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SCS 96: Generators and Weights of Completely Distributive Lattices

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TOPIC: Generators and weights of completely distributive lattices

REFERENCES: [C] Compendium

[BB] B.Banaschewski: Über den Ultrafilterraum. Math. Nachr. 13 (1955) 273-281

[R1] G.N.Raney: Completely distributive complete lattices. Proc. AMS 3 (1952) 667-680

[R2] G.N.Raney: A subdirect-union representation for completely distributive complete lattices. Proc. AMS 4 (1953) 518-522

[R3] G.N.Raney: Tight Galois connections and complete distributivity. Trans. AMS 97 (1960) 418-426

It was shown in [C, III-4.9] that a continuous lattice L has the same weight $w(L)$ as its Scott topology $\sigma(L)$, and it is easy to see that $w(L)$ also agrees with $w(\sigma(L)^{op})$, the weight of the lattice of Scott-closed sets. The observation that a complete lattice L is continuous iff $\sigma(L)$ resp. $\sigma(L)^{op}$ is completely distributive (cf. [C, II-1.14]) leads to the question whether a completely distributive (complete) lattice has always the same weight as its dual. A suitable extension of the above mentioned result to continuous *posets* gives an affirmative answer if one takes into account the well-known fact that every completely distributive lattice is isomorphic to the Scott topology of a continuous poset.

In the present Memo we study a special symmetry of completely distributive lattices which directly gives the equality $w(L) = w(L^{op})$.

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Recall that a complete lattice L is *completely distributive* iff

$$\bigwedge\{\bigvee Y_i : i \in I\} = \bigvee\{\bigwedge \xi[I] : \xi \in \prod_{i\in I} Y_i\}$$

for each family $(Y_i : i \in I)$ of subsets of L, and that complete distributivity is a self-dual property. Since Raney's pioneer work [R1,2,3] we have a broad spectrum of various characterizations of complete distributivity. In [R3] Raney used the following adjoint pair $({}^\nabla, {}^\Delta)$ of maps for an arbitrary complete lattice L:

$${}^\nabla : L \to L\ ,\ x \mapsto x^\nabla = \bigvee(L\setminus\uparrow x)\ ,$$
$${}^\Delta : L \to L\ ,\ y \mapsto y^\Delta = \bigwedge(L\setminus\downarrow y)\ .$$

By definition, we have the equivalence

$$x^\nabla \leq y \iff \uparrow x \cup \downarrow y = L \iff x \leq y^\Delta\ ,$$

which shows that ${}^\Delta$ is in fact the upper adjoint of ${}^\nabla$. In particular, ${}^\nabla$ preserves joins, ${}^\Delta$ preserves meets, $x \mapsto x^{\nabla\Delta}$ is a closure operator, and $y \mapsto y^{\Delta\nabla}$ is a kernel operator. Following an idea of V.Diercks, we shall prove that in a completely distributive lattice every $\bigvee$-generator is mapped onto a $\bigwedge$-generator by virtue of ${}^\nabla$.

In analogy to the way-below relation [C, I-1] for continuous lattices, Raney's "long way-below relation" ρ (alias $\lll$) defined by

$$x\,\rho\,y \ :\iff x \in \rho y := \bigcap\{\,Y : y \leq \bigvee Y,\ Y = \downarrow Y \subseteq L\,\}$$

plays a central rôle in the theory of completely distributive lattices. An element x with $x\,\rho\,x$ is called $\bigvee$*-prime* (*completely join-prime*), and $\bigwedge$*-primes* are defined dually. From [R3] we recall the useful equivalence

$$(1)\quad x\,\rho\,y \iff y \not\leq x^\nabla$$

which holds in *arbitrary* complete lattices, while the identity

$$y = \bigvee\{\,x : x\,\rho\,y\,\} = \bigvee\{\,x : y \not\leq x^\nabla\,\}$$

characterizes *completely distributive lattices.*

From (1) one derives the following representation of the closure operator $x \mapsto x^{\nabla\Delta}$:

$$(2)\quad x^{\nabla\Delta} = \bigwedge x\rho = \bigwedge\{\,y \in L : x\,\rho\,y\,\}\ .$$

