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SCS 95: Fixed Point Constructions for Standard Completions

Marcel Erné
Leibniz University Hannover, 30167, Hannover, Germany, erne@math.uni-hannover.de

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NAME: Marcel Erné

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TOPIC: Fixed point constructions for standard completions

REFERENCES: [C] A Compendium of Continuous Lattices.

[DGL] R.P. Dilworth and A.M. Gleason, A generalized Cantor theorem. Proc. AMS 13 (1962), 704-705.

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[ME 1] M. Erné, Adjunctions and standard constructions for partially ordered sets. In: Contributions to General Algebra 2, Proc. Klagenfurt Conf. 1982. Teubner, Stuttgart (1983), 77-106.

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For each poset P denote by $\mathcal{M}P$ the system of all principal ideals and by $\mathcal{A}P$ the system of all lower sets, i.e.

$$\mathcal{M}P = \{\downarrow y\colon y \in P\},$$
$$\mathcal{A}P = \{\downarrow Y\colon Y \subseteq P\}.$$

A *standard extension* is a function $\mathcal{Y}$ assigning to each poset P a system $\mathcal{Y}P$ of lower sets with $\mathcal{M}P \subseteq \mathcal{Y}P$. If each $\mathcal{Y}P$ is a closure system then we speak of a *standard completion* (see [ME 1]). It is obvious that every poset P is isomorphic to its least standard extensions $\mathcal{M}P$, via the isomorphism

$$\eta_P : P \to \mathcal{M}P\ ,\quad y \mapsto \downarrow y\ .$$

On the other hand, it was observed by Dilworth and Gleason [DGL] that P is never isomorphic to $\mathcal{A}P$. In a recent paper [ME 2], the following general result concerning possible isomorphisms between a poset P and certain extensions $\mathcal{Y}P$ has been established:

PROPOSITION. *If* $\mathcal{Y}$ *is a standard extension and* P *a poset such that*

(i) *for every isomorphism* $\varphi\colon P \to Q$, $Y \in \mathcal{Y}P$ *implies* $\varphi[Y] \in \mathcal{Y}Q$,

(ii) $\bigcup \mathcal{X} \in \mathcal{Y}P$ *for all* $\mathcal{X} \in \mathcal{Y}\mathcal{Y}P$,

(iii) $\mathcal{Y}P$ *contains every lower set generated by a chain*,

then P *cannot be isomorphic to* $\mathcal{Y}P$ *unless* P *satisfies the Ascending Chain Condition (ACC) and* $\mathcal{Y}P = \mathcal{M}P$.

The standard extension $\mathcal{Y}$ is called *invariant* if (i) is true, and *union complete* if (ii) holds for all posets P (see [ME 1]). Examples of union complete invariant extensions are

- $\mathcal{M}$, the *minimal extension*,
- $\mathcal{A}$, the *Alexandroff completion*,
- $\mathcal{I}$, the *ideal extension* (where $\mathcal{I}P$ consists of all *ideals*, i.e. directed lower sets),
- $\mathcal{N}$, the *completion by cuts* or *Dedekind-MacNeille completion* (where $\mathcal{N}P$ consists of all *cuts*, i.e. intersections of principal ideals),
- $\mathcal{S}$, the *Scott completion* (where $\mathcal{S}P$ consists of all *Scott-closed sets*, i.e. lower sets which are closed under directed joins).

The first four examples have the property of being invariant extensions $\mathcal{Y}$ with

$$P \simeq \mathcal{Y}P \iff \mathcal{M}P = \mathcal{Y}P,$$

while the Scott completion $\mathcal{S}$ is of completely different nature. Although it is true that P cannot be isomorphic to $\mathcal{S}P$ if $\mathcal{S}P$ contains all lower sets generated by chains (a rather rare occurence which implies $\mathcal{S}P = \mathcal{A}P$), it can happen very well that $P \simeq \mathcal{S}P$. Of course, such an isomorphism forces P to be a complete lattice. Moreover, from the main result in [ME 2] we conclude that in a poset P with $P \simeq \mathcal{S}P$ every Scott-closed set which is not a principal ideal must possess a properly ascending sequence of upper bounds. From this observation it follows easily that a chain C is isomorphic to its own Scott completion iff C is the ordinal sum of ω (the chain of natural numbers) and a complete chain. But there are also many non-linearly ordered complete lattices which are isomorphic to their Scott completion. In fact, Chapter IV of the Compendium provides the basic ideas how to construct such lattices via certain projective limits. A thorough investigation of the "fixed point construction" for certain completions like $\mathcal{S}$ is the purpose of this Memo.
One particular result states that every finite lattice L is the image of some lattice $\tilde{L}$ with $\tilde{L} \simeq \mathcal{S}\tilde{L}$ under a CL-morphism. However, on account of our previous remark, no finite lattice (and, more generally, no lattice satisfying the ACC) can be isomorphic to its own Scott completion.

