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SCS 94: Algebraic Theories for Proper Filter Monads

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TOPIC: Algebraic theories for proper filter monads

REFERENCES: [1] G. Gierz et al., A Compendium of Continuous Lattices, Springer-Verlag, 1980.

[2] K. Keimel, Continuous Lattices, General Convexity Spaces, and a Fixed Point Theorem. Notes, December 1983

[3] O. Wyler, Algebraic theories of continuous lattices.
[a] Preprint, December 1976.
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The present memo deals with the categories of algebras for the nine monads which appear in the following diagram

$$\begin{tikzcd}
\mathfrak{W}_p \arrow[rr, "i"] & & \mathfrak{W}_o & & \mathfrak{D}_o & \\
& \mathfrak{U} \arrow[ul, "r"'] \arrow[rr, "i"] \arrow[dl, "r"] & & \mathfrak{F}_o \arrow[ul, "r"'] \arrow[ur, "sup"] \arrow[dl, "r"] \arrow[dr, "adh"'] & & \mathfrak{E}_o \arrow[ll, "j"'] \\
\mathfrak{H}_p \arrow[rr, "i"] & & \mathfrak{H}_o & & \mathfrak{V}_o &
\end{tikzcd}$$

.

The monads and morphisms of monads appearing in the diagram will be described; most of them result from contravariant adjunctions. For five of the monads, the category of algebras is the category of continuous sup semilattices; the algebraic functors induced by the four morphisms with domain $\mathfrak{F}_o$ are isomorphisms of categories.

The order for continuous lattices will be that of [2] and of [3a], dual to the order of the Compendium [1] and of [3b]. In this way, we can order subsets by set inclusion, with set unions as suprema, and then deal with order preserving maps only.

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0. Categorical background

0.1. If $\mathfrak{T} = (T,\eta,\mu)$ and $\mathfrak{S} = (S,e,m)$ are monads on categories $\mathfrak{A}$ and $\mathfrak{B}$, then a morphism $(R,\pi) : \mathfrak{S} \longrightarrow \mathfrak{T}$ consists of a functor $R : \mathfrak{A} \longrightarrow \mathfrak{B}$ and a natural transformation $\pi : SR \longrightarrow RT$ such that $\pi \cdot eR = R\eta$, and $\pi \cdot mR = R\mu \cdot \pi T \cdot S\pi$. This induces an algebraic functor $(R,\pi)^* : \mathfrak{A}^{\mathfrak{T}} \longrightarrow \mathfrak{B}^{\mathfrak{S}}$ which lifts R, with $(R,\pi)^*(A,\alpha) = (RA, R\alpha \cdot \pi_A)$ for an $\mathfrak{A}$-algebra (A,α).

We have $U^{\mathfrak{S}}(R,\pi)^* = R\,U^{\mathfrak{T}}$ for the forgetful functors, and it is easily seen that π lifts to $\pi : F^{\mathfrak{S}}R \longrightarrow (R,\pi)^* F^{\mathfrak{T}}$ for the free algebra functors.

0.2. We shall deal repeatedly with a morphism of monads $(R,\pi) : \mathfrak{S} \longrightarrow \mathfrak{T}$ which satisfies the following conditions.

(i) R is faithful, and all morphisms π_A are epimorphic.

(ii) There is a functor $\Delta : \mathfrak{B}^{\mathfrak{S}} \longrightarrow \mathfrak{A}$ such that $R\Delta = U^{\mathfrak{S}}$ and $\Delta(R,\pi)^* = U^{\mathfrak{T}}$.

(iii) Every morphism $S\pi_A$ is epimorphic, and the structure of an $\mathfrak{S}$-algebra (B,β) always factors $\beta = u\,\pi_A$, with $A = \Delta(B,\beta)$ and $u : RTA \longrightarrow B$ in $\mathfrak{B}$.

THEOREM. If a morphism (R,π) of monads satisfies (i) and (ii), then the functor $(R,\pi)^*$ is full and faithful, and injective on objects. If (R,π) also satisfies (iii), then $(R,\pi)^*$ is an isomorphism of categories.

Proof. $(R,\pi)^*$ is faithful if R is, and clearly injective on objects if (i) and (ii) are valid.

If $g : (R,\pi)^*(A,\alpha) \longrightarrow (R,\pi)^*(C,\gamma)$ and $f = \Delta g : A \longrightarrow C$, then $g = Rf$, and $Rf \cdot R\alpha \cdot \pi_A = R\gamma \cdot \pi_C \cdot SRf = R\gamma \cdot RTf \cdot \pi_A$ by naturality of π. Now $f : (A,\alpha) \longrightarrow (C,\gamma)$, and $g = (R,\lambda)^* f$, if (i) as well as (ii) is valid.

For the last part, we must only show that $(R,\pi)^*$ is surjective on objects. Thus consider an $\mathfrak{S}$-algebra (B,β), and put $A = \Delta(B,\beta)$. If $\beta = u\,\pi_A$, we must show that $u = R\alpha$ for an $\mathfrak{A}$-algebra structure of A.

For this, consider the following diagram:

$$\begin{tikzcd}
SSRA \arrow[r, "S\pi_A"] \arrow[dd, "m_{RA}"] & SRTA \arrow[r, "Su"] \arrow[d, "\pi_{TA}"] & SRA \arrow[d, "\pi_A"] \\
 & RTTA \arrow[r, "RT\alpha"] \arrow[d, "R\mu_A"] & RTA \arrow[d, "u"] \\
SRA \arrow[r, "\pi_A"] & RTA \arrow[r, "u"] & B
\end{tikzcd} .$$

The outer square and the lefthand rectangle commute by hypothesis. Since $S\pi_A$ is epi, $u : (R,\pi) * F^{\mathfrak{J}} A \longrightarrow (B,\beta)$; it follows that $u = R\alpha$ for $\alpha = \Delta u : TA \longrightarrow A$. Now the upper righthand square commutes by naturality of π , and $\alpha\mu_A = \alpha \cdot T\alpha$ follows by (i). Finally, $id_{RA} = u\pi_A e_{RA} = uR\eta_A$, and $\alpha\eta_A = id_A$ follows ▯

