal{X}$, i.e. $\mathcal{M}P = \mathcal{H}_{\mathcal{Z}}\mathcal{X}$ ($\mathcal{S}_{\mathcal{Z}}\mathcal{X}$). Then we may summarize our results as follows :

THEOREM 10. *Let $\mathcal{Z}$ be a union complete stable subset selection. Then the following statements on a T_0 closure space $(P,\mathcal{X})$ and the underlying poset P are equivalent:*

(a) *P is a (strongly) $\mathcal{Z}$-continuous poset and $\mathcal{X} = \mathcal{Z}^\vee P$.*

(b) *$(P,\mathcal{X})$ is a $\mathcal{Z}$-hypersober core space.*

(c) *$(P,\mathcal{X})$ is a $\mathcal{Z}$-supersober space, i.e. $\mathcal{S}_{\mathcal{Z}}\mathcal{X} = \mathcal{M}P$.*

PROOF. (a) ⇒ (c) : See Proposition 9.

(c) ⇒ (b): Since $\mathcal{M}P$ is join-dense in the closure system $\mathcal{X}$, this is a $\mathcal{Z}$-supercompactly generated lattice; in particular, $\mathcal{X}$ is completely distributive, i.e. $(P,\mathcal{X})$ is a core space. By Lemma 2, the $\mathcal{Z}$-supercompact elements of $\mathcal{X}$ are the $\mathcal{Z}$-hypercompact ones.

(b) ⇒ (a): By Theorem 8, the inclusion $\mathcal{M}P \subseteq \mathcal{H}_{\mathcal{Z}}\mathcal{X}$ means that $\mathcal{X}$ is $\mathcal{Z}$-ary, and for $Z \in \mathcal{Z}P$, we have $\Gamma Z \in \mathcal{H}_{\mathcal{Z}}\mathcal{X} = \mathcal{S}_{\mathcal{Z}}\mathcal{X}$, whence $\Gamma Z = {\downarrow}y$ for some $y \in P$. This shows that $(P,\mathcal{X})$ is a $\mathcal{Z}$-complete core space, and Theorem 1 applies. □

It is well known and easy to see that in case of a union complete E-invariant subset selection $\mathcal{Z}$, the $\mathcal{Z}$-below relation of any $\mathcal{Z}$-continuous poset is automatically idempotent. Invoking Theorem 1 once more, we arrive at

THEOREM 11. *For any union complete stable subset selection $\mathcal{Z}$, the assignment $P \mapsto (P, \mathcal{Z}^\vee P)$ yields an isomorphism between the category of $\mathcal{Z}$-continuous posets with $\mathcal{Z}$-below interpolating (i.e. $\mathcal{Z}^\vee$-continuous) maps and the category of $\mathcal{Z}$-supersober closure spaces (i.e. $\mathcal{Z}$-hypersober core spaces) with continuous maps. Moreover, a morphism between $\mathcal{Z}$-continuous posets preserves the $\mathcal{Z}$-below relation iff it is quasi-open as a map between the corresponding core spaces.*

Notice that for a closure system $\mathcal{X}$, $\mathcal{Z}$-*hypersoberness* means $\mathcal{Z}^\vee$-*soberness* if the operator $\Delta_{\mathcal{X}}^{\mathcal{Z}}$ is idempotent, and that the latter automatically holds if $\mathcal{Z} = \mathcal{P}_m$ for some regular cardinal m, or if $\mathcal{Z} = \mathcal{D}_m$ and $\mathcal{X}$ is completely distributive (cf. [9]).

Now let us turn to the second part of the isomorphism theorem for $\mathcal{Z}$-continuous posets, generalizing the nontrivial fact that the spectrum of a completely distributive lattice L is always a continuous poset whose Scott topology is isomorphic to L (see Hoffmann [12, 13], Lawson [16] or [11, V-1.9]). In view of Example (4) above, it is evident that some restrictions have to be imposed on the subset selections in question.