Hence the corresponding closure system is given by

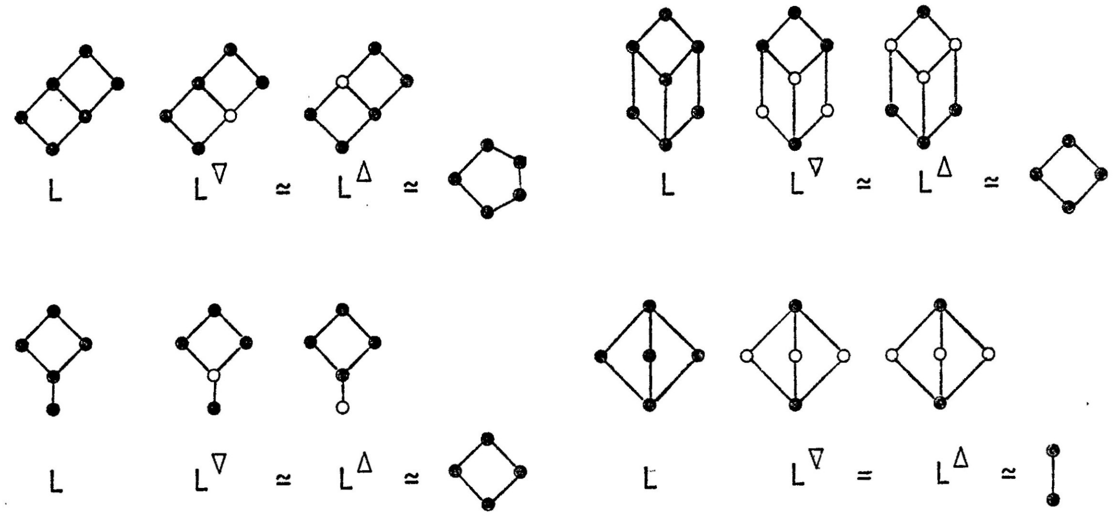
$$(3)\quad L^\Delta = \{\,y^\Delta : y \in L\,\} = \{\,x \in L : x = x^{\nabla\Delta}\,\} = \{\,x \in L : x = \bigwedge x\rho\,\}.$$

By virtue of the Galois connection, L^Δ is isomorphic to the kernel system

$$(4)\ L^\nabla = \{ x^\nabla : x \in L \} = \{ y \in L : y = y^{\Delta\nabla} \} .$$

(Caution: The dual of the long way-below relation of L is in general distinct from the long way-below relation of the dual lattice L^{op}!)

Examples 1 - 4.



These examples demonstrate that the statements " L is (completely) distributive " and " L^∇ resp. L^Δ is (completely) distributive " are independent.

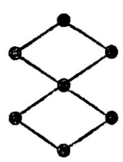
We observe that L^Δ contains all $\wedge$-prime elements and the least element, while L^Δ contains all $\vee$-prime elements and the greatest element. In fact, $^\nabla$ and $^\Delta$ induce mutually inverse isomorphisms between the poset of all $\vee$-primes and the poset of all $\wedge$-primes of L, where

$$x \text{ is } \vee\text{-prime} \iff x \,\rho\, x \iff x \not\leq x^\nabla ,$$

and clearly $x \,\rho\, x$ implies $x = \wedge x\rho = x^{\nabla\Delta} \in L$. The greatest element 1 belongs to L^Δ since $1 = \wedge\emptyset = 1^\Delta$. As our examples show, L^∇ (resp. L^Δ) need not contain all $\wedge$-(resp. $\vee$-) irreducible elements. The greatest element 1 belongs to L^∇ iff 1 is not $\vee$-irreducible (since then $1 = \vee(L \setminus \{1\})$).

In the preceding examples, $L^\nabla \setminus \{0,1\}$ is precisely the set of all $\wedge$-primes, but this is not always the case, even if L is finite and distributive:

Example 5.



$L = L^\nabla = L^\Delta$.

Notice that

x^∇ is Λ-prime iff $x\rho$ is a principal filter,
x^∇ is (finitely meet-)prime iff $x\rho$ is a filter,
x^∇ is co-compact iff $x\rho$ is dually Scott-closed.

Indeed, for $Y \subseteq L$ we have

$$\bigwedge Y \not\leq x^\nabla \iff \bigwedge Y \in x\rho,$$

$$y \not\leq x^\nabla \text{ for all } y \in Y \iff Y \subseteq x\rho.$$

A useful consequence of (1) is the following

Lemma. *Let* B *be an arbitrary subset of a complete lattice* L. *Then* $B^\nabla = \{ b^\nabla : b \in B \}$ *is a* Λ*-generator of* L *(i.e. each element is a meet of elements in* B^∇ *) iff* $\rho y \cap B \subseteq \rho z \cap B$ *implies* $y \leq z$ $(y,z \in L)$.