In order to formulate the desired results in adequate generality, we must collect a few preliminaries developed in [ME 1] and [C, IV-3/4]. Henceforth, let $\mathcal{Y}$ denote a standard extension. A map φ between posets P and Q is called $\mathcal{Y}$*-continuous* (resp. *weakly* $\mathcal{Y}$*-continuous*) if $\varphi^{-1}[Y] \in \mathcal{Y}P$ for all $Y \in \mathcal{Y}Q$ (resp. $Y \in \mathcal{M}Q$). We say $\mathcal{Y}$ to be *compositive* if the composition of any two weakly $\mathcal{Y}$-continuous maps is again weakly $\mathcal{Y}$-continuous. From [ME 1] we recall the following facts:

(1) $\mathcal{Y}$ is compositive iff every weakly $\mathcal{Y}$-continuous map is already $\mathcal{Y}$-continuous.

(2) If $\mathcal{Y}$ is compositive then every lower adjoint map and, in particular, every isomorphism is $\mathcal{Y}$-continuous.

In fact, lower adjoint (= residuated) maps may be characterized by the property that inverse images of principal ideals are principal ideals (cf. [C, O-3]).

(3) The canonical embedding

$$\eta_P : P \to \mathcal{Y}P \ , \ y \mapsto \ \downarrow y$$

is always weakly $\mathcal{Y}$-continuous. It is $\mathcal{Y}$-continuous iff $\mathcal{X} \in \mathcal{Y}\mathcal{Y}P$ implies $\bigcup\mathcal{X} \in \mathcal{Y}P$.

From (1), (2) and (3) we derive the following consequence:

(4) Every compositive standard extension is invariant and union complete. □

Each of the standard extensions $\mathcal{M}$, $\mathcal{A}$, $\mathcal{I}$, $\mathcal{N}$ and $\mathcal{S}$ is compositive. Notice that "$\mathcal{A}$-continuous" means "isotone", while the $\mathcal{S}$-continuous ("Scott continuous") maps are precisely those which preserve directed joins (see [C,I-2]). Similarly, a map between *complete lattices* is $\mathcal{N}$-continuous iff it preserves arbitrary joins. Another compositive standard completion is the *minimal topological completion* $\mathcal{U}$, where $\mathcal{U}P$ denotes the least topological closure system containing all principal ideals of P. Thus $\mathcal{U}P$ consists of arbitrary intersections formed by finite unions of principal ideals. In [C] the corresponding system of open sets is referred to as the *upper topology*.

Let us return to the general situation of an arbitrary standard *completion* $\mathcal{Y}$. We have to consider the following categories:

category	objects	morphisms
$P_{\mathcal{Y}}$	posets	$\mathcal{Y}$-continuous maps
SUP	complete lattices	sup-preserving maps
INF	complete lattices	inf-preserving maps
$INF_{\mathcal{Y}}$	complete lattices	inf-preserving $\mathcal{Y}$-continuous maps
C	complete lattices	complete homomorphisms (i.e. sup- and inf-preserving maps)

(Notice that $C = INF_{\mathcal{M}} = INF_{\mathcal{N}}$).

We may regard $\mathcal{Y}$ as a functor from $P_{\mathcal{Y}}$ to SUP by assigning to each $\mathcal{Y}$-continuous map $\varphi: P \to Q$ the "lifted" map

$$\mathcal{Y}\varphi:\ \mathcal{Y}P \to \mathcal{Y}Q\ ,\quad Y \mapsto \overline{\varphi[Y]} = \bigcap\{Z \in \mathcal{Y}Q:\ \varphi[Y] \subseteq Z\}.$$

In order to see that $\mathcal{Y}\varphi$ actually preserves arbitrary sups, we only have to observe that

$$\varphi^{-1}: \mathcal{Y}Q \to \mathcal{Y}P\ ,\ Z \mapsto \varphi^{-1}[Z]$$

is the upper adjoint of φ . Indeed, we have the equivalence

$$\overline{\varphi[Y]} \subseteq Z \iff \varphi[Y] \subseteq Z \iff Y \subseteq \varphi^{-1}[Z] \qquad (Y \in \mathcal{Y}P,\ Z \in \mathcal{Y}Q).$$

Of particular use is the following

LEMMA 1. *A compositive standard completion $\mathcal{Y}$ preserves adjoint pairs; i.e., if $\varphi: P \to Q$ is $\mathcal{Y}$-continuous and has a lower adjoint $\phi : Q \to P$ then $\mathcal{Y}\phi$ is the lower adjoint of $\mathcal{Y}\varphi$. In particular, $\mathcal{Y}$ induces a functor from the category $INF_{\mathcal{Y}}$ to the subcategory* C .

PROOF. The assumption that $\mathcal{Y}$ be compositive is only needed in order to guarantee that the lower adjoint ϕ be a $P_{\mathcal{Y}}$-morphism (see (2)). By definition of a lower adjoint, we have

$$\phi(z) \leq y \iff z \leq \varphi(y) \qquad (y \in P,\ z \in Q),$$

and for $Y \in \mathcal{Y}P$ it follows that

$$\begin{aligned} z \in \phi^{-1}[Y] &\iff \phi(z) \in Y \\ &\iff \phi(z) \leq y \ \text{ for some } y \in Y \\ &\iff z \leq \varphi(y) \ \text{ for some } y \in Y\ , \end{aligned}$$

whence $\quad {\downarrow}\varphi[Y] = \phi^{-1}[Y] \in \mathcal{Y}Q \quad$ and consequently

$$\downarrow\varphi[Y] = \overline{\varphi[Y]} = \mathcal{Y}\varphi(Y).$$

But, on the other hand, we have seen that ϕ^{-1} is the upper adjoint of $\mathcal{Y}\phi$. In particular, this shows that for any $\mathbf{INF}_{\mathcal{Y}}$-morphism φ, the lifted map $\mathcal{Y}\varphi$ has both an upper and a lower adjoint; in other words, φ is a $\mathbf{C}$-morphism. □