<u>0.3</u>. A (non-commutative) diagram

$$\begin{tikzcd}
\mathfrak{B}^{op} \arrow[r, "S^{op}"] \arrow[d, shift left, "G"] & \mathfrak{B}_1^{op} \arrow[d, shift left, "G_1"] \\
\mathfrak{A} \arrow[u, shift left, "F^{op}"] \arrow[r, "R"] & \mathfrak{A}_1 \arrow[u, shift left, "F_1^{op}"]
\end{tikzcd}$$

of categories and functors, with contravariant adjunctions for the vertical arrows, results in a bijective correspondence between natural transformations $\kappa : RG \longrightarrow G_1 S^{op}$ and $\lambda : SF \longrightarrow F_1 R^{op}$, as follows. κ and λ correspond to each other, and are called <u>adjoint</u>, if $\lambda_A \cdot Sf$ and $\kappa_B \cdot Rg$ are adjoint for $F_1^{op} \dashv G_1$ whenever $f : B \longrightarrow FA$ and $g : A \longrightarrow GB$ are adjoint for $F \dashv G$.

Let $\mathfrak{T}$ on $\mathfrak{A}$ and $\mathfrak{S}$ on $\mathfrak{A}_1$ be the monads induced by the contravariant adjunctions, and let $K : \mathfrak{B}^{op} \longrightarrow \mathfrak{A}^{\mathfrak{T}}$ and $K_1 : \mathfrak{B}_1{}^{op} \longrightarrow \mathfrak{A}_1{}^{\mathfrak{S}}$ be the comparison functors, with $U^{\mathfrak{T}} K = G$ and $U^{\mathfrak{S}} K_1 = G_1$, and with $KB = (GB, G\epsilon_B)$ and $K_1 B = (G_1 B, G_1 \epsilon_B)$ for objects. There are two situations in which adjoint natural transformations produce a morphism of monads.

(i) If all $G_1 \lambda_A$ factor $G_1 \lambda_A = \kappa_{FA} \cdot \pi_A$, with every κ_{FA} monomorphic, then the π_A define a morphism $(R,\pi) : \mathfrak{S} \longrightarrow \mathfrak{T}$, and κ lifts to a natural transformation $\kappa : (R,\pi)*K \longrightarrow K_1 S^{op}$.

(ii) If $\mathfrak{A}_1 = \mathfrak{A}$ and $R = Id_{\mathfrak{A}}$, and if all κ_{FA} factor $\kappa_{FA} = G_1 \lambda_A \cdot \pi_A$, with every $G_1 \lambda_A$ monomorphic, then the π_A define a morphism $(Id,\pi) : \mathfrak{T} \longrightarrow \mathfrak{S}$. In this situation, we have $\kappa : K \longrightarrow (Id,\pi)*K_1 S^{op}$ at the level of $\mathfrak{T}$-algebras.

We omit the diagram-chasing proofs.

1. The proper filter monad on sets

1.1. We denote by MSL_o the category of meet semilattices with 0 (and 1). Morphisms of MSL_o preserve finite meets, and 0. The contravariant powerset functor on sets obviously lifts to $P_o : ENS^{op} \longrightarrow MSL_o$, with $P_o A$ the powerset of A for a set A, ordered by set inclusion and regarded as meet semilattice with 0.

If $f : A \longrightarrow PL$ and $g : L \longrightarrow PA$ are exponentially adjoint, i.e. always $a \in f(x) \iff x \in g(a)$ if $x \in A$ and $a \in L$, for a set A and an object L of MSL_o, then g is a morphism $g : L \longrightarrow P_o A$ of MSL_o iff every $f(x)$ is a proper filter in L. Thus we have a functor $G_o : MSL_o{}^{op} \longrightarrow ENS$, adjoint on the right to P_o, with $G_o L$ the set of all proper filters in L for a meet semilattice L with 0, and $(G_o f)(\psi) = f^{\leftarrow}(\psi)$ for $f : L \longrightarrow M$ in MSL_o and a proper filter ψ in M.

We denote by $\mathfrak{F}_o$ the monad on sets obtained from this adjunction; this is the proper filter monad. The proper filter functor $F_o = G_o P_o{}^{op}$ assigns to every set A the set of all proper filters on A.

1.2. We denote by LAT the category of lattices, with 0 and 1, and by $P_p : ENS^{op} \longrightarrow LAT$ the functor which assigns to every set its powerset, ordered by inclusion and considered as a lattice. This functor has as adjoint on the right, with exponential adjunction, the functor $G_p : LAT^{op} \longrightarrow ENS$ which assigns to every lattice L the set of all prime filters in L.

The resulting monad on sets is the ultrafilter monad, denoted by $\mathfrak{U}$ in this paper, with functor part $U = G_p P_p^{op}$, the ultrafilter functor on sets. As is well known, $\mathfrak{U}$-algebras are compact Hausdorff spaces; the $\mathfrak{U}$-algebra structure of a compact Hausdorff space X assigns to every ultrafilter on X its limit for X.

If $S : LAT \longrightarrow MSL_o$ is the inclusion functor, then clearly $S P_p = P_o$. Adjoint to the resulting identity natural transformation is $\kappa : G_p \longrightarrow G_o S^{op}$ given by inclusions. By 0.3.(ii), this produces a morphism $i = (Id, \kappa P_p^{op}) : \mathfrak{U} \longrightarrow \mathfrak{F}_o$. Thus every $\mathfrak{F}_o$-algebra (L,α) has an underlying compact space $i*(L,\alpha)$, with the restriction of α to ultrafilters as convergence of ultrafilters. Morphisms of $\mathfrak{F}_o$-algebras are continuous maps for the underlying compact topologies.

1.3. We define a sup semilattice as an ordered set L such that every non-empty subset of L has a supremum in L. Morphisms of sup semilattices preserve suprema of non-empty subsets.

We denote by $\underline{E}_o$ the free sup semilattice functor on sets. It is well known that $\underline{E}_o A$, for a set A, is the set of non-empty subsets of A, with set unions as suprema. $\underline{E}_o$ is left adjoint to the forgetful functor $|\ | : SSL \longrightarrow ENS$, with SSL the category of sup semilattices. The unit s of this adjunction is given by $s_A(x) = \{x\}$, for $x \in A$.

