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# CALCULATING INFINITESIMAL GENERATORS

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Abstract. Itô's lemma is a standard approach for calculating infinitesimal generators for stochastic process models. We present alternate ad-hoc methods based on elementary calculations for many commonly considered stochastic models. The only tools that we employ are binomial expansion, Taylor series expansion and solution methods for elementary differential equations. Our alternate approaches are especially useful for processes which cannot be expressed as stochastic differential equations, for which Itô's lemma cannot be applied.

## 1. Introduction

Typically infinitesimal generator for a diffusion process or a jump-diffusion process that can be written in stochastic differential equation (SDE) form, is calculated by an application of Itô's lemma. However for many of the common processes seen in literature, it is also possible to calculate the infinitesimal generator directly from the limit definition by calculating the conditional moments or by using the conditional distribution function (that is, the transition density function). Although the latter method is cumbersome compared to Itô's formula for processes which has an SDE form, it is probably the only way to calculate for processes without an SDE form. In this paper, we illustrate our calculations for a number of examples. The only tools that we employ are binomial theorem, Taylor series of a function and solution methods for elementary differential equations, all of which are typically part of the repertoire of a first-year graduate student. In this regard, hopefully, the paper has intuitive and pedagogical appeal.

For the record and for comparison with the alternate calculations, we first present the standard calculation for a one-dimensional homogenous diffusion which has the following SDE form:

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x.$$

Here $B_t$ is the standard Brownian motion. For convenience, we will assume that $b$ and $\sigma$ are bounded continuous functions (or we may need functions in $C^2$). However the calculations below may probably be extended for processes for which the above conditions are not met.

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The infinitesimal generator in this case is

$$Af(x) = b(x)f'(x) + \frac{\sigma^2(x)}{2} f''(x). \tag{1.1}$$

We get this using Itô's lemma (which was originally obtained from Taylor's expansion [5, 9])

$$\begin{aligned} df(X_t) &= f'(X_t)dX_t + \frac{1}{2} f''(X_t) d\langle X\rangle_t \\ &= \left[ b(X_t)f'(X_t) + \frac{\sigma^2(X_t)}{2} f''(X_t) \right] dt + f'(X_t)\sigma(X_t)dB_t. \end{aligned}$$

Integrating, noting that stochastic integral is a martingale, fundamental theorem of calculus and taking limits finally will give the desired result, (1.1). (Standard textbooks in stochastic calculus, for example [5, 6, 10], present this derivation).

The basic idea in our method is also based on Taylor series expansion of $f(X_t)$:

$$\begin{aligned} f(X_t) - f(X_0) = f'(X_0)(X_t - X_0) &+ \frac{1}{2} f''(X_0)(X_t - X_0)^2 \\ &+ \frac{1}{6} f'''(X_0)(X_t - X_0)^3 + \dots \end{aligned}$$

But instead of applying Itô's lemma, we calculate explicitly $\mathbb{E}_x(X_t - X_0)$ and $\mathbb{E}_x(X_t - X_0)^2$ and show that $\mathbb{E}_x(X_t - X_0)^k$ is $o(t^2)$ for $k > 2$. Regarding notation, throughout this paper by $o(t^2)$ we mean terms involving $t^k$, $k \geq 2$, and $\mathbb{E}_x(\cdot)$ denotes conditional expectation given $X_0 = x$. We also note that, although adhoc and elementary, many of our calculations have not appeared in prior literature.

## 2. Diffusion Examples

### 2.1. Brownian motion with drift.

$$dX_t = \mu dt + \sigma dB_t$$

Using (1.1) we know

$$Af(x) = \mu f'(x) + \frac{\sigma^2}{2} f''(x)$$

We will derive this using the alternate method. There are two different ways to calculate the conditional moments in this case (as for many other diffusion process as well) - 1) either work with the solution, or 2) use the conditional density. In the present case, both will work:

1) The solution to the SDE, for this diffusion is

$$X_t = X_0 + \mu t + \sigma B_t$$

from which it is easy to see that

$$\mathbb{E}_x(X_t - X_0) = \mu t,$$

since $B_t$ has mean zero. Similarly,

$$\mathbb{E}_x(X_t - X_0)^2 = o(t^2) + \sigma^2 t.$$

Using the fact that the higher moments $B_t$ is $o(t^2)$ and taking the limit $t \to 0$, after dividing by $t$, we get the required result.

2) The transition density $p_x^t(y)$ in this case is

$$p_x^t(y) = \frac{1}{\sigma\sqrt{2\pi t}} \exp\left[-\frac{1}{2\sigma^2 t}(y-x-\mu t)^2\right]$$

from which it is easy to see that $\mathbb{E}_x(X_t - X_0)$ is

$$\int_{-\infty}^{\infty} (y-x)p_x^t(y)dy = \mu t$$

and $\mathbb{E}_x(X_t - X_0)^2$ is

$$\int_{-\infty}^{\infty} (y-x)^2 p_x^t(y)dy = \sigma^2 t,$$

using the substitution $z = y - x - \mu t$. For $k \geq 3$,

$$\frac{1}{\sigma\sqrt{2\pi t}} \int_{-\infty}^{\infty} (y-x)^k \exp\left\{-\frac{(y-x-\mu t)^2}{2\sigma^2 t}\right\} dy$$

$$= \frac{1}{\sigma\sqrt{2\pi t}} \int_{-\infty}^{\infty} (z+\mu t)^k e^{-z^2/2\sigma^2 t} dz$$

$$= \frac{1}{\sigma\sqrt{2\pi t}} \int_{-\infty}^{\infty} z^k e^{-z^2/2\sigma^2 t} dz + \frac{k\mu t}{\sigma\sqrt{2\pi t}} \int_{-\infty}^{\infty} z^{k-1} e^{-z^2/2\sigma^2 t} dz + o(t^2) = o(t^2),$$

since the odd moments of N(0, $\sigma^2 t$) are zero, and the even moments for $k \geq 2$ is constant $\times$ $(\sigma^2 t)^k = o(t^2)$.