LEMMA 2. *If* Q *is a poset with* $\mathcal{Y}Q \subseteq \mathcal{S}Q$ *then the restriction of the functor* $\mathcal{Y}$ *to the set of all* $\mathcal{Y}$*-continuous maps from a fixed poset* P *into* Q *is* $\mathcal{S}$*-continuous (i.e. preserves directed joins).*

PROOF. Let $\{\varphi_i : i \in I\}$ be a directed family of $\mathbf{P}_{\mathcal{Y}}$-morphisms $\varphi_i : P \to Q$ which have a point-wise supremum $\varphi: P \to Q$. Then we compute for $Y \in \mathcal{Y}P$:

$$\mathcal{Y}\varphi(Y) = \overline{\varphi[Y]} = \overline{\{\bigvee\{\varphi_i(y) : i \in I\} : y \in Y\}} =: B \in \mathcal{Y}Q \subseteq \mathcal{S}Q.$$

$$\bigvee\{\mathcal{Y}\varphi_i : i \in I\}(Y) = \bigvee\{\overline{\varphi_i[Y]} : i \in I\} = \overline{\bigcup\{\varphi_i[Y] : i \in I\}}$$

$$= \overline{\{\varphi_i(y) : i \in I,\ y \in Y\}} =: B' \in \mathcal{Y}Q \subseteq \mathcal{S}Q.$$

Since a set $A \subseteq Q$ is Scott closed iff

$$(\,D \subseteq A \iff \bigvee D \in A\,) \text{ for all directed sets } D \subseteq Q,$$

we see that B coincides with B'. □

Now we are ready to apply the results of [C,IV-3] (in particular 3.13 and the subsequent remarks).

THEOREM 1. *Let* $\mathcal{Y}$ *be a compositive completion with* $\mathcal{Y}L \subseteq \mathcal{S}L$ *for all complete lattices* L. *Then the functor* $\mathcal{Y}: \mathbf{INF}_{\mathcal{Y}} \to \mathbf{C}$ *preserves projective limits, surjectivity and injectivity of morphisms.*

If $\mathcal{Y}L$ is a topological closure system then the inclusion $\mathcal{Y}L \subseteq \mathcal{S}L$ means that the corresponding topology is *order consistent* in the sense of [C, II-1.16], i.e. every directed set in L converges to its supremum. Notice that $\mathcal{U}L$ is the smallest and $\mathcal{S}L$ is the largest topological closure system with this property.

Now, suppose L is an arbitrary complete lattice, and there is given a $\mathbf{C}$-morphism $\varphi: \mathcal{Y}L \to L$. Then the pair (L,φ) (or simply φ itself) is usually referred to as a $\mathcal{Y}$*-algebra* over $\mathbf{C}$ (see [C, IV-4.3]). The induced "inverse sequence"

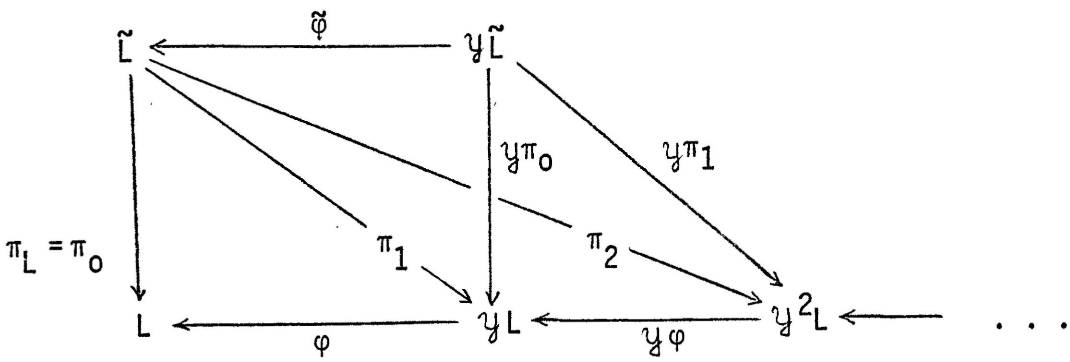
$$L \xleftarrow{\ \varphi\ } \mathcal{Y}L \xleftarrow{\ \mathcal{Y}\varphi\ } \mathcal{Y}^2L \xleftarrow{\ \mathcal{Y}^2\varphi\ } \mathcal{Y}^3L \longleftarrow \cdots$$

may be considered as a projective system.