Every sup semilattice L has a one-point extension to a complete lattice $\tilde{L}$, obtained by adding a zero o_L to L, and a map $f : L \longrightarrow L'$ of SSL extends to $\tilde{f} : \tilde{L} \longrightarrow \tilde{L}'$ with $f(o_L) = o_{L'}$. We obtain a functor $D_o : SSL^{op} \longrightarrow MSL_o$ by letting $D_o L = \tilde{L}$, considered as object of MSL_o, with the same order, and putting $x \leq (D_o f)(x') \Longleftrightarrow \tilde{f}(x) \leq x'$, for $(x,x') \in L \times L'$. Then $D_o \underline{E}_o^{op}$ is naturally isomorphic to P_o. Adjoint to this isomorphism is $\kappa : |\ | \longrightarrow G_o D_o^{op}$ with $\kappa_L(x) = \uparrow x$ for $x \in L$.

We denote by $\mathcal{E}_o = (E_o, s, u)$ the <u>powerset monad</u> on sets which results from the adjunction $\underline{E}_o \dashv |\ |$. Algebras for $\mathcal{E}_o$ are sup semilattices; the $\mathcal{E}_o$-algebra structure of a sup semilattice is given by suprema.

By 0.3.(ii), the natural transformation $\kappa : |\ | \longrightarrow G_o D_o^{op}$ induces a morphism $j = (\mathrm{Id}, \kappa \underline{E}_o) : \mathcal{E}_o \longrightarrow \mathfrak{F}_o$ of monads, and hence an algebraic functor j^*, from $\mathfrak{F}_o$-algebras to SSL, which preserves underlying sets and mappings. Thus an $\mathfrak{F}_o$-algebra (L,α) has an underlying sup semilattice $j^*(L,\alpha)$, with $\sup S = \alpha(\uparrow S)$ for a non-empty subset S of L, and homomorphisms of $\mathfrak{F}_o$-algebras preserve these suprema.

<u>1.4</u>. The functor $G_o : MSL_o^{op} \longrightarrow ENS$ lifts to a comparison functor K_o to $\mathfrak{F}_o$-algebras, with $K_o L = (G_o L, G_o \epsilon_L)$ for an object L, and $K_o f = G_o f : K_o M \longrightarrow K_o L$ for $f : L \longrightarrow M$.

One sees easily (see e.g. [3]) that suprema in $j^* K_o L$ are set intersections; thus the order of filters in $K_o L$ is the natural order of filters, dual to set inclusion. The limit in $i^* K_o L$ of an ultrafilter Ψ on $G_o L$ consists of all $a \in L$ with $\epsilon_L(a) \in \Psi$; it follows that $i^* K_o L$ is a Stone space, with the coarsest topology for which the sets $\epsilon_L(a)$ are clopen.

There is also a comparison functor $K_p : LAT^{op} \longrightarrow CH$ which assigns to a lattice L its Stone space of prime filters in L. By 0.3.(ii), the inclusion $G_p L \longrightarrow G_o L$ lifts to a closed embedding $K_p L \longrightarrow i^* K_o L$, for every lattice L.

<u>1.5</u>. PROPOSITION. <u>If</u> (L,α) <u>is an</u> $\mathfrak{F}_o$<u>-algebra, with underlying compact space</u> $X = i^*(L,\alpha)$, <u>then</u> $\alpha(\Phi) = \sup \mathrm{adh}_X \Phi$ <u>for a filter</u> Φ <u>on</u> L, <u>with supremum in</u> $j^*(L,\alpha)$. <u>Morphisms of</u> $\mathfrak{F}_o$<u>-algebras are all morphisms of the underlying sup semilattices which are continuous for the underlying compact topologies</u>.

<u>Proof</u>. All morphisms of $\mathfrak{F}_o$-algebras, including α, are continuous and preserve suprema as stated. In $K_o P_o L$, a filter Φ on L is the supremum of all finer ultrafilters Ψ; thus $\alpha(\Phi)$ is the supremum of the limits $\alpha(\Psi)$ of these ultrafilters. These limits form the adherence $\mathrm{adh}_X \Phi$; this proves our formula. Now the last part of 1.5 follows immediately from the fact that maps of compact Hausdorff spaces preserve filter adherences.▯

1.6. PROPOSITION. $\mathfrak{F}_o$-algebras can be embedded into compact join semilattices (without 0) as a full subcategory. If (L,α) is an $\mathfrak{F}_o$-algebra, then $\alpha(\Phi) = \inf\{\sup S : S \in \Phi\}$ for a proper filter Φ on L.

Proof. We have $\vee \cdot (\alpha \times \alpha) = \alpha \cdot \vee$ for joins in $j^*(L,\alpha)$ an and in $j^* K_o P_o L$. Since joins in $K_o P_o L$ are set intersections, we have $\overset{\leftarrow}{\vee}(S^{\#}) = S^{\#} \times S^{\#}$ for $S \subset L$ and $S^{\#} = \epsilon_{PL}(S)$, with $\Phi \in S^{\#} \iff S \in \Phi$. This shows that $\vee$ is continuous for the compact topology of $K_o P_o L$. As α and $\alpha \times \alpha$ are continuous and surjective for the compact topology, hence topological quotient maps, it follows that $\vee$ for L is continuous for $i^*(L,\alpha)$. Thus $\mathfrak{F}_o$- algebras are compact join semilattices.

Compact join semilattices are compact ordered spaces; thus filter bases have infima, and dual filter bases suprema, which are topological limits. It follows that maps of compact join semilattices preserve infima of filter bases, and suprema of dual filter bases. Since these maps preserve finite non-empty joins, they preserve all joins. With 1.5, it follows that $\mathfrak{F}_o$-algebras and their induced compact join semilattices have the same maps.