### 2.2. Geometric Brownian Motion.

$$dX_t = \theta X_t dt + aX_t dB_t, \quad X_0 = x > 0$$

Infinitesimal generator based on Itô's lemma calculation is

$$Af(x) = \theta x f'(x) + \frac{a^2x^2}{2} f''(x).$$

We know the solution to be

$$X_t = X_0 \exp\left[\left(\theta - \frac{a^2}{2}\right)t + aB_t\right], \text{ so that}$$

$$\mathbb{E}_x(X_t - X_0) = x\mathbb{E}(e^{aB_t}) \exp\left(\theta - \frac{a^2}{2}\right)t - x = x(e^{\theta t} - 1) = x(\theta t + o(t^2))$$

Taking $r = \theta - \frac{a^2}{2}$, we have

$$\begin{aligned}
\mathbb{E}_x(X_t - X_0)^2 &= \mathbb{E}\left[xe^{(rt+aB_t)} - x\right]^2 \\
&= \mathbb{E}\left[x^2 e^{2rt} e^{2aB_t} - 2x^2 e^{rt} e^{aB_t}\right] + x^2 \\
&= x^2 e^{2rt} e^{2a^2 t} - 2x^2 e^{rt} e^{\frac{a^2 t}{2}} + x^2 \\
&= x^2 \left[e^{(2\theta + a^2)t} - 2e^{\theta t} + 1\right] \\
&= x^2 \left[(1 + (2\theta + a^2)t + o(t^2)) - 2(1 + \theta t + o(t^2)) + 1\right] \\
&= x^2 \left[a^2 t + o(t^2)\right].
\end{aligned}$$

Dividing by $t$ and taking the limit as $t \to 0$ we get the desired result. Similarly, using binomial expansion, it is easy to show that $\mathbb{E}_x(X_t - X_0)^k = o(t^2)$, for $k > 2$.

**2.3. Ornstein-Uhlenbeck process.**

$$dX_t = (a - bX_t)dt + \sigma dB_t, X_0 = x,\ a \in R,\ \ b, \sigma > 0$$

Based on Itô's lemma we know that

$$Af(x) = (a - bx)f'(x) + \frac{\sigma^2}{2} f''(x).$$

The transition density in this case is

$$p_x^t(y) = N\left(xe^{-bt} + \frac{a}{b}(1 - e^{-bt}), \frac{\sigma^2}{2b}(1 - e^{-2bt})\right)$$

from which we get

$$\mathbb{E}_x(X_t - X_0) = xe^{-bt} + \frac{a}{b}(1 - e^{-bt}) - x = \left(\frac{a}{b} - x\right)(1 - e^{-bt}) = \left(\frac{a}{b} - x\right)(bt + o(t^2))$$

Dividing by $t$ and taking the limit as $t \to 0$ this becomes $a - bx$ as desired. Also,

$$\mathbb{E}_x(X_t - X_0)^2 = \int_{-\infty}^{\infty} (y - x)^2 p_x^t(y)dy = \frac{\sigma^2}{2b}(1 - e^{-2bt}) = \sigma^2 t + o(t^2),$$

which yields $\sigma^2$ after division by $t$ and taking the limit as $t \to 0$. If we substitute $z = y - xe^{-bt} - \frac{a}{b}(1 - e^{-bt})$, then $y - x = z + u$, where

$$u = \frac{(bx - a)(e^{-bt} - 1)}{b}$$

so that

$$u^k = (a - bx)^k (t + o(t^2))^k = o(t^2) \text{ for } k \geq 2. \tag{2.1}$$

Hence, for $k \geq 2$,

$$\begin{aligned}
\mathbb{E}_x(X_t - X_0)^k &= \frac{1}{\sqrt{2\pi s^2}} \int_{-\infty}^{\infty} (z + u)^k e^{-z^2/2s^2} dz, \text{ where } s^2 = \frac{\sigma^2(1 - e^{-2bt})}{2b} \\
&= \frac{1}{\sqrt{2\pi s^2}} \int_{-\infty}^{\infty} z^k e^{-z^2/2s^2} dz \\
&\quad + \frac{ku}{\sqrt{2\pi s^2}} \int_{-\infty}^{\infty} z^{k-1} e^{-z^2/2s^2} dz + o(t^2), \text{ using (2.1)} \\
&= o(t^2),
\end{aligned}$$

since the odd moments of N(0, $s^2$) are zero, and the even moments for $k \geq 2$ is

$$\begin{aligned}
\text{constant} \times (s^2)^k &= \text{constant} \times \left(\frac{\sigma^2[1 - e^{-2bt}]}{2b}\right)^k \\
&= \text{constant} \times (t + o(t^2))^k \\
&= o(t^2).
\end{aligned}$$

**2.4. Cox-Ingersoll-Ross process.**

$$dX_t = (a + bX_t)dt + \sigma\sqrt{X_t}\, dB_t, \quad X_0 = x,\ a, b \in R,\ \sigma, X_t > 0. \tag{2.2}$$

We know

$$Af(x) = (a + bx)f'(x) + \frac{\sigma^2 x}{2} f''(x),$$

which we now verify by our ad-hoc methods. Schematically proceeding based on the steps outlined in [1], p.40, we will get a differential equation corresponding to $\mathbb{E}_x(X_t^k)$, for each $k$ (= 1,2,3, ...), whose solution will be $\mathbb{E}_x(X_t^k)$; then using binomial expansion, we will be able to compute $\mathbb{E}_x(X_t - X_0)^k$.