Proof. B^∇ is a Λ-generator of L

$$\iff \forall\, y,z \in L \;(\, y \not\leq z \implies \exists\, b \in B: y \not\leq b^\nabla \text{ and } z \leq b^\nabla \,)$$

$$\iff \forall\, y,z \in L \;(\, y \not\leq z \implies \exists\, b \in B: b\,\rho\, y \text{ but not } b\,\rho\, z \,)$$

$$\iff \forall\, y,z \in L \;(\, y \not\leq z \implies \rho y \cap B \not\subseteq \rho z \cap B \,). \;\square$$

Of course, the implication $y \leq z \implies \rho y \cap B \subseteq \rho z \cap B$ is always true.

After these preliminaries we are in position to supplement the long list of characterizations for complete distributivity by certain conditions on special $\bigvee$-, resp., Λ-generators.

Theorem. *For a complete lattice* L, *the following conditions are equivalent:*

(a) L *is completely distributive.*
(b) $y = \bigvee \rho y$ *for all* $y \in L$.
(c) *For all* $y,z \in L$ *with* $y \not\leq z$, *there is an* $x \in L$ *with* $x\,\rho\, y$ *and* $x \not\leq z$.
(d) *If* B *is a* $\bigvee$*-generator of* L *then* $y = \bigvee(\rho y \cap B)$ *for all* $y \in L$.
(e) *If* B *is a* $\bigvee$*-generator of* L *then* B^∇ *is a* Λ*-generator of* L, *and* ρ *is idempotent.*
(f) L^∇ *is a* Λ*-generator of* L, *and* ρ *is idempotent.*

Each of these conditions implies that there is a one-to-one correspondence between the $\bigvee$*-generators of* L *which are contained in* L^Δ *and the* Λ*-generators of* L *which are contained in* L^∇.

Proof. The equivalence of (a), (b) and (c) is due to Raney [R2] who also has shown that these statements imply idempotency of ρ.

(c) $\Longrightarrow$ (d): Clearly $z := \bigvee(\rho y \cap B) \leq y$. Assume $y \not\leq z$. Then we may choose $x \in L$ with $x \rho y$ and $x \not\leq z$. As B is a $\bigvee$-generator, we find a $b \in B$ with $b \leq x$ and $b \not\leq z$. But $b \leq x \rho y$ leads to $b \in \rho y \cap B$, which is excluded by the inequality $b \not\leq z = \bigvee(\rho y \cap B)$.

(d) $\Longrightarrow$ (e): $y = \bigvee(\rho y \cap B)$, $z = \bigvee(\rho z \cap B)$ and $\rho y \cap B \subseteq \rho z \cap B$ implies $y \leq z$. Hence, by the Lemma, B^{∇} is a $\bigwedge$-generator of L. Idempotency of ρ follows from (b) which is a trivial consequence of (d).

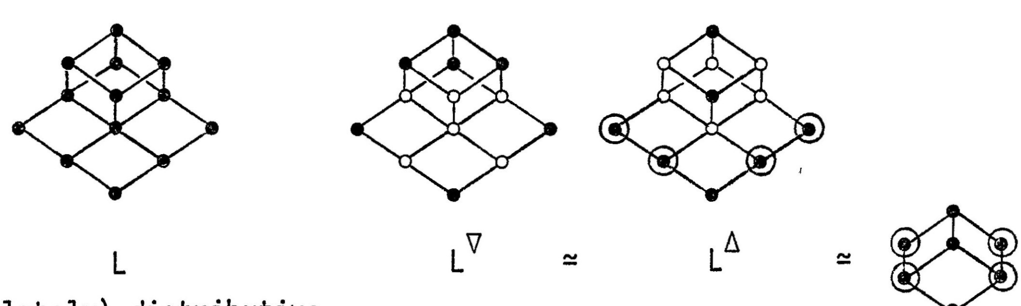
(e) $\Longrightarrow$ (f): L is a $\bigvee$-generator of L.

(f) $\Longrightarrow$ (b): By the Lemma, we have $y \leq z \iff \rho y \subseteq \rho z$. $(B = L)$.
In other words, the map $\varphi : L \to \mathcal{P}L, \ y \mapsto \rho y$ is an order embedding. Now idempotency of ρ yields $\bigcup \varphi[\rho y] = \{x \in L : x \rho z \text{ for some } z \in \rho y\} = \rho y = \varphi(y)$, and as φ induces an isomorphism between L and $\varphi[L]$, it follows that $\bigvee \rho y = y$.