Now if (L,α) is an $\mathfrak{F}_o$-algebra and Φ a proper filter on L, then Φ is the infimum of all filters $\uparrow S$ with $S \in \Phi$. These filters $\uparrow S$ form a filter basis in $K_o P_o L$; thus $\alpha(\Phi)$ is the infimum of all $\alpha(\uparrow S) = \sup S$ with $S \in \Phi$ □

2. Continuous sup semilattices

2.1. A sup semilattice L (1.3) is called complete if every filter base in L has an infimum in L.

For elements a, b of a complete sup semilattice L, we say that b is way above a, and we write $b \gg a$, if b is in every filter φ in L with $\inf \varphi \leq a$. The elements way above a in L form a filter which we denote by $\Uparrow a$, with $\inf \Uparrow a \geq a$. We say that L is a continuous sup semilattice if $\inf \Uparrow a = a$ for all $a \in L$.

Morphisms of complete and continuous sup semilattices are mappings which preserve non-empty suprema and infima of filter bases, or equivalently of filters.

2.2. We use again the notations of 1.3. For a sup semilattice L, putting $q_L(x) = {\downarrow}x$ for $x \in L$ provides a map $q_L : L \longrightarrow P|L|$. With $q_L(o_L) = \emptyset$, this is clearly a morphism $q_L : D_o L \longrightarrow P_o|L|$ in MSL_o, and it is easily verified that q_L as just shown is natural in L.

Proper filters in a meet semilattice M with 0, ordered dually to set inclusion, form a sup semilattice, with intersections as suprema. Maps $G_o f$, for morphisms f of MSL_o, clearly preserve these suprema; thus we can lift G_o to a functor $\underline{G}_o : MSL_o \longrightarrow SSL$, with $|\ |\underline{G}_o = G_o$.

PROPOSITION. <u>The functors</u> D_o <u>and</u> $\underline{G}_o$ <u>are adjoint on the right, with</u> $q : D_o \longrightarrow P_o|\ |^{op}$ <u>and</u> $id(G_o)$ <u>adjoint</u>.

<u>Proof</u>. For objects L of SSL and M of MSL_o, it is well known that $a \in f(x) \Longleftrightarrow x \in g_1(a)$, for $x \in L$ and $a \in M$, provides a bijection between morphisms $g_1 : M \longrightarrow P_o|L|$ of MSL_o and mappings $f : |L| \longrightarrow G_o M$. In this situation, $a \in f(\sup x_i)$ $\Longleftrightarrow \sup x_i \in g_1(a)$, and $a \in \sup f(x_i) \Longleftrightarrow x_i \in g_1(a)$ for all i. Thus f preserves suprema iff each $g_1(a)$ is a principal dual filter ${\downarrow}g(a)$, i.e. iff g_1 factors $g_1 = q_L \cdot g$. Since q_L is a natural embedding, this gives the desired bijection between $f : L \longrightarrow \underline{G}_o M$ in SSL and $g : M \longrightarrow D_o L$ in MSL_o, and this bijection is natural in L as well as M. The proof also shows that q and $id(G_o)$ are adjoint ▯

2.3. We denote by $\mathfrak{Q}_o$ the monad on SSL obtained from the adjunction $D_o^{op} \dashv \underline{G}_o$ of 2.2, with functor part $Q_o = \underline{G}_o D_o^{op}$. Since $f : L \longrightarrow \underline{G}_o M$ and $g : M \longrightarrow D_o L$ are adjoint iff always $a \in f(x) \Longleftrightarrow x \leq g(a)$, both units of this adjunction are principal filter maps. We note that $Q_o L$ is the sup semilattice of all filters in L; these are the proper filters in $D_o L$.

THEOREM. $SSL^{\mathfrak{Q}}$ <u>is the category of continuous sup semilattices, with algebra structures</u> $\inf : Q_o L \longrightarrow L$.

<u>Proof</u>. If (L,α) is a $\mathfrak{Q}_o$-algebra, then $\alpha(\varphi) \leq \alpha({\uparrow}a) = a$ for a filter φ in L and $a \in \varphi$. On the other hand, if $x \leq a$ for all $a \in \varphi$, then ${\uparrow}x \leq \varphi$, and $x \leq \alpha(\varphi)$ follows. Thus L is complete, and $\alpha(\varphi) = \inf \varphi$ for all φ in $Q_o L$.

Now a $\mathfrak{D}_o$-algebra is a complete sup semilattice L such that $\inf : Q_o L \longrightarrow L$ preserves suprema and satisfies the formal laws. The formal laws require that always $\inf \uparrow x = x$, which is valid, and $\inf \inf \Phi = \inf (Q_o \inf)(\Phi)$ for a filter Φ in $Q_o \Phi$. This is also valid since $\inf \Phi$ is the set union of all $\varphi \in \Phi$, and the filter $(Q_o \inf)(\Phi)$ has the elements $\inf \varphi$, $\varphi \in \Phi$, as a basis.

The map $\inf : Q_o L \longrightarrow L$ preserves suprema iff there is a mapping $\iota : L \longrightarrow Q_o L$ such that always $\inf \varphi \leq x \Longleftrightarrow \varphi \leq \iota(x)$. $\iota(x)$ must be the supremum of all φ with $\inf \varphi \leq x$, i.e. $\iota(x) = \Uparrow x$, and this must satisfy $\inf \iota(x) \leq x$. This is the case iff L is a continuous sup semilattice.

Morphisms of $\mathfrak{D}_o$-algebras must preserve suprema of non-empty subsets, and algebra structures $\inf_L$; thus they are the maps of continuous sup semilattices ▯

2.4. By 2.2 and 0.3.(i), we have a morphism of monads $\sup = (|\ |, G_o q^{op}) : \mathfrak{F}_o \longrightarrow \mathfrak{D}_o$. For a proper filter Φ on a sup semilattice L, the filter $(G_o q_L)(\Phi)$ consists of all $x \in L$ with $\downarrow x \in \Phi$, and hence of all $\sup S$ with $S \in \Phi$.

THEOREM. <u>The algebraic functor</u> $\sup^*$ <u>from continuous sup semilattices to</u> $\mathfrak{F}_o$<u>-algebras is an isomorphism of categories, preserving underlying sets and mappings, and with</u> $j^* \sup^*$ <u>the forgetful functor from continuous sup semilattices to</u> SSL.