Cox-Ingersoll-Ross process in integral form is

$$X_t = X_0 + \int_0^t (a + bX_s)ds + \int_0^t \sigma\sqrt{X_s}\, dB_s. \tag{2.3}$$

Taking expectations on both sides, we get

$$\mathbb{E}_x(X_t) = x + \int_0^t (a + b\mathbb{E}_x(X_s))ds,$$

where we used the fact the expectation of a stochastic integral is zero and also took the expectation inside the first integral in (2.3). The above equation can be written as a differential equation

$$\frac{du}{dt} = a + bu,\ u(0) = x,$$

whose solution is

$$u(t) = \mathbb{E}_x(X_t) = xe^{bt} + \frac{a}{b}\left(e^{bt} - 1\right).$$

Hence

$$\mathbb{E}_x\left(X_t - X_0\right) = x\left(e^{bt} - 1\right) + \frac{a}{b}\left(e^{bt} - 1\right),$$

$$\lim_{t\to 0} \frac{1}{t}\mathbb{E}_x\left(X_t - X_0\right) = bx + a.$$

For a general $k$, we may write

$$X_t^k = X_0^k + \int_0^t \left[ka + \frac{\sigma^2}{2}k(k-1)\right] X_s^{k-1} ds + bk\int_0^t X_s^k ds + \text{stochastic integral}. \tag{2.4}$$

Although Itô's lemma is usually applied to derive (2.4), it could also be derived from the scratch by using Taylor series expansion to the function $f(x) = x^k$ and using (2.2). Taking expectations on both sides of (2.4) and differentiating with respect to $t$, we get a differential equation for $u(t) = \mathbb{E}_x(X_t^k)$:

$$\frac{du}{dt} = \left[ka + \frac{\sigma^2 k(k-1)}{2}\right]\mathbb{E}_x(X_t^{k-1}) + bku,$$
$$u(0) = x^k.$$

Iteratively, if we know $\mathbb{E}_x(X_t^{k-1})$, we can solve the above differential equation to obtain $u(t) = \mathbb{E}_x(X_t^k)$ as the solution. We illustrate the calculations for $k = 2$ and 3 in the Appendix A and based on those calculations we get

$$\lim_{t\to 0} \frac{\mathbb{E}_x(X_t - X_0)^2}{t} = \sigma^2 x \quad \text{and} \quad \lim_{t\to 0} \frac{\mathbb{E}_x(X_t - X_0)^3}{t} = 0.$$

For a general $k$,

$$\begin{aligned}
\lim_{t\to 0} \frac{\mathbb{E}_x(X_t - X_0)^k}{t} &= \frac{d}{dt}\Big|_{t=0} \mathbb{E}_x(X_t - X_0)^k \\
&= \frac{d}{dt}\Big|_{t=0} \sum_{m=0}^{k} (-1)^{k-m} \binom{k}{m} \mathbb{E}_x(X_t^m) x^{k-m} \\
&= \sum_{m=0}^{k} (-1)^{k-m} \binom{k}{m} \frac{d}{dt}\Big|_{t=0} \mathbb{E}_x(X_t^m) x^{k-m}.
\end{aligned}$$

Denote by $u_m(t) = \mathbb{E}_x(X_t^m)$, $u_m(0) = x^m$ and so

$$\frac{d}{dt} u_m(t) = \left[ ma + \frac{\sigma^2 m(m-1)}{2} \right] u_{m-1}(t) + bmu_m(t).$$

Thus

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} u_m(t) &= \left[ ma + \frac{\sigma^2 m(m-1)}{2} \right] u_{m-1}(0) + bmu_m(0) \\
&= \left[ ma + \frac{\sigma^2 m(m-1)}{2} \right] x^{m-1} + bmx^m \\
&= x^{m-1} \left[ m(a + bx) + \frac{\sigma^2 m(m-1)}{2} \right].
\end{aligned}$$

Plugging this into the limit expression we get

$$\begin{aligned}
&\sum_{m=0}^{k} (-1)^{k-m} \binom{k}{m} \frac{d}{dt}\Big|_{t=0} \mathbb{E}_x(X_t^m) x^{k-m} \\
= &\sum_{m=0}^{k} (-1)^{k-m} \binom{k}{m} x^{m-1} \left[ m(a + bx) + \frac{\sigma^2 m(m-1)}{2} \right] x^{k-m} \\
= &(-1)^k x^{k-1} \left[ (a + bx) \left( \sum_{m=0}^{k} (-1)^m \binom{k}{m} m \right) + \frac{\sigma^2}{2} \left( \sum_{m=0}^{k} (-1)^m \binom{k}{m} m(m-1) \right) \right]
\end{aligned}$$

which is 0 if $k \geq 3$.

## 3. Jump Processes

**3.1. Poisson process.** For a Poisson process $X_t$ with parameter $\lambda$, the standard calculation does not involve Itô's lemma. In a small interval $[0, t]$, the probability of more than one jump is $o(t^2)$ and so we consider only scenarios with 0 or 1 jump in that interval. Then

$$\mathbb{E}_x[f(X_t)] = \lambda t \mathbb{E}_x[f(X_0 + 1)] + (1 - \lambda t + o(t^2))\mathbb{E}_x[f(X_0)] + o(t^2)$$

$$= \lambda t\,[f(x+1) - f(x)] + \mathbb{E}_x[f(X_0)] + o(t^2)$$

so that

$$Af(x) = \lim_{t\downarrow 0} \frac{1}{t}\mathbb{E}_x\,[f(X_t) - f(X_0)] = \lambda\,[f(x+1) - f(x)]$$

is the standard calculation.

**3.2. Compound Poisson process.** Similarly for a compound Poisson process with jump rate $\lambda$ and the jump size $Y$ distributed with law $\nu$,

$$\begin{aligned}
\mathbb{E}_x[f(X_t)] &= \lambda t\mathbb{E}\,[f(X_0+Y)|_{X_0=x})] + (1-\lambda t + o(t^2))\mathbb{E}_x[f(X_0)] \\
&= \lambda t \left\{ \int_{-\infty}^{\infty} \mathbb{E}\,[f(X_0+Y)|_{X_0=x,Y=y}]\,d\nu(y) \right\} \\
&\qquad + (1-\lambda t)\mathbb{E}_x[f(X_0)] + o(t^2) \\
&= \lambda t \int_{-\infty}^{\infty} f(x+y)d\nu(y) - \lambda t f(x) \int_{-\infty}^{\infty} d\nu(y) + \mathbb{E}_x[f(X_0)] + o(t^2)
\end{aligned}$$

so that

$$Af(x) = \lim_{t\downarrow 0} \frac{1}{t}\mathbb{E}_x\,[f(X_t) - f(X_0)] = \lambda \int_{-\infty}^{\infty} [f(x+y) - f(x)]\,d\nu(y),$$

is the standard calculation.

We could also use Itô's lemma for jump processes to do the above calculations ([2], p. 272):

$$f(X_t) - f(X_0) = M_t + \int_0^t \lambda ds \int_{-\infty}^{\infty} [f(X_{s-}+y) - f(X_{s-})]d\nu(y),$$

where $M_t$ is a martingale whose expectation is zero.

