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THE n-DIMENSIONAL QUADRATIC HEISENBERG ALGEBRA AS A "NON–COMMUTATIVE" $\mathrm{sl}(2,\mathbb{C})$

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Abstract. We prove that the commutation relations among the generators of the quadratic Heisenberg algebra of dimension $n \in \mathbb{N}$, look like a kind of *non-commutative extension* of $\mathrm{sl}(2,\mathbb{C})$ (more precisely of its unique 1–dimensional central extension), denoted $\mathrm{heis}_{2;\mathbb{C}}(n)$ and called the complex n–dimensional quadratic Boson algebra. This *non-commutativity* has a different nature from the one considered in quantum groups. We prove the exponentiability of these algebras (for any n) in the Fock representation. We obtain the group multiplication law, in coordinates of the first and second kind, for the quadratic Boson group and we show that, in the case of the adjoint representation, these multiplication laws can be expressed in terms of a generalization of the Jordan multiplication. We investigate the connections between these two types of coordinates (disentangling formulas). From this we deduce a new proof of the expression of the vacuum characteristic function of homogeneous quadratic boson fields.

1. Introduction

The **complex n–dimensional Heisenberg algebra**, denoted $\mathrm{heis}_{1,\mathbb{C}}(n)$, is the complex $*$–Lie algebra with generators

$$\mathbf{1} \text{ (central element)} \ , \ a_j^\dagger, a_k, j,k \in \{1,\dots,n\} \ ,$$

commutation relations

$$[a_k, a_j^\dagger] = \delta_{j,k}\mathbf{1} \ ; \ [a_k,\mathbf{1}] = [a_j^\dagger,\mathbf{1}] = [a_k,a_j] = [a_k^\dagger,a_j^\dagger] = 0 \ , \tag{1.1}$$

and involution given by

$$a_j^* = a_j^\dagger \qquad ; \qquad \left(a_j^\dagger\right)^* = a_j \ . \tag{1.2}$$

The n–**dimensional Heisenberg algebra**, denoted $\mathrm{heis}_1(n)$, is the real $*$–Lie sub–algebra of $\mathrm{heis}_{1,\mathbb{C}}(n)$ consisting of its skew–adjoint elements. The associated local $*$–Lie groups are denoted respectively $\mathrm{Heis}_{1,\mathbb{C}}(n)$ and $\mathrm{Heis}_1(n)$. The universal enveloping algebra of the n–dimensional Heisenberg $*$–algebra, also called Boson or CCR algebra, contains as a $*$–Lie sub–algebra the complex linear span of the

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normally ordered products of pairs of creation or annihilation operators that we denote $\mathrm{heis}_{2;\mathbb{C}}(n)$ (see Section 2).

This $*$–Lie algebra can be considered as a representation of an abstract finite–dimensional complex $*$–Lie algebra that we call the **complex n–dimensional quadratic Boson algebra** (Definition 2.5) and, in this paper we denote it with the same symbol $\mathrm{heis}_{2;\mathbb{C}}(n)$ used for its Boson representation (which is faithful by construction). It is known that $\mathrm{heis}_{2;\mathbb{C}}(1)$ is isomorphic to a 1–dimensional (necessarily trivial) central extension of $\mathrm{sl}(2,\mathbb{C})$. This fact was exploited in [6], [11] to construct a projective representation of $\mathrm{heis}_{2;\mathbb{C}}(1)$ (the quadratic analogue of the Weyl representation) and to identify the composition law of the associated Lie group (the quadratic analogue of the Heisenberg group).

The structure of this composition law was simplified in [5] and recently a substantial step forward in this direction has been achieved in [12] with the identification of the one–mode quadratic Heisenberg group with the projective group $PSU(1,1)$ and an explicit realization of its holomorphic representation. The multi–dimensional extension of these results is essential to extend the presently available results concerning the **vacuum distributions of the Virasoro fields** which, in their truncated form (see [4]), are elements of the (non–homogeneous) quadratic algebra. The attempt to solve this problem has been the starting point of the present paper. As it happens for the theory of orthogonal polynomials, difficulties undergo a veritable phase transition in the passage from dimension 1 to dimensions ≥ 2. These difficulties disappear in the product (or factorizable) case where calculations can be reduced to the 1–dimensional case up to a linear transformation in the 1–particle space. However, as shown in [3], there exist quadratic fields for which such a reduction is impossible. Furthermore there are many indications that the truncated Virasoro fields generically (i.e. for most choices of the parameters defining them) belong to this class, even if a proof of this fact is at the moment not available.

With these motivations we have carried out in the past years a systematic investigation of different aspects of quadratic boson fields [1], [2], [3], [4].