Its limit can be constructed explicitly in the usual way:

$$\tilde{L} := L_\varphi := \{x \in \prod_{n\in\omega} \mathcal{Y}^n L : \mathcal{Y}^n\varphi(x_{n+1}) = x_n \text{ for all } n \in \omega\}$$

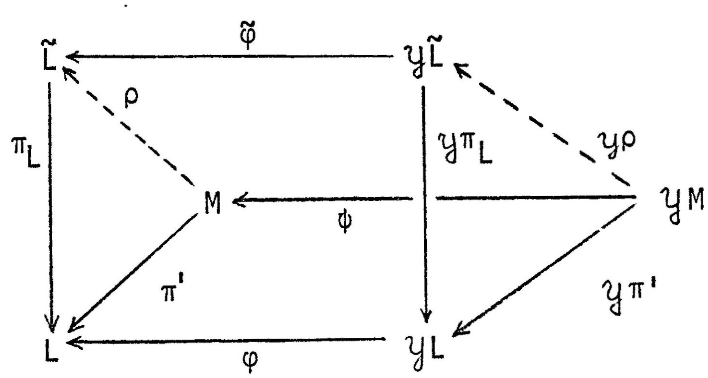
(where $\mathcal{Y}^0 L = L$ and $\mathcal{Y}^0\varphi = \varphi$).
The projections $\pi_n : \tilde{L} \to \mathcal{Y}^n L$ are C-morphisms possessing the universal property required for a limit. In particular, $\pi_L := \pi_0$ is a C-morphism from $\tilde{L}$ onto L.



The statement "$\mathcal{Y}$ preserves projective limits" means that the unique C-morphism $\tilde{\varphi}$ making the diagram commute is in fact an isomorphism. As was explained in [C, IV-4], the assignment $(L,\varphi) \mapsto (\tilde{L},\tilde{\varphi})$ extends to a functor from the category of $\mathcal{Y}$-algebras (L,φ) to the full subcategory of those $\mathcal{Y}$-algebras (M,ϕ) for which ϕ is an isomorphism. Moreover, this functor turned out to be a co-reflection (see [C, IV-4.9]).

Let us summarize:

THEOREM 2. *Suppose $\mathcal{Y}$ is a compositive standard completion with $\mathcal{Y}L \subseteq \mathcal{S}L$ for all complete lattices L (e.g. $\mathcal{Y} = \mathcal{N}, \mathcal{S}$, or $\mathcal{U}$). Then for every surjective $\mathcal{Y}$-morphism $\varphi : \mathcal{Y}L \to L$ there is a complete lattice $\tilde{L}$, an isomorphism $\tilde{\varphi} : \mathcal{Y}\tilde{L} \to \tilde{L}$ and a surjective C-morphism $\pi_L : \tilde{L} \to L$ such that $\pi_L\tilde{\varphi} = \varphi\mathcal{Y}\pi_L$. Moreover, this construction is universal in the following sense. If $\phi : \mathcal{Y}M \to M$ is an isomorphism and π' is a C-morphism from M onto L such that $\pi'\phi = \varphi\mathcal{Y}\pi'$ then there is a unique C-morphism ρ from M to $\tilde{L}$ with $\pi' = \pi_L\rho$ and $\rho\phi = \tilde{\varphi}\mathcal{Y}\rho$.*



The same statements hold when C *is replaced with the larger category* $INF_{\mathcal{Y}}$.

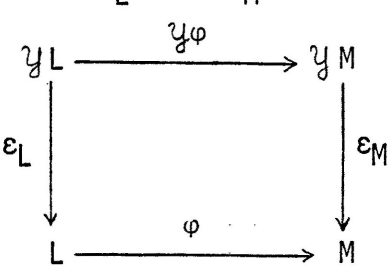
It remains to settle the problem how to find a complete homomorphism (or at least an $\mathrm{INF}_{\mathcal{Y}}$-morphism) φ from $\mathcal{Y}L$ onto a given complete lattice L. Indeed, it is not even clear whether such a morphism does exist at all. If L is completely distributive then the join map

$$\varepsilon_L : \mathcal{Y}L \to L \ , \quad Y \mapsto \bigvee Y$$

will do the job; in fact, for ε_L to be a complete homomorphism, it is necessary and sufficient that L be $\mathcal{Y}$-*distributive*, i.e.

$$\bigwedge\{\bigvee Y_i : i \in I\} = \bigvee\{\bigwedge \xi[I] : \xi \in \prod_{i\in I} Y_i\}$$

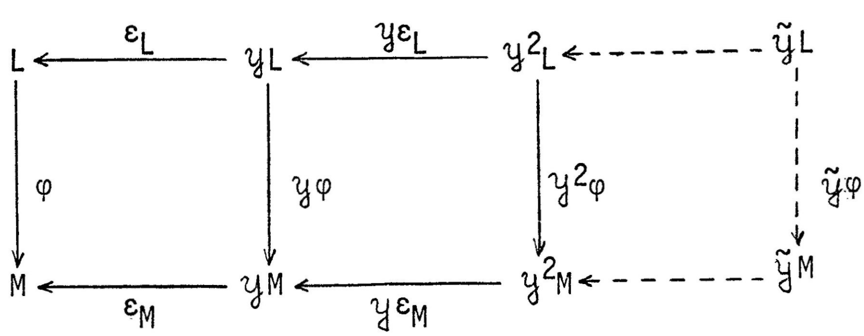
for every family $(Y_i : i \in I)$ of sets in $\mathcal{Y}L$. It can be shown that the $\mathcal{Y}$-distributive complete lattices form a complete full subcategory $\mathbf{CD}_{\mathcal{Y}}$ of $\mathbf{C}$ whenever $\mathcal{Y}$ is a compositive standard completion. If $\mathcal{Y}$ is only assumed to be union complete then $\mathcal{Y}P$ is always a $\mathcal{Y}$-distributive complete lattice because $\mathcal{Y}$-joins are unions and meets are intersections. Hence $\mathcal{Y}$ may be viewed as a functor from $\mathbf{C}$ to $\mathbf{CD}_{\mathcal{Y}}$. Moreover, given a $\mathbf{C}$-morphism $\varphi : L \to M$, it is easy to see that the join maps ε_L and ε_M make the following diagram commute:



Hence we arrive at

COROLLARY 1. *Let $\mathcal{Y}$ be a compositive standard completion. Then the $\mathcal{Y}$-distributive complete lattices form a complete subcategory $\mathbf{CD}_{\mathcal{Y}}$ of $\mathbf{C}$, and $\mathcal{Y}$ gives rise to a functor from $\mathbf{C}$ to $\mathbf{CD}_{\mathcal{Y}}$. The join maps $\varepsilon_L : \mathcal{Y}L \to L$ define a natural transformation. If $\mathcal{Y}L \subseteq \mathcal{S}L$ for all complete lattices L then there is a self-functor $\tilde{\mathcal{Y}}$ of $\mathbf{CD}_{\mathcal{Y}}$ such that each $\mathcal{Y}$-distributive lattice L is the complete homomorphic image of $\tilde{L} = \tilde{\mathcal{Y}}L$ and $\tilde{\mathcal{Y}}L \simeq \mathcal{Y}\tilde{\mathcal{Y}}L \simeq \tilde{\mathcal{Y}}\tilde{\mathcal{Y}}L$.*

For any $\mathbf{CD}_{\mathcal{Y}}$-morphism $\varphi : L \to M$, $\tilde{\mathcal{Y}}\varphi$ is the unique $\mathbf{CD}_{\mathcal{Y}}$-morphism making the following diagram commute:



This corollary applies, for example, to the standard completions $\mathcal{N}$, $\mathcal{S}$ and $\mathcal{U}$. However, for $\mathcal{N}$ it does not give any interesting information since $CD_{\mathcal{N}} = C$ and $L \simeq \mathcal{N}L \simeq \tilde{\mathcal{N}}L$ for each complete lattice L.

It should be mentioned that the isomorphism $\tilde{\varepsilon}_L: \mathcal{S}\tilde{L} \to \tilde{L}$ (see Theorem 2) is always distinct from the join map $\varepsilon_{\tilde{L}}$ which never can be an isomorphism because $\varepsilon_{\tilde{L}}(\emptyset) = \varepsilon_{\tilde{L}}(\{0\}) = 0$.

Notice that $CD_{\mathcal{A}}$ is the category of completely distributive lattices; but the second part of Corollary 1 does not apply to the Alexandroff completion $\mathcal{A}$ since $\mathcal{A}L \nsubseteq \mathcal{S}L$ unless L satisfies the ACC. However, we have the following result:

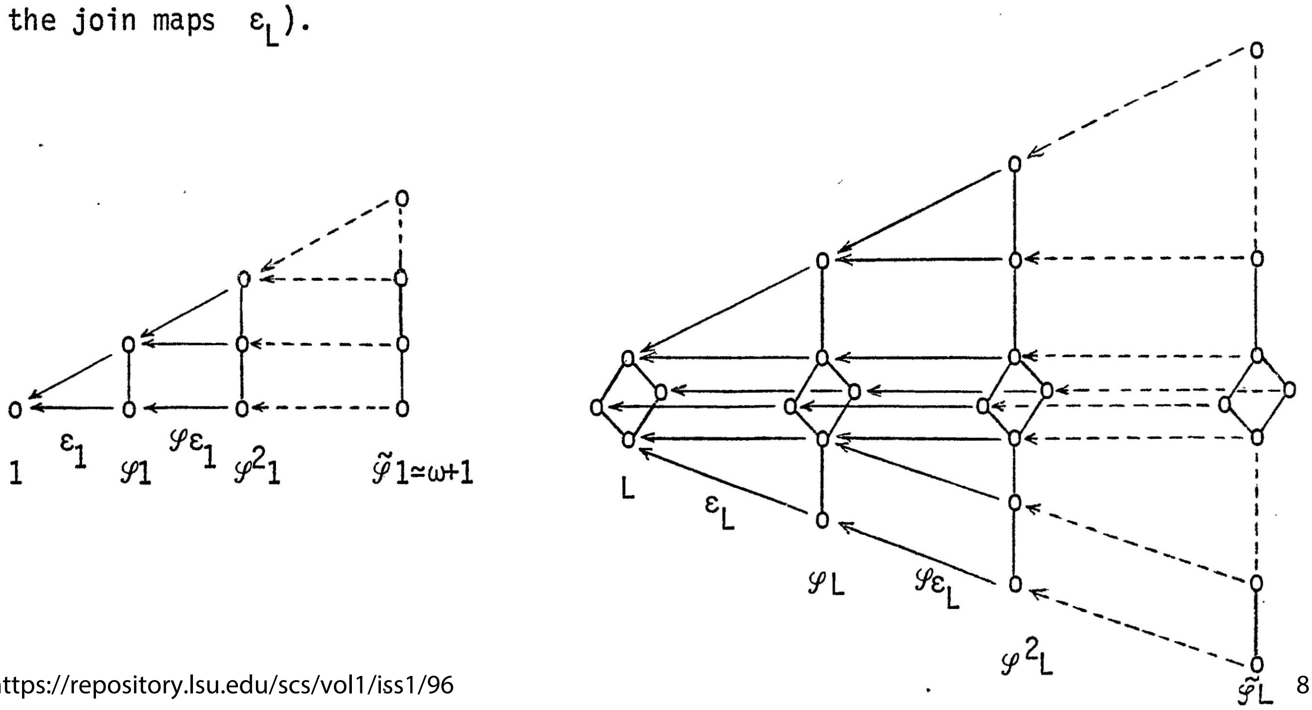
COROLLARY 2. *Every completely distributive lattice is the complete homomorphic image of a completely distributive lattice which is isomorphic to its own Scott completion.*