## 4. Jump Diffusion Processes

We consider the jump diffusion process where the jump is independent of diffusion. This process is the sum of two independent Lévy processes, where the first and second Lévy components correspond to the diffusion and jump parts, respectively. Lévy processes are processes with independent and stationary increments with right continuity and left limits, so that all the diffusion and jump processes considered above are examples of Lévy processes. It is a theorem (actually an exercise in [8]) that the infinitesimal generator of the sum of two independent Lévy processes is the sum of the corresponding infinitesimal generators. The standard way to calculate the infinitesimal generator for a jump diffusion process, with jump part independent of diffusion, is to calculate generators separately and then apply the above theorem. Using this standard approach it is easy to see that the infinitesimal generator of

$$X_t = \mu t + \sigma B_t + \sum_{i=1}^{N_t} Y_i, \tag{4.1}$$

where $Y_i$'s are i.i.d jump sizes with distribution $\nu$ and $N_t$ is a Poisson process with rate $\lambda$ is

$$Af(x) = \mu f'(x) + \frac{\sigma^2}{2} f''(x) + \lambda \int_{-\infty}^{\infty} [f(x+y) - f(x)]\, d\nu(y). \tag{4.2}$$

We will arrive at the same formula using an ad-hoc approach.

**4.1. Jump diffusion process with Poisson jump part.** First we illustrate the ad-hoc method for the simpler jump-diffusion process, where the jump part is just the Poisson process (that is, jump size is always equal to 1):

$$X_t = \mu t + \sigma B_t + N_t.$$

Again, in a small interval $[0, t]$ we consider only the scenarios with 0 or 1 jump. In this case, we can write,

$$X_t = \begin{cases} X_0 + \mu t + \sigma B_t, & \text{with prob. } (1 - \lambda t + o(t^2)), \\ X_0 + \mu t + \sigma B_t + 1, & \text{with prob. } \lambda t. \end{cases}$$

Let $\Delta X_t$ denote $X_t - X_0$ when there is no jump. That is,

$$\Delta X_t = (X_t - X_0)|_{N_t=0} = \mu t + \sigma B_t, \text{ so that}$$

$$\mathbb{E}_x(\Delta X_t) = \mu t,\ \mathbb{E}_x(\Delta X_t)^2 = \sigma^2 t,\ \mathbb{E}_x(\Delta X_t)^k = o(t^2), \text{ when } k > 2, \tag{4.3}$$

for the current example. Now using Taylor's series we can write in the first case (i.e. when there is no jump),

$$f(X_t)|_{N_t=0} = f(X_0) + f'(X_0)\Delta X_t + \frac{1}{2} f''(X_0)(\Delta X_t)^2 + \dots$$

and

$$f(X_t)|_{N_t=1} = f(X_0+1) + f'(X_0+1)\Delta X_t + \frac{1}{2} f''(X_0+1)(\Delta X_t)^2 + \dots$$

so that

$$\begin{aligned} \mathbb{E}_x(f(X_t)|_{N_t=0}) &= \mathbb{E}[f(X_t)|_{X_0=x, N_t=0}] \\ &= \mathbb{E}_x(f(X_0)) + \mathbb{E}_x(f'(X_0)\Delta X_t) + \frac{1}{2}\mathbb{E}_x(f''(X_0)(\Delta X_t)^2) + \dots \\ &= f(x) + f'(x)\mathbb{E}_x(\Delta X_t) + \frac{1}{2} f''(x)\mathbb{E}_x(\Delta X_t)^2 + \dots \end{aligned}$$

and

$$\mathbb{E}_x(f(X_t)|_{N_t=1}) = f(x+1) + f'(x+1)\mathbb{E}_x(\Delta X_t) + \frac{1}{2} f''(x+1)\mathbb{E}_x(\Delta X_t)^2 + \dots$$

Combining, we get

$$\begin{aligned} \mathbb{E}_x(f(X_t)) &= (1 - \lambda t + o(t^2))\mathbb{E}_x(f(X_t)|_{N_t=0}) + (\lambda t)\mathbb{E}_x(f(X_t)|_{N_t=1}) \\ &= \mathbb{E}_x(f(X_t)|_{N_t=0}) \\ &\quad + \lambda t \left\{\mathbb{E}_x(f(X_t)|_{N_t=1}) - \mathbb{E}_x(f(X_t)|_{N_t=0})\right\} + o(t^2) \qquad (4.4) \\ &= \mathbb{E}_x(f(X_0)) + f'(x)\mathbb{E}_x(\Delta X_t) + \frac{1}{2} f''(x)\mathbb{E}_x(\Delta X_t)^2 + \dots \\ &\quad + \lambda t \left\{[f(x+1) - f(x)] + [f'(x+1) - f'(x)]\mathbb{E}_x(\Delta X_t)\right. \end{aligned}$$

$$+\frac{1}{2}[f''(x+1)-f''(x)]\mathbb{E}_x(\Delta X_t)^2+\ldots\}.$$

Thus, essentially, knowing the conditional moments $\mathbb{E}_x(\Delta X_t)^k,\ k=0,1,\ldots$ we can calculate the infinitesimal generator. In the current example, since we know these conditional moments (4.3), we complete the calculation:

$$\begin{aligned}\mathbb{E}_x(f(X_t)-f(X_0)) = \{f'(x)(\mu t)+\frac{1}{2}(\sigma^2 t)+o(t^2)\} \\ +\lambda t\,\{[f(x+1)-f(x)]+[f'(x+1)-f'(x)](\mu t) \\ +\frac{1}{2}[f''(x+1)-f''(x)](\sigma^2 t)+o(t^2)\}.\end{aligned}$$

Dividing by $t$ and taking the limit as $t\to 0$, we get

$$Af(x)=\mu f'(x)+\frac{1}{2}\sigma^2 f''(x)+\lambda[f(x+1)-f(x)].$$

We get the correct formula for $Af(x)$ although the ad-hoc process was very cumbersome. Using similar ad-hoc approach, we can calculate the $Af(x)$ for the jump-diffusion process given in (4.1).