Finally, observe that the adjoint pair $(^{\nabla}, ^{\Delta})$ induces mutually inverse isomorphisms between L^{Δ} and L^{∇}, and that complete distributivity is a selfdual property. Hence the last statement of the Theorem follows from the implication (a) $\Longrightarrow$ (e). □

Remark. If B is a $\bigvee$-generator of the completely distributive lattice L with $B \subseteq L^{\Delta}$ then B^{∇} is a $\bigwedge$-generator of L contained in L^{∇}, and in particular, B^{∇} is a $\bigwedge$-generator of L^{∇}. Moreover, the isomorphism $^{\Delta}: L^{\nabla} \to L^{\Delta}$ maps B^{∇} onto the initial set B, and consequently, B is not only a $\bigvee$-generator but also a $\bigwedge$-generator of L^{Δ}. However, a $\bigvee$-generator of L^{Δ} need not be a $\bigvee$-generator of L, although L^{Δ} is a $\bigvee$-generator of L (provided L is completely distributive). The reason is that joins in L^{Δ} may differ from those in L (while meets agree in both lattices).

Example 6.



L (completely) distributive $\qquad L^{\nabla} \simeq L^{\Delta} \simeq$

⦿ (minimal) $\bigvee$-generator of L^{Δ}, but not of L.

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A complete lattice L is said to be a *normal completion* of a poset P iff there exists a join- and meet-dense embedding φ of P in L (i.e. φ[P] is both, a ⋁- and a ⋀-generator of L). It is well known that L is a normal completion of P iff L is isomorphic to the *completion by cuts* of P which consists of all intersections formed by principal ideals of P.

The above remark may now be summarized as follows:

Corollary 1. *If* B *is a* ⋁*-generator of the completely distributive lattice* L *with* $B \subseteq L^{\Delta}$ *then* L^{Δ} *and* L^{∇} *are normal completions of* B.

From the Lemma and the Theorem, we infer:

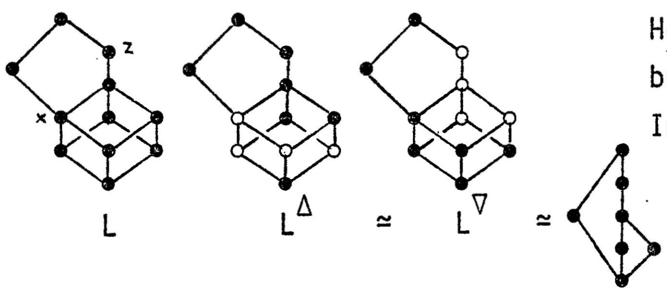
Corollary 2. *The following two conditions are equivalent for a complete lattice* L*:*

(a) L^{Δ} *is a* ⋀*-generator of* L.

(b) $\rho y \subseteq \rho z$ *implies* $y \leq z$.

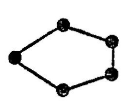
Each of these conditions is necessary, but not sufficient for complete distributivity.

Example 7. The following finite lattice L fails to be (completely) distributive, although L^{Δ} is a ⋀-generator of L:



Here we have $x \ \rho \ z$,
but there is no $y \in L$ with $x \ \rho \ y \ \rho \ z$.
In other words, ρ is not idempotent.

It was already observed by Raney [R2] that idempotency of ρ alone is not sufficient for complete distributivity, as a five-element nonmodular lattice shows.



In the Compendium [C], the *weight* $\mathcal{W}(L)$ of a complete lattice L has been defined to be the smallest cardinality of a "base", i.e. a ⋁-generator which is closed under finite joins. Now from the Theorem we infer (using the fact that complete distributivity is a self-dual property):

Corollary 3. *Every completely distributive lattice has the same weight as its dual.*

This equality cannot be extended to, say, algebraic distributive lattices. For example, if X is an infinite set then the lattice F(X) of all set-theoretical filters on X has weight $2^{|X|}$ (the principal filters form the smallest "base"), while $F(X)^{op}$ has weight $2^{2^{|X|}}$ since there are $2^{2^{|X|}}$ ultrafilters on X (see, for example, Banaschewski [BB]), and these form the smallest $\wedge$-generator of F(X).

<u>Problem</u>. Is it always true that $w(L) = w(L^{op})$ if L and L^{op} are continuous (algebraic) lattices?
By Corollary 3, the answer is in the affirmative if L is also assumed to be distributive.