<u>Proof</u>. $\sup^*$ clearly preserves underlying sets and mappings. For a continuous sup semilattice L and $\alpha = \inf_L \cdot G_o q_L$, we have $(G_o q_L)(\uparrow S) = \uparrow \sup S$ for non-empty $S \subset L$, hence $\alpha(\uparrow S) = \sup S$. Thus $j^* \sup^* L$ is the underlying sup semilattice of L.

Now $\sup^*$ has the factorization property of 0.2.(ii), with $\Delta = j^*$, and 0.2.(i) is also satisfied since every filter φ in a sup semilattice L satisfies $\varphi = (G_o q_L)(\Phi)$ for the filter Φ on L with the sets $\downarrow x$, $x \in \varphi$, as filter base. 0.2.(iii) is satisfied by 1.6; thus $\sup^*$ is an isomorphism by 0.2 ▯

3. The proper Vietoris monad

3.1. For a compact Hausdorff space X, we denote by V_oX the set of non-empty closed subsets of X, ordered by set inclusion and provided with the coarsest topology such that the sets $\downarrow S = \{A \in V_oX : A \subset S\}$ are closed for $S \subset X$ closed, and open for $S \subset X$ open. This is the Vietoris space of X.

V_oX is a Hausdorff space, for if A, B are closed in X with $A \not\subset B$, then there are disjoint open sets R and S with $B \subset S$ and R and A not disjoint. Then $V_oX \setminus \downarrow(X \setminus R)$ and $\downarrow S$ are disjoint neighborhoods of A and B in V_oX.

For $f : X \longrightarrow Y$ in CH, let V_of be the restriction of $f^{\rightarrow}$ to V_oX and V_oY. Since $(V_of)^{\leftarrow}(\downarrow T) = \downarrow f^{\leftarrow}(T)$ for $T \subset Y$, the map $V_of : V_oX \longrightarrow V_oY$ is continuous, and V_o is a functor, to CH since we shall see in 3.2 that V_oX is compact.

3.2. We denote by CH the category of compact Hausdorff spaces, and by $|\ | : CH \longrightarrow ENS$ the underlying set functor. For a compact Hausdorff space X, adherences of proper filters on X provide a mapping $adh_X : F_o|X| \longrightarrow |V_oX|$. One sees easily that adh_X is natural in X, and we have shown in [3], 5.2.2, that adh_X preserves suprema of sets of filters.

PROPOSITION. $adh_X : i^*K_oP_o|X| \longrightarrow V_oX$ is continuous.

COROLLARY. V_oX is a compact Hausdorff space.

Proof. For $S \subset X$ closed and a filter Φ on X, we have $adh_X \Phi \subset S$ iff all closed neighborhoods of S in X are in Φ. Thus $adh_X^{\leftarrow}(\downarrow S) = \bigcap R^{\#}$, with $R^{\#} = \epsilon_L(R)$ for $L = P_o|X|$ (see 1.4), for all closed neighborhoods R of S in V_oX. This set is closed in $i^*K_oP_o|X|$.

For $S \subset X$ open, we have $adh_X \Phi \subset S$ iff S contains a closed neighborhood R of $adh_X \Phi$, with $R \in \Phi$. Thus $adh_X^{\leftarrow}(\downarrow S) = \bigcup R^{\#}$ for closures R of open sets contained in S. This set is open in $i^*K_oP_o|X|$.

Since $adh_X \uparrow S = cl_X S$, the closure of S in X, the map adh_X is surjective. Thus V_oX is compact $\square$

3.3. For a compact Hausdorff space X, we denote by $s_X : X \longrightarrow V_o X$ the singleton map, with $s_X(x) = \{x\}$ for $x \in X$. Since $\overleftarrow{s_X}(\downarrow S) = S$ for $S \subset X$, the map s_X is continuous.

It is well known (see e.g. [3b], 5.4) that the set union $\bigcup K$ is closed in X for K closed in $V_o X$. Thus set unions define $u_X : V_o V_o X \longrightarrow V_o X$. Clearly $\overleftarrow{u_X}(\downarrow S) = \downarrow\downarrow S$ for $S \subset X$; thus u_X is continuous.

It is easily seen that s_X and u_X are natural in X.

PROPOSITION. <u>The functor</u> V_o, <u>and the natural transformations obtained above</u>, <u>define a monad</u> $\mathfrak{V}_o = (V_o, s, u)$ <u>on</u> CH, <u>and a morphism</u> $(|\ |, \mathrm{adh}) : \mathfrak{F}_o \longrightarrow \mathfrak{V}_o$ <u>of monads</u>.

We call $\mathfrak{V}_o$ the <u>proper Vietoris monad</u>, and we use adh as abbreviation for $(|\ |, \mathrm{adh})$ when this is convenient.

<u>Proof</u>. The monadic identites for s and u are easily verified; we omit details.

We must show that $\mathrm{adh} \cdot \eta |\ | = |\ | s$, and

$$\mathrm{adh} \cdot \mu |\ | = |\ | u \cdot \mathrm{adh}\, V_o \cdot F_o\, \mathrm{adh} .$$

The first of these is obvious; the point filter $\eta_X(x)$ has $\{x\}$ as its adherence. For the second one, put $Z = i^* K_o P_o X$. Since $\mu_{|X|}$ is the $\mathfrak{F}_o$-algebra structure of $K_o P_o |X|$, we have a diagram

$$\begin{tikzcd}
F_o|Z| \arrow[r, "\mathrm{adh}_Z"] \arrow[d, "F_o \mathrm{adh}_X"] & |V_o Z| \arrow[r, "\sup"] \arrow[d, "V_o \mathrm{adh}_X"] & |Z| \arrow[d, "\mathrm{adh}_X"] \\
F_o|V_o X| \arrow[r, "\mathrm{adh}_{V_o X}"] & |V_o V_o X| \arrow[r, "u_X"] & |V_o X|
\end{tikzcd},$$

with the factorization of $\mu_{|X|}$ by 1.5 on top. The lefthand square commutes by naturality of adh_X, and the righthand square since adh_X preserves suprema as remarked above. Thus the diagram commutes $\square$

3.4. THEOREM. <u>The algebraic functor</u> $(|\ |, \mathrm{adh})^*$, <u>from</u> $\mathfrak{V}_o$<u>-algebras to</u> $\mathfrak{F}_o$<u>-algebras</u>, <u>is an isomorphism of categories</u>, <u>preserving underlying sets and mappings</u>, <u>with</u> $i^* (|\ |, \mathrm{adh})^*$ <u>the forgetful functor from</u> $\mathfrak{V}_o$<u>-algebras to</u> CH.