**4.2. Jump diffusion process with Compound Poisson jump part.** We start with

$$X_t=\begin{cases} X_0+\mu t+\sigma B_t, & \text{with prob. } (1-\lambda t+o(t^2)), \\ X_0+\mu t+\sigma B_t+Y, & \text{with prob. } \lambda t. \end{cases},$$

where $Y$ has distribution $\nu$. Again, we denote $\Delta X_t = X_t - X_0$ when there is no jump. Then

$$\begin{aligned}\mathbb{E}_x(f(X_t)|_{N_t=0}) &= \mathbb{E}_x(f(X_0))+\mathbb{E}_x(f'(X_0)\Delta X_t)+\frac{1}{2}\mathbb{E}_x(f''(X_0)(\Delta X_t)^2)+\ldots \\ &= \int_{-\infty}^{\infty} f(x)d\nu(y)+\int_{-\infty}^{\infty} f'(x)\mathbb{E}_x(\Delta X_t)d\nu(y)+\frac{1}{2}\int_{-\infty}^{\infty} f''(x)\mathbb{E}_x(\Delta X_t)^2 d\nu(y)+\ldots\end{aligned} \tag{4.5}$$

where we used the fact that

$$\int_{-\infty}^{\infty} d\nu(y)=1$$

and that the integrands in (4.5) do not depend on $y$ and hence can be pulled outside the integrals.

$$\begin{aligned}\mathbb{E}_x(f(X_t)|_{N_t=1}) &= \int_{-\infty}^{\infty} \mathbb{E}\left[f(X_t)|_{N_t=1,X_0=x,Y=y}\right] d\nu(y) \\ &= \int_{-\infty}^{\infty} f(x+y)d\nu(y)+\int_{-\infty}^{\infty} f'(x+y)\mathbb{E}_x(\Delta X_t)d\nu(y) \\ &\quad +\frac{1}{2}\int_{-\infty}^{\infty} f''(x+y)\mathbb{E}_x(\Delta X_t)^2 d\nu(y)+\ldots\end{aligned}$$

Again, as before (see (4.4)),

$$\begin{aligned}\mathbb{E}_x(f(X_t)) &= (1-\lambda t+o(t^2))\mathbb{E}_x(f(X_t)|_{N_t=0})+(\lambda t)\mathbb{E}_x(f(X_t)|_{N_t=1}) \\ &= \mathbb{E}_x(f(X_t)|_{N_t=0})+\lambda t\,\{\mathbb{E}_x(f(X_t)|_{N_t=1})-\mathbb{E}_x(f(X_t)|_{N_t=0})\}+o(t^2)\end{aligned}$$

which for our current example becomes

$$\begin{aligned}\mathbb{E}_x(f(X_t)) = &\left\{\mathbb{E}_x(f(X_0)) + (\mu t)f'(x) + \frac{\sigma^2 t}{2}f''(x) + o(t^2)\right\}\\ &+ (\lambda t)\left\{\int_{-\infty}^{\infty}[f(x+y)-f(x)]d\nu(y) + \mu t\int_{-\infty}^{\infty}[f'(x+y)-f'(x)]d\nu(y)\right.\\ &\left.+\frac{\sigma^2}{2}t\int_{-\infty}^{\infty}[f''(x+y)-f''(x)]d\nu(y) + o(t^2)\right\}\end{aligned}$$

so that

$$Af(x) = \lim_{t\to 0}\frac{1}{t}\mathbb{E}_x[f(X_t)-f(X_0)] = \mu f'(x)+\frac{\sigma^2}{2}f''(x)+\lambda\int_{-\infty}^{\infty}[f(x+y)-f(x)]d\nu(y),$$

same as in (4.2).

**4.3. Log-normal jump diffusions.** Next we consider the following SDE:

$$dS_t = \mu S_t dt + \sigma S_t dB_t + S_{t-}dQ_t, \quad \mu \in R,\ \sigma > 0,$$

where $B$ is the Brownian motion independent of $Q$, which is a compound Poisson process with jump rate $\lambda$ and jump size $Y$ distributed $\nu$. We may write it alternatively as

$$S_t = S_0 \exp\left(\mu t + \sigma B_t + \sum_{i=1}^{N_t} Y_i\right)$$

where $N$ is a Poisson process with rate $\lambda$ and $Y_i$'s are i.i.d $\nu$. Again, ignoring the scenarios with more than one jump in a small interval $[0,t]$ (since they have probability $o(t^2)$), we write

$$S_t = \begin{cases} S_0\exp(\mu t + \sigma B_t), & \text{with prob. } (1-\lambda t + o(t^2)),\\ S_0\exp(\mu t + \sigma B_t + Y), & \text{with prob. } \lambda t. \end{cases},$$

and as before, we will be able to write

$$\mathbb{E}_x(f(S_t)) = \mathbb{E}_x(f(S_t)|_{N_t=0}) + \lambda t\left\{\mathbb{E}_x(f(S_t)|_{N_t=1}) - \mathbb{E}_x(f(S_t)|_{N_t=0})\right\} + o(t^2) \quad (4.6)$$

Using the Taylor series expansion, we get

$$\begin{aligned}\mathbb{E}_x(f(S_t)|_{N_t=0}) &= \mathbb{E}(f(S_t)|_{N_t=0,S_0=x})\\ &= \mathbb{E}_x(f(S_0)) + \mathbb{E}_x(f'(S_0)\Delta S_t)\\ &\quad + \frac{1}{2}\mathbb{E}_x(f''(S_0)(\Delta S_t)^2) + \frac{1}{6}\mathbb{E}_x(f'''(S_0)(\Delta S_t)^3) + \dots\\ &= f(x) + f'(x)\mathbb{E}_x(\Delta S_t)\\ &\quad + \frac{1}{2}f''(x)\mathbb{E}_x(\Delta S_t)^2 + \frac{1}{6}f'''(x)\mathbb{E}_x(\Delta S_t)^3 + \dots\end{aligned}$$

where

$$\Delta S_t = (S_t - S_0)|_{N_t=0} \ \ (= [S_0\exp(\mu t + \sigma B_t) - S_0],\ \text{in the current example}).$$