In particular, in Lemma 6 of [3] it was remarked that, in dimensions ≥ 2, the commutation relations among generators of the quadratic Heisenberg algebra (see Definition 2.5) look like a **a kind of non–commutative** extension of the relations defining $\mathrm{sl}(2,\mathbb{C})$ (more precisely of its unique 1–dimensional central extension), even if this terminology might seem weird since $\mathrm{sl}(2,\mathbb{C})$ is itself non–commutative.

When $n = 1$ this analogy becomes strict in the sense that, as already mentioned,

$$\mathrm{sl}(2,\mathbb{R}) \equiv \mathrm{heis}_2(1) \ , \tag{1.3}$$

where $\equiv$ means **$*$–isomorphism of $*$–Lie algebras**.

The complex n–dimensional quadratic Boson algebra was implicitly introduced, with different notations and for different purposes, in Section 4 of the deep paper [9], where a matrix representation for it was constructed. However the *non–commutative* $\mathrm{sl}(2,\mathbb{C})$ was not considered in that paper, where the analogy with $\mathrm{sl}(N,\mathbb{R})$ (for some Natural integer N) was emphasized. We prove (see Lemma 2.9) that this analogy in general does not hold. In fact, recent results obtained after the completion of this paper show that it holds only in the case of $\mathrm{sl}(2,\mathbb{C})$, i.e. in the case of 1 degree of freedom.

The additional non–commutativity, arising in $\mathrm{heis}_2(n)$ (and more generally in $\mathrm{heis}_{2,\mathbb{C}}(n)$) with respect to $\mathrm{sl}(2,\mathbb{R})$, makes formulas more implicit due to the fact that, even in the finite dimensional case, there is no explicit form for the exponential (2.34) except for very special cases. This exponential plays a crucial role in the Feinsilver–Pap splitting Lemma that is one of the main tools used in this paper. In fact, in the first part of this paper (sections 4, 5), we use this lemma to **determine the composition law** of the quadratic Heisenberg group, both in first and second type Lie–coordinates. These results are used in section 6 to give a **new deduction of the vacuum characteristic function** of hermitian homogeneous quadratic fields (vacuum expectation of quadratic Weyl operators). In the same section we find an inductive relation for the scalar product of a special class of n–particle vectors. Notice that, even in the 1–dimensional case, it took several years to find the explicit form of the scalar product of two homogeneous quadratic n–particle vectors (see Lemma 2.2 in [8]). In Proposition 6.6, **we give an inductive formula for this scalar product**.

In section 7 we prove that the adjoint action of the quadratic Lie group on the quadratic $*$–Lie algebra has a simple and explicit expression. This result, combined with the Feinsilver–Pap splitting Lemma, is used to put products of quadratic Weyl operators in *semi–normal form*. This differs from usual normal form because between an exponential of creators and an exponential of annihilators one finds, instead of a single exponential of a number type operator, a product of such exponentials. Since the vacuum vector is left invariant by exponentials of number type operators, this gives a tool to compute scalar products of quadratic coherent vectors much more explicitly than with the Baker–Campbell–Hausdorff formula.

Finally, in section 8 (Theorem 8.5), we extend to the multi–dimensional case the estimates, proved in [6] for the 1–dimensional case. These allow to prove that, in the Fock representation, the number vectors are analytic vectors for the elements of the quadratic algebra. Thus, by Nelson's theorem, the hermitian elements of this algebra are essentially self–adjoint and their exponential series converges strongly on the domain of number vectors. From this the existence and unitarity of the quadratic Weyl operators follows.

2. The Complex n–dimensional Quadratic Boson Algebra

In this section, we identify the complex n–dimensional quadratic Boson algebra with a non–commutative version of $\mathrm{sl}(2,\mathbb{C})$.

Denote $M_n(\mathbb{C})$ (resp. $M_{n,sym}(\mathbb{C})$) the algebra of $n\times n$ complex matrices (resp. symmetric matrices) and, for $M\equiv(M_{j,k})\in M_n(\mathbb{C})$, define the transpose, conjugate and adjoint of M in the standard way:

$$(M^T)_{j,k} := M_{k,j} \ ; \ (\overline{M})_{j,k} := \overline{M_{j,k}} \ ; \ (M^*)_{j,k} := (\overline{M})^T_{j,k} = \overline{(M)^T}_{j,k} = \overline{M_{k,j}} \ .$$

We identify $\mathrm{heis}_{1,\mathbb{C}}(n)$ to a sub–algebra of its universal enveloping $*$–algebra and consider the space of all **homogeneous quadratic expressions** in the generators (1.1):

$$a^\dagger A a^\dagger := \sum_{j,k=1}^n A_{j,k} a_j^\dagger a_k^\dagger \ ,$$
$$a^\dagger B a := \sum_{j,k=1}^n B_{j,k} a_j^\dagger a_k \ ,$$
$$aCa := \sum_{j,k=1}^n C_{j,k} a_j a_k \ .$$