Proof. It is clear that $(|\ |, \mathrm{adh})^*$ preserves underlying sets and mappings. If (X,ξ) is a $\mathfrak{V}_o$-algebra and Ψ an ultrafilter on X, with limit x, then $\mathrm{adh}_X \Psi = \{x\}$, and $\xi(\{x\}) = x$. Thus x is the limit of Ψ in $i^*(|\ |, \mathrm{adh})^*(X,\xi)$; it follows that $i^*(|\ |)^*(X,\xi) = X$. Since $i^*(|\ |,\mathrm{adh})^*$ preserves underlying mappings, it is the forgetful functor from $\mathfrak{V}_o$-algebras to CH. As $|\ |i^*$ also preserves underlying sets and mappings, 0.2.(ii) is satisfied with $\Delta = i^*$. Since $\mathrm{adh}_X {\uparrow} S = S$ for S closed in X, 0.2.(i) is satisfied. Finally, 0.2.(iii) is satisfied by 1.5, so that 0.2 applies ▯

4. The closed proper filter monad

4.1. We denote by TOP the category of topological spaces and continuous maps, and by $R : \mathrm{TOP} \longrightarrow \mathrm{ENS}$ the underlying set functor. For purposes of this paper, objects of TOP could be restricted to be T_o spaces or sober spaces, or the super-sober spaces of the Compendium [1].

For a topological space X, an object L of MSL_o, and adjoint maps $f : RX \longrightarrow G_o L$ and $g : L \longrightarrow P_o R_o X$, we have $g(a) = f^{\leftarrow}(a^{\#})$ for $a \in L$ and $a^{\#} = \epsilon_L(a)$. If $\Gamma_o X$ is the meet semilattice of closed sets of X, ordered by set inclusion, and $\Sigma_o L$ the set $G_o L$ with the coarsest topology such that $a^{\#}$ is closed for all $a \in L$, it follows that $f : X \longrightarrow \Sigma_o L$ is continuous iff g maps L into $\Gamma_o X$. Thus we have contravariant functors $\Gamma_o : \mathrm{TOP}^{op} \longrightarrow \mathrm{MSL}_o$ and $\Sigma_o : \mathrm{MSL}_o{}^{op}$, adjoint on the right.

Clearly $R\Sigma_o = G_o$, and the natural transformation $\lambda : \Gamma_o \longrightarrow P_o R^{op}$ adjoint to $\mathrm{id}(G_o)$ is given by inclusions. We denote by $\mathfrak{W}_o$ the closed proper filter monad on TOP, with functor part $W_o = \Sigma_o \Gamma_o{}^{op}$, obtained from the adjunction $\Gamma_o{}^{op} \dashv \Sigma_o$.

4.2. By 0.2.(i), we have a morphism $r = (R, G_o\lambda^{op}) : \mathfrak{F}_o \longrightarrow \mathfrak{W}_o$ of monads, with $(G_o\lambda_X)(\Phi)$ the restriction of Φ to closed sets for a proper filter Φ on a topological space X.

If L is a lattice, then prime filters in L form a subspace $\Sigma_p L$ of $\Sigma_o L$; this defines a functor $\Sigma_p : LAT^{op} \longrightarrow TOP$, adjoint on the right to the functor $\Gamma_p : TOP^{op} \longrightarrow LAT$ with $\Gamma_p X$ the lattice of closed sets of a space X. If $S : LAT \longrightarrow MSL_o$ is the inclusion functor, then $\Gamma_o = S\Gamma_p$, and subspace inclusions define a natural transformation $\kappa : \Sigma_p \longrightarrow \Sigma_o S^{op}$, with κ and $id(\Gamma_o)$ clearly adjoint.

We denote by $\mathfrak{W}_p$ the monad on TOP resulting from the adjunction $\Gamma_p{}^{op} \dashv \Sigma_p$, with functor part $W_p = \Sigma_p\Gamma_p{}^{op}$. This is the <u>prime closed filter monad</u> on TOP, and W_pX is the <u>prime Wallman compactification</u> of X for a topological space X.

$\mathfrak{W}_p$-algebras were studied in [4]; they are compact ordered spaces. If $(Z,\leq)$ is a compact ordered space, then $Z = r^*(X,\alpha)$ for a unique $\mathfrak{W}_p$-algebra (X,α), where r^* is the algebraic functor induced by the restriction morphism $r : \mathfrak{U} \longrightarrow \mathfrak{W}_p$. In this situation, X is Z with the upper topology, i.e. open sets of X are increasing open sets of Z, the topology of Z is the patch topology of X, and Z has the induced order of X, i.e. $x \leq y$ in the order of Z iff $cl_X\{x\} \subset cl_X\{y\}$.

4.3. By 4.2, a $\mathfrak{W}_o$-algebra (X,α) has an induced compact ordered space $i^*(X,\alpha)$ and an induced continuous sup semilattice, or $\mathfrak{F}_o$-algebra, $r^*(X,\alpha)$. Since the diagram on p.1 of this paper commutes, both have the same compact topology, the patch topology of X. They also have the same order:

PROPOSITION. <u>If</u> (X,α) <u>is a</u> $\mathfrak{W}_o$<u>-algebra, then the order of the induced continuous sup semilattice</u> $r^*(X,\alpha)$ <u>is the induced order of</u> X.

<u>Proof</u>. We note first that the induced order of a space $\Sigma_o L$ is the natural order of filters in L, since $\psi \leq \varphi$ for that order iff $\psi \in \cup a_i^{\#}$ for every basic closed set with $\varphi \in \cup a_i^{\#}$.