Using the fact that for a random variable $W$ distributed $N(a,b^2)$, $\mathbb{E}(e^W)=e^{a+\frac{b^2}{2}}$, we get

$$\begin{aligned}
\mathbb{E}_x(\Delta S_t) &= x\mathbb{E}_x(e^{\mu t+\sigma B_t}-1) = x(e^{\mu t+\frac{\sigma^2 t}{2}}-1) = x(\mu t+\frac{\sigma^2 t}{2})+o(t^2)\\
\mathbb{E}_x(\Delta S_t)^2 &= x^2\mathbb{E}_x(e^{\mu t+\sigma B_t}-1)^2 = x^2[e^{2\mu t+\frac{4\sigma^2 t}{2}}-2e^{\mu t+\frac{\sigma^2 t}{2}}+1]\\
&= x^2[(2\mu t+2\sigma^2 t)-2(\mu t+(\sigma^2 t/2))]+o(t^2) = x^2\sigma^2 t+o(t^2)\\
\mathbb{E}_x(\Delta S_t)^3 &= x^3(e^{3\mu t+\frac{9\sigma^2}{2}}-3e^{2\mu t+\frac{4\sigma^2}{2}}+3e^{\mu t+\frac{\sigma^2}{2}}-1) = o(t^2).
\end{aligned}$$

We may show in general,

$$\mathbb{E}_x(\Delta S_t)^k = o(t^2), \quad k\geq 3$$

using binomial expansion of $[\exp(\mu t+\sigma B_t)-1]^k$:

$$[\exp(\mu t+\sigma B_t)-1]^k = \sum_{r=0}^{k}\binom{k}{r}(-1)^r\exp\{(k-r)[\mu t+\sigma B_t]\},$$

so that, for $k\geq 3$,

$$\begin{aligned}
\mathbb{E}_x[\exp(\mu t+\sigma B_t)-1]^k &= \sum_{r=0}^{k}\binom{k}{r}(-1)^r\left\{1+(k-r)\mu t+\frac{(k-r)^2\sigma^2 t}{2}\right\}+o(t^2)\\
&= [1+(-1)]^k+(-1)^k k\mu t[1+(-1)]^{k-1}\\
&\qquad +\frac{\sigma^2 t}{2}\sum_{r=0}^{k-1}\binom{k}{r}(-1)^r(k-r)^2+o(t^2)\\
&= o(t^2)
\end{aligned}$$

using the fact that

$$\sum_{u=1}^{k}\binom{k}{u}(-1)^u u^2 = 0.$$

Thus we get

$$\mathbb{E}_x(f(S_t)|_{N_t=0}) = \mathbb{E}_x(f(S_0))+f'(x)[x(\mu t+\frac{\sigma^2 t}{2})]+\frac{1}{2}f''(x)[x^2\sigma^2 t]+o(t^2).$$

Also

$$\begin{aligned}
&\lambda t\left\{\mathbb{E}_x(f(S_t)|_{N_t=1})-\mathbb{E}_x(f(S_t)|_{N_t=0})\right\}\\
&\quad = \lambda t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[f(xe^{\mu t+\sigma\sqrt{t}b+y})-f(xe^{\mu t+\sigma\sqrt{t}b})\right]\varphi_{0,1}(b)dbd\nu(y),
\end{aligned}$$

where $\varphi_{0,1}$ is the standard normal density function. Putting it all together, we get

$$\begin{aligned}
\mathbb{E}_x(f(S_t))-\mathbb{E}_x(f(S_0)) &= f'(x)[x(\mu t+\frac{\sigma^2 t}{2})]+\frac{1}{2}f''(x)[x^2\sigma^2 t]+o(t^2)\\
&\quad +\lambda t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[f(xe^{\mu t+\sigma\sqrt{t}b+y})-f(xe^{\mu t+\sigma\sqrt{t}b})\right]\varphi_{0,1}(b)dbd\nu(y),
\end{aligned}$$

so that

$$Af(x) = \lim_{t\to 0} \frac{1}{t}\mathbb{E}_x[f(S_t) - f(S_0)] = x(\mu + \frac{\sigma^2}{2})f'(x) + \frac{x^2\sigma^2}{2} f''(x)$$
$$+ \lim_{t\to 0} \lambda \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[f(xe^{\mu t+\sigma\sqrt{t}b+y}) - f(xe^{\mu t+\sigma\sqrt{t}b})\right] \varphi_{0,1}(b)db\, d\nu(y).$$

Assuming that the limit can be taken inside the integral term in the right-hand side above (that is, assuming for example that the conditions for dominated convergence theorem or monotone convergence theorem are met), we get, using the continuity of $f$,

$$Af(x) = x\left(\mu + \frac{\sigma^2}{2}\right) f'(x) + \frac{x^2\sigma^2}{2} f''(x) + \lambda \int_{-\infty}^{\infty} [f(xe^y) - f(x)]\, d\nu(y),$$

same as the one obtained using standard methods.

$2^{nd}$ *calculation*: We present a slight variation of the above calculation. The expectation in the $\lambda$ term in (4.6) may be re-written as

$$[\mathbb{E}_x(f(S_t)|_{N_t=1}) - \mathbb{E}_x(f(S_0))] - [\mathbb{E}_x(f(S_t)|_{N_t=0}) - \mathbb{E}_x(f(S_0))].$$

Focusing on the first term, $\lim_{t\downarrow 0}[\mathbb{E}_x(f(S_t)|_{N_t=1}) - \mathbb{E}_x(f(S_0))]$

$$= \lim_{t\downarrow 0} \mathbb{E}\left[f(x\exp(\mu t + \sigma B_t + Y)) - f(x)\right]$$
$$= \lim_{t\downarrow 0} \mathbb{E}\left\{\sum_{k=1}^{\infty} f^{(k)}(x)\left[x\{\exp(\mu t + \sigma B_t + Y) - 1\}\right]^k\right\}$$
$$= \lim_{t\downarrow 0} \int_{-\infty}^{\infty} \sum_{k=1}^{\infty} f^{(k)}(x) \left\{x^k\left[\exp\left(y + \mu t + \frac{\sigma^2 t}{2}\right) - 1\right]^k\right\} d\nu(y)$$
$$= \int_{-\infty}^{\infty} \left(\sum_{k=1}^{\infty} f^{(k)}(x)\left[xe^y - x\right]^k\right) d\nu(y) = \int_{-\infty}^{\infty} [f(xe^y) - f(x)]\, d\nu(y).$$

Here again we assume that $f$ is such that the interchange of integral and the infinite sum in one step and the interchange of integral and the limit in another step are justified. The main difference with the first calculation is that we take the expectation of $\exp(\mu t + \sigma B_t + Y)$ before taking the limit as $t \to 0$. Using similar steps as above it is easy to show that $\lim_{t\downarrow 0}[\mathbb{E}_x(f(S_t)|_{N_t=0}) - \mathbb{E}_x(f(S_0))]$ equals zero.