First, let us recall a few general remarks on functions between arbitrary (closure) spaces from [7]. We have a functor G from the category of closure spaces with continuous maps to the category of complete lattices with join-preserving maps; on the morphism level, it sends a continuous map $\varphi : (P,\mathfrak{X}) \to (Q,\mathfrak{Y})$ to the "lifted" map

$$G\varphi : \mathfrak{X} \to \mathfrak{Y},\ X \mapsto cl(\varphi[X]) = \bigcap\{Y \in \mathfrak{Y} : \varphi[X] \subseteq Y\}.$$

But we have also a contravariant functor, mapping φ to the upper adjoint of $G\varphi$, which is given by the inverse image map

$$G^{+}\varphi : \mathfrak{Y} \to \mathfrak{X},\ Y \mapsto \varphi^{-1}[Y].$$

The equivalence of the following conditions on such a morphism φ is straightforward:

(a) φ is quasiopen

(b) $G^{+}\varphi$ is a complete homomorphism

(c) $G\varphi$ is doubly residuated, i.e. lower adjoint to a complete homomorphism.

Moreover, if φ is a continuous map between core spaces then the previous conditions are also equivalent to the following two:

(d) $G\varphi$ preserves joins and the $\mathcal{A}$-below relations

(e) $G\varphi$ preserves and interpolates the $\mathcal{A}$-below relations.

The key result is now the following special instance of a representation theorem in [7]:

THEOREM 12. *For any I-invariant subset selection $\mathfrak{Z}$, the functor G induces an equivalence between the category of $\mathfrak{Z}$-(hyper-, super-)sober spaces and the category of $\mathfrak{Z}$-(hyper-, super-)compactly generated complete lattices with maps preserving joins and $\mathfrak{Z}$-(hyper-, super-)compactness. Under this equivalence, the continuous quasiopen maps correspond to the doubly residuated maps. Hence, the category of $\mathfrak{Z}$-(hyper-, super-)sober spaces with continuous quasiopen maps is dually equivalent to the category of $\mathfrak{Z}$-(hyper-, super-)compactly generated lattices and complete homomorphisms.*

Putting all pieces together, we obtain the desired equivalence theorem for $\mathfrak{Z}$-continuous posets and the completion functor $\mathfrak{Z}^{\vee}$:

THEOREM 13. *For any union complete stable subset selection $\mathfrak{Z}$, the join-ideal completion $\mathfrak{Z}^{\vee}$ induces an equivalence between the category of $\mathfrak{Z}$-continuous posets with $\mathfrak{Z}^{\vee}$-continuous (i.e. $\mathfrak{Z}$-below interpolating isotone) maps and the category of $\mathfrak{Z}$-supercompactly generated lattices with maps preserving joins and $\mathfrak{Z}$-supercompactness. Furthermore, the category of $\mathfrak{Z}$-continuous posets with $\mathfrak{Z}$-below preserving and interpolating maps is equivalent to the category of $\mathfrak{Z}$-supercompactly generated lattices with $\mathcal{A}$-below preserving and interpolating maps, and dual to the category of $\mathfrak{Z}$-supercompactly generated lattices with complete homomorphisms. The inverse equivalence resp. duality is obtained by restricting the morphisms to the subposets of $\mathfrak{Z}$-supercompact elements.*

COROLLARY 14. *Let $\mathfrak{Z}$ be a union complete stable subset selection. Then the $\mathfrak{Z}$-supercompact elements of a $\mathfrak{Z}$-supercompactly generated lattice L form a (strongly) $\mathfrak{Z}$-continuous poset $P = \mathcal{S}_{\mathfrak{Z}}L$, and L is isomorphic to the $\mathfrak{Z}^{\vee}$-ideal completion $\mathfrak{Z}^{\vee}P$.*

Consequently, a complete lattice is $\mathfrak{Z}$-supercompactly generated iff it is isomorphic to the closure system of a $\mathfrak{Z}$-ary core space.