Now let (X,α) be a $\mathfrak{W}_o$-algebra, with induced $\mathfrak{F}_o$-algebra structure $\alpha \cdot cl_X$, where $cl_X \Phi$ is the restriction of a filter Φ to closed sets. If $x \leq y$ in the induced order of X, then

$x \vee y = \alpha(\uparrow cl_X\{x,y\}) = \alpha(\uparrow cl_X\{y\}) = y$ in $r^*(X,\alpha)$. Conversely, $\uparrow cl_X\{x\} \leq \uparrow cl_X\{x,y\}$ in the induced order of $W_o X$. The continuous map α preserves the induced order; thus $x \leq x \vee y$ in the induced order of X, and $x \leq y$ in this order if $x \vee y = y$ ▯

4.4. For an $\mathfrak{F}_o$-algebra (A,α), let $U(A,\alpha)$ be A provided with the upper topology for the induced compact ordered space of (A,α). This clearly defines a functor U which preserves underlying sets and mappings, from $\mathfrak{F}_o$-algebras to TOP, with RU the forgetful functor from $\mathfrak{F}_o$-algebras to sets. Note that our upper topology is the lower topology of the Compendium.

THEOREM. The algebraic functor r^* from $\mathfrak{W}_o$-algebras to $\mathfrak{F}_o$-algebras is an isomorphism of categories, preserving underlying sets and mappings, and with $U r^*$ the forgetful functor from $\mathfrak{W}_o$-algebras to TOP.

Proof. r^* clearly preserves underlying sets and mappings. If (X,α) is a $\mathfrak{W}_o$-algebra, then X has the upper topology of the induced compact ordered space which by 4.3 is also the induced compact ordered space of $r^*(X,\alpha)$. Thus $U r^*(X,\alpha) = X$. Since $U r^*$ preserves underlying mappings, it follows that $U r^*$ is the forgetful functor for $\mathfrak{W}_o$-algebras. Thus 0.2.(ii) is satisfied with $\Delta = U$.

0.2.(i) is valid; every filter of closed sets of a space X is the restriction of a filter on RX to closed set.

It remains to verify the factorization of 0.2.(iii), i.e. if (L,α) is an $\mathfrak{F}_o$ algebra with $X = i^*(L,\alpha)$, then $\alpha(\Phi) = \sup adh_X \Phi$ depends only on the decreasing closed sets in Φ, i.e. the sets closed for $U(L,\alpha)$. Restricting Φ to these sets can only increase $\alpha(\Phi)$. On the other hand, if $x \not\leq \alpha(\Phi)$, then $\uparrow x$ and $adh_X \Phi$ are disjoint; thus x has an increasing neighborhood V in X with $X \setminus V$ a neighborhood of $adh_X \Phi$ and thus in Φ. This shows that restricting Φ to increasing closed sets does not change $\sup adh_X \Phi$ ▯

5. The open proper filter monad

Due to the duality between open and closed sets, the developments of this section are closely parallel to those of Section 4.

5.1. We denote by $\mathfrak{G}_o X$ and $\mathfrak{G}_p X$ the set of open sets of a topological space X, ordered by set inclusion and regarded as an object of MSL_o and LAT respectively. In the other direction, we denote by $\Pi_o L$ the set $G_o L$ of proper filters in a meet semilattice L with 0, provided with the topology for which the sets $a^{\#} = \epsilon_L(a)$, for $a \in L$, form a basis of open sets. If L is a lattice, then $\Pi_p L$ is the subspace of $\Pi_o L$ consisting of all prime filters in L.

As in Section 4, this defines functors $\mathfrak{G}_p : TOP^{op} \longrightarrow LAT$ and $\mathfrak{G}_o : TOP^{op} \longrightarrow MSL_o$, with $S\,\mathfrak{G}_p = \mathfrak{G}_o$ for the forgetful functor $S : LAT \longrightarrow MSL_o$, and with adjoints on the right $\Pi_p : LAT^{op} \longrightarrow TOP$ and $\Pi_o : MSL_o{}^{op} \longrightarrow TOP$.

We denote by $\mathbb{H}_p$ the <u>prime open filter monad</u> on TOP and by $\mathbb{H}_o$ the <u>open proper filter monad</u> on TOP which result from the adjunctions discussed above, with functor parts $H_p = \Pi_p \mathfrak{G}_p{}^{op}$ and $H_o = \Pi_o \mathfrak{G}_o{}^{op}$.

5.2. We have $R\,\Pi_p = G_p$ and $R\,\Pi_o = G_o$ for the underlying set functor $R : TOP \longrightarrow ENS$, and natural inclusions $\lambda : \mathfrak{G}_p \longrightarrow P_p R^{op}$ and $\lambda : \mathfrak{G}_o \longrightarrow P_o R^{op}$ adjoint to identity transformations. By 0.3.(i), we get morphisms $r = (R,\pi)$ of monads, with $\pi_X = G_p \lambda_X$ or $\pi_X = G_o \lambda_X$ reducing ultrafilters or proper filters to their open sets. These maps π_X are surjective.

Subspace inclusions provide $\kappa : \Pi_p \longrightarrow \Pi_o S^{op}$ for the forgetful functor S, adjoint to $id(\mathfrak{G}_o)$. Thus we have an inclusion morphism $i = (Id, \kappa\,\mathfrak{G}_p{}^{op}) : \mathbb{H}_p \longrightarrow \mathbb{H}_o$.

We denote by $D : LAT \longrightarrow LAT$ the dual lattice functor which reverses order in every lattice, preserving underlying sets and mappings. Complements of closed sets define a natural isomorphism $\rho : D\,\Gamma_p \longrightarrow \mathfrak{G}_p$. For a lattice L, the complement $L \setminus \varphi$ of a prime filter φ in L is a prime filter in $D\,L$; it is easily seen that complements of prime filters define a natural isomorphism

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$\sigma : \Sigma \longrightarrow \Pi D^{op}$. By 0.3, $\sigma \Gamma^{op} = \Pi \rho^{op} \cdot \pi$ for a natural isomorphism of monads $(\mathrm{Id}, \pi) : \mathfrak{W}_p \longrightarrow \mathfrak{H}_p$.

5.3. If $h : \mathrm{Id} \longrightarrow H$ is the unit of $\mathfrak{H}_p$ or $\mathfrak{H}_o$, then $h_X(x)$ is the filter of open neighborhoods of x, for a space X and $x \in X$. We define the dual induced order of X, dual to the induced order of X, by putting $x \leq y$ iff $h_X(x) \leq h_X(y)$.