PROOF. Starting with a completely distributive lattice L, we obtain a projective system

$$L \xleftarrow[\varepsilon_L]{} \mathcal{S}L \xleftarrow[\mathcal{S}\varepsilon_L]{} \mathcal{S}^2L \longleftarrow \cdots$$

of completely distributive lattices and complete homomorphisms because a complete lattice L is continuous iff $\mathcal{S}L$ is completely distributive (see [C, II-1.14]) and every completely distributive lattice L is continuous (see [C, I-3.15]). Hence the projective limit $\tilde{\mathcal{S}}L$ is also completely distributive. The rest is clear by Corollary 1. □

EXAMPLES 1-2. Two fixed point constructions for the Scott completion (via the join maps ε_L).

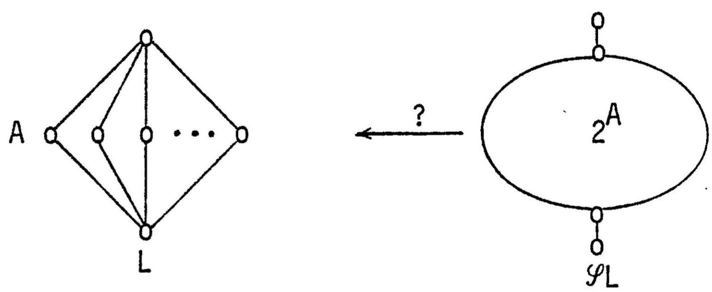


For certain purposes the restriction to $\mathcal{Y}$-distributive lattices will be too strong. For example, $\mathcal{U}$-distributivity entails the infinite distributive law

$$x \vee \bigwedge Y = \bigwedge\{x \vee y : y \in Y\}$$

(but not its dual). Accordingly, if we want to apply the fixed point construction $L \mapsto \tilde{L}$ to non-distributive complete lattices L, we must look for another $\mathrm{INF}_{\mathcal{Y}}$-morphism from $\mathcal{Y}L$ to L because the join map ε_L will not work in general. Unfortunately, it can happen that there is no $\mathrm{INF}_{\mathcal{Y}}$-morphism from $\mathcal{Y}L$ onto L at all.

EXAMPLE 3. If L denotes the normal completion of an infinite antichain A then there is no $\mathrm{INF}_{\mathcal{S}}$-morphism form $\mathcal{S}L$ onto L.



More generally, if L is a continuous lattice but not *hypercontinuous* (cf. [C, III-3.22],[GL]) then L cannot be the image of $\mathcal{S}L$ under an $\mathrm{INF}_{\mathcal{S}}$-morphism. In fact, if L is continuous then $\mathcal{S}L$ is completely distributive, and the following statements are equivalent for L (cf. [ME 0],[GL]):

(a) L is hypercontinuous.
(b) L is the image of a completely distributive lattice under an $\mathrm{INF}_{\mathcal{S}}$-morphism.
(c) L satisfies the infinite distributive law

$$\bigvee\{\bigwedge F : F \in \mathcal{F}\} = \bigwedge\{\bigvee \xi[\mathcal{F}] : \xi \in \prod_{F \in \mathcal{F}} F\}$$

for each set-theoretical filter $\mathcal{F}$ on L.

(d) $\mathcal{S}L = \mathcal{U}L$.
(e) $\mathcal{U}L$ is completely distributive.

If $\mathcal{Y}$ is an arbitrary invariant completion and L a complete lattice then the "dualized meet map"

$$\mu_L : \mathcal{Y}L \to L, \quad Y \mapsto \bigwedge(L \setminus Y)$$

preserves arbitrary meets, but in general it is not a $\mathrm{INF}_{\mathcal{Y}}$-morphism (see the above example). However, if $\mathcal{Y}\mathcal{Y}L$ coincides with $\mathcal{A}\mathcal{Y}L$ then μ_L is trivially $\mathcal{Y}$-continuous, being isotone. Thus we have shown:

COROLLARY 3. *Let* $\mathcal{Y}$ *be a compositive standard completion. Then for every complete lattice* L *with* $\mathcal{A}\mathcal{Y}L = \mathcal{Y}\mathcal{Y}L$ *there is an* $\mathrm{INF}_{\mathcal{Y}}$*-morphism from a complete lattice* $\tilde{L}$ *with* $\mathcal{Y}\tilde{L} \cong \tilde{L}$ *onto* L.