If a continuous map $\varphi : P \to Q$ is lower adjoint to a continuous map $\psi : Q \to P$ (with respect to specialization) then $G\varphi$ is lower adjoint to $G\psi$ (cf. [15]).

COROLLARY 15. *Let $\mathfrak{Z}$ be an O-invariant, union complete and stable subset selection. Then an isotone map between $\mathfrak{Z}$-continuous posets is $\mathfrak{Z}$-interpolating iff it preserves $\mathfrak{Z}$-joins. Hence, the category of $\mathfrak{Z}$-continuous posets with $\mathfrak{Z}$-join preserving residual (i.e. upper adjoint) maps is equivalent to the category of $\mathfrak{Z}$-supercompactly generated lattices with complete homomorphisms preserving $\mathfrak{Z}$-supercompactness.*

On the other hand, these categories are dually equivalent to the category of $\mathfrak{Z}$-continuous posets with residuated (i.e. lower adjoint) maps preserving the $\mathfrak{Z}$-below relation.

References

1. B. Banaschewski and R.-E. Hoffmann (eds.), *Continuous lattices. Bremen 1979.* Lecture Notes in Math. **871**, Springer-Verlag, Berlin - Heidelberg - New York 1981
2. H.-J. Bandelt and M. Erné, *The category of Z-continuous posets.* J. Pure and Appl. Algebra **30** (1983) 219-226
3. H.-J. Bandelt and M. Erné, *Representations and embeddings of M-distributive lattices.* Houston J. Math. **10** (1984) 315-324
4. M. Erné, *Scott convergence and Scott topologies on partially ordered sets II.* In [1], 61-96
5. M. Erné, *Homomorphisms of M-distributive and M-generated posets.* Tech. Report **125**, Institut für Mathematik, Universität Hannover 1981
6. M. Erné, *Adjunctions and standard constructions for partially ordered sets.* In: G.Eigenthaler et al. (eds.), *Contributions to General Algebra 2. Proc. Klagenfurt Conf. 1982.* Hölder - Pichler - Tempsky, Wien 1983, 77-106
7. M. Erné, *Lattice representations of closure spaces.* In: L. Bentley et al. (eds.), *Categorical Topology. Conf. Toledo, Ohio 1983.* Heldermann, Berlin 1984, 197-222
8. M. Erné, *Order extensions as adjoint functors.* Quaestiones Math. (to appear)
9. M. Erné, *Z-continuous posets, Z-ary closure spaces and generalized soberness.* SCS Memo 10-5-1985
10. M. Erné and G. Wilke, *Standard completions for quasiordered sets.* Semigroup Forum **27** (1983) 351-376
11. G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, D.S. Scott, *A Compendium of Continuous Lattices.* Springer-Verlag, Berlin - Heidelberg — New York 1980
12. R.-E. Hoffmann, *Projective sober spaces.* In [1], 125-158
13. R.-E. Hoffmann, *Continuous posets, prime spectra of completely distributive complete lattices, and Hausdorff compactifications.* In [1], 159-208
14. R.-E. Hoffmann and K. H. Hofmann (eds.), *Continuous lattices and their applications. Bremen 1982.* Lecture Notes in Pure and Appl. Math. **101**, Marcel Dekker, New York 1985
15. K. H. Hofmann and M. Mislove, *Free objects in the category of completely distributive lattices.* In [14], 129-151
16. J. D. Lawson, *The duality of continuous posets.* Houston J. Math. **5** (1979) 357-386
17. D. Novak, *Generalization of continuous posets.* Trans. Amer. Math. Soc. **272** (1982) 645-667
18. J. B. Wright, E. G. Wagner and J. W. Thatcher, *A uniform approach to inductive posets and inductive closure.* Theor. Comp. Sci. **7** (1978) 57-77