For φ in a space $\Pi_p L$ or $\Pi_o L$, the sets $a^{\#}$ with $a \in \varphi$ form a base of neighborhoods of φ; it follows that the dual induced order of $\Pi_p L$ or $\Pi_o L$ is the natural order for filters, dual to set inclusion.

By 5.2, we have algebraic functors r^*, from $\mathfrak{H}_p$-algebras to compact Hausdorff spaces and from $\mathfrak{H}_o$-algebras to $\mathfrak{F}_o$-algebras. These functors preserve underlying sets and mappings.

THEOREM. If (X,α) is an $\mathfrak{H}_p$-algebra, then $r^*(X,\alpha)$, provided with the dual induced order of X, is a compact ordered space, and X has the lower topology of this compact ordered space. Every $\mathfrak{H}_p$-algebra is obtained in this way, and morphisms of $\mathfrak{H}_p$-algebras are morphisms of the corresponding compact ordered spaces.

Proof. If $Z = r^*(X,\alpha)$ for an $\mathfrak{H}_p$-algebra (X,α), then it is seen as in [4], 1.7, that the topology of Z is finer than the topology of X. If φ is an ultrafilter on X and $\bar{\varphi}$ the prime filter of open sets in φ, then $\alpha(\bar{\varphi})$ is the limit of φ for Z; thus φ converges to all $x \geq \alpha(\bar{\varphi})$ for the dual induced order of X. Conversely, if φ converges to x for X, then $h_X(x) \geq \bar{\varphi}$ in $H_p X$; thus $x \geq \alpha(\bar{\varphi})$ for the dual induced order of X since the continuous map α preserves this order.

If $x \not\leq y$ in X, so that $y \notin \mathrm{cl}_X\{x\}$, then $h_X(y)$ is not in the closed set $\overleftarrow{\alpha}(\mathrm{cl}_X\{x\})$ in $H_p X$; thus there is a basic open set $V^{\#}$ in $H_p X$, with V an open neighborhood of x in X, disjoint from $\overleftarrow{\alpha}(\mathrm{cl}_X\{x\})$. It follows that $X \setminus V$ is in every ultrafilter with limit x for Z; thus $X \setminus V$ is a neighborhood of x in Z. Now $X \setminus V \times V$ is a neighborhood of (x,y) in $Z \times Z$, disjoint from the graph of $\leq$ since V is decreasing, and $(Z,\leq)$ is a compact ordered space. Ultrafilters have the same limits for X as for the lower topology of $(Z,\leq)$; thus X has this lower topology.

The remainder of the proof follows the proof of the corresponding results of [4], with only minor changes.∎

REMARK. Since the monads $\mathbb{W}_p$ and $\mathbb{H}_p$ are isomorphic, the same spaces have algebra structures for the two monads. These spaces are the super-sober spaces of the Compendium [1]. The compact ordered spaces obtained in this way from a super-sober space X are dual. They have the same topology, the patch topology of X [1; 4], but dual orders.

5.4. The morphisms of monads of 5.2 provide algebraic functors i^* and r^*, from $\mathbb{H}_o$-algebras to $\mathbb{H}_p$-algebras and to $\mathfrak{F}_o$-algebras. If (X,α) is an $\mathbb{H}_o$-algebra, then the $\mathfrak{F}_o$-algebra $r^*(X,\alpha)$ and the compact ordered space obtained from $i^*(X,\alpha)$ provide us with the same compact Hausdorff topology; we now show that they also provide us with the same order.

PROPOSITION. <u>If</u> (X,α) <u>is an</u> $\mathbb{H}_o$<u>-algebra, then the order of the</u> $\mathfrak{F}_o$<u>-algebra</u> $r^*(X,\alpha)$ <u>is the dual induced order of</u> X.

<u>Proof</u>. Let $r = (R,\pi)$, with π_X restricting a filter on X to its open sets. If (X,α) is an $\mathbb{H}_o$-algebra, then $x \vee y = \alpha(\pi_X(\uparrow\{x,y\}))$ in $r^*(X,\alpha)$. If $x \leq y$ in the dual induced order of X, then $\pi_X(\uparrow\{x,y\}) = h_X(y)$; thus $x \vee y = y$. Conversely, we have $h_X(x) \leq \pi_X(\uparrow\{x,y\})$ in the dual induced order of $H_o X$, and α preserves this order. Thus $x \leq y$ in the dual induced order of X if $x \vee y = y$ in $r^*(X,\alpha)$ ∎

5.5. The lower topology of a continuous sup semilattice L is the Scott topology, with $U \subset L$ open iff U is decreasing and meets every filter φ in L with $\inf\varphi$ in U. Scott topologies provide a functor S, from $\mathfrak{F}_o$-algebras to TOP, which preserves underlying sets and mappings. It follows that RS is the forgetful functor from $\mathfrak{F}_o$-algebras to sets.

THEOREM. <u>The algebraic functor</u> r^* <u>from</u> $\mathbb{H}_o$<u>-algebras to</u> $\mathfrak{F}_o$<u>-algebras is an isomorphism of categories, preserving underlying sets and mappings, and with</u> $S\,r^*$ <u>the forgetful functor from</u> $\mathbb{H}_o$<u>-algebras to</u> TOP.

Proof. 0.2.(i) and 0.2.(ii), with $\Delta = U$, are verified as in the proof of 4.4. To obtain 0.2.(iii) for an $\mathfrak{F}_o$-algebra (L,α), we must show that $\alpha(\Phi) = \sup \mathrm{adh}_X \Phi$, for a filter Φ on L and $X = i^*(L,\alpha)$, depends only on the decreasing open sets in Φ. Restricting Φ to these sets can only increase $\alpha(\Phi)$. On the other hand, if $x \not\leq \alpha(\Phi)$, then $\downarrow\alpha(\Phi)$, and hence also $\mathrm{adh}_X \Phi$, has a decreasing neighborhood V with $X \setminus V$ a neighborhood of x. Then $V \in \Phi$; thus restricting Φ to its decreasing open sets cannot increase $\sup \mathrm{adh}_X \Phi$ $\square$