Of course, such an $\mathrm{INF}_{\mathcal{Y}}$-morphism can be a complete homomorphism only if L is $\mathcal{Y}$-distributive (because complete homomorphisms preserve $\mathcal{Y}$-distributivity and $\mathcal{Y}\tilde{L}$ is always $\mathcal{Y}$-distributive).
For the Scott completion, we obtain the following application:

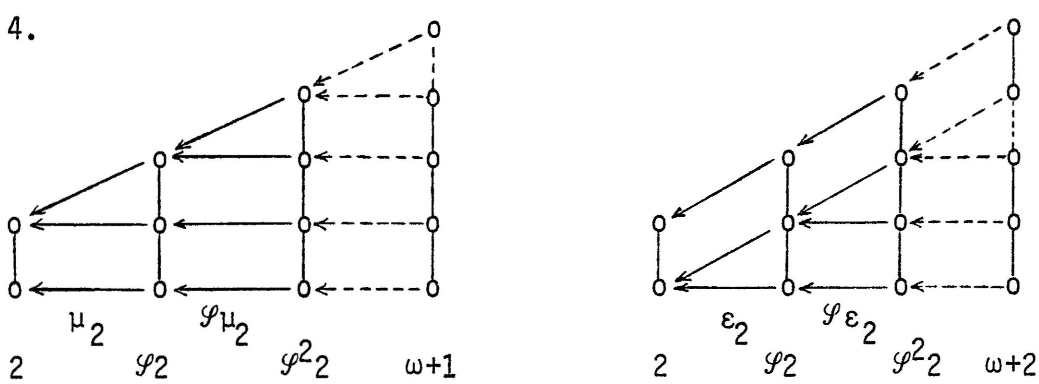
COROLLARY 4. *For every complete lattice* L *such that* $\mathcal{S}L$ *satisfies the* ACC *– in particular, for every finite lattice – there is an* $\mathrm{INF}_{\mathcal{S}}$*-morphism (but no isomorphism!) from a complete lattice* $\tilde{L}$ *with* $\mathcal{S}\tilde{L} \cong \tilde{L}$ *onto* L. *If* L *is continuous then* $\tilde{L}$ *is completely distributive.*

The following problem remains open:

Is a complete lattice L with $\mathcal{S}L \cong L$ necessarily completely distributive? (We know that it must be $\mathcal{S}$-distributive, and if L would be continuous then complete distributivity would follow).

The class of complete lattices whose Scott completion satisfies the ACC includes all lattices L whose dual is partially well-ordered, i.e. L satisfies the ACC and does not contain infinite antichains. Alternatively, such lattices can be characterized by the property that their Alexandroff completion $\mathcal{A}L$ satisfies the ACC.
As Example 3 demonstrates, Corollary 4 cannot be extended to all lattices satisfying the ACC in their own right (while the ACC for $\mathcal{S}L$ entails that for L).
Recall that the construction of the limit $\tilde{L}$ depends on the choice of the $\mathrm{INF}_{\mathcal{Y}}$-morphism $\varphi: \mathcal{Y}L \to L$. For example, starting with the two-element chain 2 and the corresponding meet map μ_2, iteration of $\mathcal{S}$ leads to the limit $\omega+1$, while starting with the join map ε_2 gives the limit $\omega+2$.

EXAMPLE 4.



It also should be mentioned that in the frequently cited Chapter IV-4 of [C], a fixed point construction has been given for the *Scott topology functor* σ which associates to a complete lattice L the lattice of *Scott-open sets* (i.e. $\sigma L = \{L \setminus Y: Y \in \mathcal{S}L\}$) and to an $\mathrm{INF}_{\mathcal{S}}$-morphism $\varphi: L \to M$ the map

$$\sigma\varphi: \sigma L \to \sigma M, \; Y \mapsto \hat{\varphi}^{-1}[Y],$$

where $\hat{\varphi}$ denotes the lower adjoint of φ. Starting with a one-point lattice, the fixed point construction for σ leads to the chain $\omega+1+\omega^*$ (where ω^* is the dual of ω). As was mentioned in [C, IV-4], there is no natural construction associating with each continuous lattice L a lattice $\tilde{L}$ with $\sigma\tilde{L} \simeq \tilde{L}$ and L as $\mathrm{INF}_{\mathcal{S}}$-homomorphic image (see Example 2!). Compare this observation with Corollaries 1-4!

It was proposed in the Compendium to investigate whether the fixed point constructions $L \mapsto \tilde{L}$ would preserve *weights*, where the *weight* $w(L)$ of a complete lattice L is the smallest cardinality of a basis, i.e. a $\vee$-o-subsemilattice B such that each element of L may be represented as a join of elements from B. By [C, IV-3.24 and 4.15], an infinite continuous lattice L has the same weight as σL and $\sigma\tilde{L}$. It is also true that $\mathcal{S}$ does not enlarge the weight:

REMARK. *Under the hypothesis of Theorem 2 we have* $w(L) = w(\mathcal{Y}L)$ *for each infinite complete lattice* L, *and if* $\varphi: \mathcal{Y}L \to L$ *is a surjective* $\mathrm{INF}_{\mathcal{Y}}$ *-morphism then the corresponding limit* $\tilde{L}$ *has the same weight as* L.