$3^{rd}$ *calculation*: We present yet another calculation which is more aligned to the original scheme that we proposed: using the first two conditional moments of $S_t - S_0$ and further showing that all the higher moments are zero. In the first two calculations, we were able to complete the calculations by focusing on zero or one jump. In the third calculation, we will take into consideration all possible number of jumps, which makes the calculation more interesting.

If there are $N_t$ jumps in the interval $[0, t]$, then

$$S_t = S_0 \exp[\mu t + \sigma B_t] \prod_{i=1}^{N_t} J_i,$$

where $Y_i = \log(J_i) \sim \nu$ and $N_t \sim \text{Poisson}(\lambda t)$. Thus for any non-negative integer $m$,

$$S_t^m = S_0^m \exp[m\mu t + m\sigma B_t] \prod_{i=1}^{N_t} J_i^m.$$

Then

$$\mathbb{E}_x\left(S_t^m|_{N_t=N}\right) = x^m \exp\left[(m\mu + \frac{1}{2}\sigma^2 m^2)t\right] \left[\mathbb{E}(J^m)\right]^N, \text{ where } Y = \log(J) \sim \nu.$$

$$\begin{aligned}\mathbb{E}_x\left(S_t^m\right) &= x^m \exp\left[(m\mu + \frac{1}{2}\sigma^2 m^2)t\right] e^{-\lambda t} \exp\left[\lambda t(\mathbb{E}(J^m))\right] \\ &= x^m \left[1 + t\left(m\mu + \frac{1}{2}\sigma^2 m^2 + \lambda(\mathbb{E}(J^m) - 1)\right)\right] + o(t^2).\end{aligned}$$

To order $o(t^2)$

$$\begin{aligned}\mathbb{E}_x\left((S_t - S_0)^n\right) &= \sum_{m=0}^{n} (-1)^{n-m} \frac{n!}{m!(n-m)!} \mathbb{E}_x(S_t^m) S_0^{n-m} \\ &= x^n \sum_{m=0}^{n} (-1)^{n-m} \frac{n!}{m!(n-m)!} \\ &\quad \times \left[1 + t\left(m\mu + \frac{1}{2}\sigma^2 m^2 + \lambda(\mathbb{E}(J^m) - 1)\right)\right].\end{aligned}$$

Using

$$\sum_{m=0}^{n} (-1)^{n-m} \frac{n!}{m!(n-m)!} = 0,$$

$$\sum_{m=0}^{n} (-1)^{n-m} \frac{n!\, m}{m!(n-m)!} = \begin{cases} 1 & \text{for } n = 1 \\ 0 & \text{for } n \geq 2, \end{cases}$$

$$\sum_{m=0}^{n} (-1)^{n-m} \frac{n!\, m^2}{m!(n-m)!} = \begin{cases} 1 & \text{for } n = 1 \\ 2 & \text{for } n = 2 \\ 0 & \text{for } n \geq 3, \end{cases}$$

$$\begin{aligned}\sum_{m=0}^{n} (-1)^{n-m} \frac{n!}{m!(n-m)!} \mathbb{E}(J^m) &= \mathbb{E}\left(\sum_{m=0}^{n} (-1)^{n-m} \frac{n!}{m!(n-m)!} J^m\right) \\ &= \mathbb{E}((J-1)^n)\end{aligned}$$

we have, to order $o(t^2)$

$$\begin{aligned}\mathbb{E}_x((S_t - S_0)^n) &= \lambda t \mathbb{E}_x((S_0 J - S_0)^n) + I_{(n=1)} S_0 \left(\mu t + \frac{1}{2}\sigma^2 t\right) + I_{(n=2)} S_0^2 \sigma^2 t \\ &= \lambda t \mathbb{E}_x((S_0 e^Y - S_0)^n) + I_{(n=1)} S_0 \left(\mu t + \frac{1}{2}\sigma^2 t\right) + I_{(n=2)} S_0^2 \sigma^2 t.\end{aligned}$$

Note that the summation formulas above come from $\phi(1)$, $\phi'(1)$ and $\phi''(1)$ where $\phi(x) = (1-x)^n$.

Finally, to order $o(t^2)$

$$\begin{aligned}
\mathbb{E}(f(S_t) - f(S_0)) &= \sum_{n=1}^{\infty} \frac{1}{n!} f^{(n)}(S_0)\mathbb{E}_x((S_t - S_0)^n) \\
&= x\left(\mu t + \frac{1}{2}\sigma^2 t\right) f'(x) + \frac{1}{2}x^2\sigma^2 t f''(x) \\
&\quad + \lambda t \sum_{n=1}^{\infty} \frac{1}{n!} f^{(n)}(x)\mathbb{E}_x((S_0 J - S_0)^n) \\
&= x\left(\mu t + \frac{1}{2}\sigma^2 t\right) f'(x) + \frac{1}{2}x^2\sigma^2 t f''(x) \\
&\quad + \lambda t \mathbb{E}_x\left(\sum_{n=1}^{\infty} \frac{1}{n!} f^{(n)}(S_0)(S_0 J - S_0)^n\right) \\
&= x\left(\mu t + \frac{1}{2}\sigma^2 t\right) f'(x) + \frac{1}{2}x^2\sigma^2 t f''(x) \\
&\quad + \lambda t \mathbb{E}_x(f(S_0 J) - f(S_0)) \\
&= x\left(\mu t + \frac{1}{2}\sigma^2 t\right) f'(x) + \frac{1}{2}x^2\sigma^2 t f''(x) \\
&\quad + \lambda t \mathbb{E}_x(f(S_0 e^Y) - f(S_0)) \\
&= x\left(\mu t + \frac{1}{2}\sigma^2 t\right) f'(x) + \frac{1}{2}x^2\sigma^2 t f''(x) \\
&\quad + \lambda t \int_{-\infty}^{\infty} [f(xe^y) - f(x)] d\nu(y),
\end{aligned}$$

so that

$$Af(x) = x\left(\mu + \frac{\sigma^2}{2}\right) f'(x) + \frac{x^2\sigma^2}{2} f''(x) + \lambda \int_{-\infty}^{\infty} [f(xe^y) - f(x)]\, d\nu(y).$$