PROOF. Concerning the inequality $w(\mathcal{Y}L) \leq w(L)$, it suffices to show that for a basis B of L the system

$$\mathcal{B}_B = \{\downarrow F : F \subseteq B, \; F \text{ finite}\}$$

is a basis of $\mathcal{Y}L$. Consider two sets $A, A' \in \mathcal{Y}L \subseteq \mathcal{S}L$ with $A \nsubseteq A'$, and choose $x \in A \setminus A'$. Since B is a basis, we have $x = \bigvee(\downarrow x \cap B)$, and consequently, we find an $b \in B$ with $b \leq x$ and $b \notin A'$ (otherwise $x = \bigvee(\downarrow x \cap B) \in A'$ because $\downarrow x \cap B$ is directed and A' is Scott closed). But then $b \leq x \in A$ yields $b \in A$, whence $\downarrow b \subseteq A$, $\downarrow b \nsubseteq A'$, and clearly $\downarrow b \in \mathcal{B}_B$.

The converse inequality $w(L) \leq w(\mathcal{Y}L)$ follows from the observation that every basis $\mathcal{B}$ of $\mathcal{Y}L$ gives rise to a basis $\{\bigvee A : A \in \mathcal{B}\}$ of L. Now it follows by induction that $w(L) = w(\mathcal{Y}^n L)$ for all $n \in \omega$ and then $w(L) = w(\tilde{L})$ (see [C,IV-3.25]). □

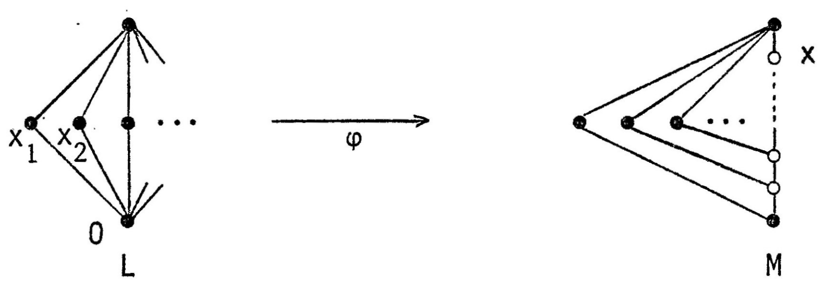
Finally, an additional comment is in order concerning a fixed point construction for the minimal topological completion $\mathcal{U}$ which cannot be obtained directly from Theorem 2 but follows from the results in [C, IV-3/4] by suitable dualization.

Let CL* denote that category whose objects are dually continuous lattices and whose morphisms preserve arbitrary joins and filtered meets. Obviously CL* is isomorphic to the usual category CL of continuous lattices à la Compendium. Furthermore, $\mathcal{U}L$ is the system of all Lawson-closed upper (!) sets of the dual lattice L^{op} (cf. [C, III-3.16]). Hence, dualizing the results in [C, IV-3.22, 4.10(i) and 4.13], we obtain:

COROLLARY 5. *The minimal topological completion* $\mathcal{U}$ *induces a self-functor of the category* CL* *which preserves projective limits, injectivity and surjectivity of morphisms. The join maps* $\varepsilon_L : \mathcal{U}L \to L$ *define a natural transformation. Moreover, there is a self-functor* $\tilde{\mathcal{U}}$ *of* CL* *such that each dually continuous lattice* L *is the image of* $\tilde{\mathcal{U}}L$ *under a natural* CL*-*morphism and*
$\tilde{\mathcal{U}}L \simeq \mathcal{U}\tilde{\mathcal{U}}L \simeq \tilde{\mathcal{U}}\tilde{\mathcal{U}}L$.

If $\varphi : L \to M$ is a CL*-morphism then the lifted CL*-morphism $\mathcal{U}\varphi : \mathcal{U}L \to \mathcal{U}M$ maps $Y \in \mathcal{U}L$ onto $\downarrow\varphi[Y]$. An analogous statement for $\mathcal{S}$ instead of $\mathcal{U}$ fails in general.

EXAMPLE 5 .



In this example L and M are algebraic lattices satisfying the descending chain condition. In particular, L and M are continuous and dually continuous. The map φ preserves arbitrary joins and filtered meets. But for the Scott-closed sets

$$Y_n = \{0\} \cup \{x_k : k \in \omega, k > n\}$$

the prolongated images $\downarrow\varphi[Y_n]$ are not Scott closed in M. Although $\mathcal{S}L$ and $\mathcal{S}M$ are completely distributive, the lifted map $\mathcal{S}\varphi : \mathcal{S}L \to \mathcal{S}M$, $Y \mapsto \overline{\varphi[Y]}$ fails to be a CL*-morphism. Indeed, we have

$$\mathcal{S}\varphi(\bigcap\{Y_n : n \in \omega\}) = \{\varphi(0)\} ,$$

while

$$\bigcap\{\mathcal{S}\varphi(Y_n) : n \in \omega\} = \downarrow x .$$