## 5. Conclusions

Calculating infinitesimal generators has applications in many fields including, for example, genetics (see [4, 3]). We illustrated how infinitesimal generators may be calculated based on the limit definition using elementary methods such as binomial theorem and Taylor's expansion. The alternate approaches presented in this paper are useful in the cases where Itô's lemma cannot be applied. Non-Markovian diffusions are examples where a simple application of Itô's lemma may not work. Fokker-Planck equations can still be written for such diffusions using a memory kernel [7]. An interesting topic for future work could be to extend the elementary methods outlined in this paper for non-Markovian diffusions, including a recently proposed jump-diffusion process with non-independent jumps [11].

## 6. Appendix

Here we illustrate the calculations outlined for Cox-Ingersoll-Ross process, for the cases $k = 2$ and $k = 3$.

$\underline{k=2}$: The differential equation in this case is

$$\frac{du}{dt} = (2a+\sigma^2)\left[xe^{bt} + \frac{a}{b}(e^{bt}-1)\right] + 2bu,\ u(0) = x^2.$$

Multiplying throughout with the integrating factor $e^{-2bt}$ and then integrating, we may re-write the equation as

$$ue^{-2bt} = (2a+\sigma^2)\left\{-\frac{x}{b}e^{-bt} - \frac{a}{b^2}e^{-bt} + \frac{a}{2b^2}e^{-2bt}\right\} + C.$$

After applying the initial condition, we get the solution to be

$$(u =)\,\mathbb{E}_x(X_t^2) = (2a+\sigma^2)\left\{\frac{a}{2b^2}\left(1-e^{2bt}\right) - \frac{a}{b^2}\left(e^{bt}-e^{2bt}\right) - \frac{x}{b}(e^{bt}-e^{2bt})\right\} + x^2e^{2bt}.$$

Also, based on previous calculation of $\mathbb{E}_x(X_t)$, we have

$$-2x\mathbb{E}_x(X_t) + x^2 = -2x^2e^{bt} - \frac{2xa}{b}(e^{bt}-1) + x^2,$$

so that

$$\lim_{t\to 0}\frac{\mathbb{E}_x(X_t-X_0)^2}{t} = (2a+\sigma^2)\left\{\frac{a}{2b^2}(-2b) - \frac{a}{b^2}(b-2b) - \frac{x}{b}(b-2b)\right\} - \frac{2xa}{b}(b) + \lim_{t\to 0}\frac{x^2}{t}\left[e^{2bt} - 2e^{bt} + 1\right] = \sigma^2 x.$$

$\underline{k=3}$: The differential equation in this case is

$$\frac{du}{dt} = 3(a+\sigma^2)\left\{x^2e^{2bt} + \frac{(2a+\sigma^2)(e^{bt}-1)}{2b}\left[\frac{a}{b}\left(e^{bt}-1\right) + 2xe^{bt}\right]\right\} + 3bu,$$

whose solution is

$$\mathbb{E}_x(X_t^3) = 3e^{3bt}(a+\sigma^2)(2a+\sigma^2)\left\{\frac{a}{2b^2}\left(t - \frac{e^{2bt}}{2b} + \frac{1}{2b}\right) - \frac{a}{b^2}\left(\frac{e^{bt}}{b} - \frac{e^{2bt}}{2b} - \frac{1}{2b}\right) - \frac{x}{b}\left(\frac{e^{bt}}{b} - \frac{e^{2bt}}{2b} - \frac{1}{2b}\right)\right\} + x^3e^{3bt} + \frac{3(a+\sigma^2)x^2}{2b}\left(e^{2bt}-1\right),$$

so that

$$\lim_{t\to 0}\frac{\mathbb{E}_x(X_t-X_0)^3}{t} = 3(a+\sigma^2)x^2 - 3x^2(2a+\sigma^2) + 3x^2a + \lim_{t\to 0}\frac{x^3}{t}\left(e^{bt}-1\right)^3 = 0.$$

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## References


1. Björk, T.: *Arbitrage theory in continuous time*, Oxford: Oxford University Press, 1998.
2. Cont, R., and Tankov, P.: *Financial Modelling with Jump Processes*, Chapman and Hall/CRC, 2004.
3. Durrett, R.: *Probability Models for DNA Sequence Evolution*. Springer Science and Business Media, New York, NY, 2008.
4. Etheridge, A.: *Some Mathematical Models from Population Genetics*: École D'Été de Probabilités de Saint-Flour XXXIX-2009, Vol. 39. Springer Science and Business Media, Berlin, Germany, 2011.
5. Kuo, H.-H.: *Introduction to Stochastic Integration*, New York: Springer Verlag, 2006.
6. Lawler, G. F.: *Introduction to Stochastic Calculus with Applications*, Chapman and Hall/CRC, 2017.
7. Mura, A., Taqqu, M.S., Mainardi, F.: Non-Markovian diffusion equations and processes: Analysis and simulations, *Physica A: Statistical Mechanics and its Applications* **387** (2008), 5033–5064.
8. Sato, K.: *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press; $1^{st}$ edition, 1999.
9. Sengupta, A.: *Pricing Derivatives*, McGraw-Hill, 2005.
10. Steele, J.-M.: *Stochastic Calculus and Financial Applications*, New York: Springer, 2010.
11. Wu, Y. and John, M.: A jump-diffusion process for asset price with non-independent jumps, *Available at SSRN: https://ssrn.com/abstract=3089996*, 2020.



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