Note that, since creators (resp. annihilators) mutually commute, one has

$$a^\dagger A a^\dagger = \sum_{j,k=1}^n \frac{1}{2}\left(A_{j,k} + A_{k,j}\right) a_j^\dagger a_k^\dagger \ ; \ aCa = \sum_{j,k=1}^n \frac{1}{2}\left(C_{j,k} + C_{k,j}\right) a_j a_k \ , \tag{2.1}$$

i.e. the expressions $a^\dagger A a^\dagger$ and aCa are parametrized by **symmetric matrices**, $A^T = A$ and $C^T = C$. Denote

$$\mathrm{heis}_{2;\mathbb{C}}(n) := \mathbb{C}\cdot \mathbf{1} \oplus \mathbb{C}\text{–linear span of}$$
$$\left\{a^\dagger A a^\dagger \, , \, a^\dagger B a \, , \, aCa \ : \ B \in M_n(\mathbb{C}) \, , \, A, C \in M_{n,sym}(\mathbb{C})\right\} \ .$$

Because of the linear independence of the set $\{\mathbf{1} \, , \, a_j^\dagger a_k^\dagger \, , \, a_j^\dagger a_k \, , \, a_j a_k\}$, $j, k \in \{1 \ldots , n\}$, $\mathrm{heis}_{2;\mathbb{C}}(n)$ is the range of the vector space isomorphism

$$\begin{aligned}(c1_{M_n}, A, B, C) \in \mathbb{C}\cdot 1_{M_n} \times M_{n,sym}(\mathbb{C}) \times M_n(\mathbb{C}) \times M_{n,sym}(\mathbb{C}) \\ \mapsto c\mathbf{1} + a^\dagger A a^\dagger + a^\dagger B a + aCa \in \mathrm{heis}_{2;\mathbb{C}}(n) \ .\end{aligned} \tag{2.2}$$

In the following we will use the notation

$$c\mathbf{1} \equiv c \qquad ; \qquad c \in \mathbb{C} \ .$$

The involution (1.2) on $\mathrm{heis}_{1,\mathbb{C}}(n)$ induces an involution on $\mathrm{heis}_{2;\mathbb{C}}(n)$ given by

$$(a^\dagger A a^\dagger)^* = aA^*a \ ; \ (a^\dagger B a)^* = a^\dagger B^* a \ ; \ (aCa)^* = a^\dagger C^* a^\dagger \ .$$

With this involution (2.2) is a $*$–isomorphism of $*$–vector spaces.

Lemma 2.1. *$\mathrm{heis}_{2;\mathbb{C}}(n)$ is a $*$–Lie algebra with involution given by* (1.2), *central element* $\mathbf{1}$ *and with the following commutation relations:*

$$[aMa, a^\dagger N a^\dagger] = 2\, Tr(NM) + 4\, a^\dagger NMa \ ; \ M, N \in M_{n,sym}(\mathbb{C}) \ , \tag{2.3}$$
$$[aMa, a^\dagger N a] = a\left(MN + (MN)^T\right) a \ ; \ M \in M_{n,sym}(\mathbb{C}) \ , \ N \in M_n(\mathbb{C}) \ , \tag{2.4}$$
$$[a^\dagger M a, a^\dagger N a] = a^\dagger [M, N] a \ ; \ M, N \in M_n(\mathbb{C}) \ , \tag{2.5}$$
$$[a^\dagger M a^\dagger, a^\dagger N a^\dagger] = [aMa, aNa] = 0 \ ; \ M, N \in M_{n,sym}(\mathbb{C}) \ , \tag{2.6}$$
$$[a^\dagger M a, a^\dagger N a^\dagger] = a^\dagger \left(MN + (MN)^T\right) a^\dagger \quad ; \ N \in M_{n,sym}(\mathbb{C}) \, , \, M \in M_n(\mathbb{C}) \ . \tag{2.7}$$

Proof. Throughout this proof, summation over repeated indexes is understood. We will use the algebraic identities

$$\begin{aligned}[][a_i a_j, a_h^\dagger a_k^\dagger] =& [a_i a_j, a_h^\dagger]a_k^\dagger + a_h^\dagger[a_i a_j, a_k^\dagger] \\ =& a_i[a_j, a_h^\dagger]a_k^\dagger + [a_i, a_h^\dagger]a_j a_k^\dagger + a_h^\dagger a_i[a_j, a_k^\dagger] + a_h^\dagger[a_i, a_k^\dagger]a_j \\ =& a_i a_k^\dagger \delta_{jh} + a_j a_k^\dagger \delta_{ih} + a_h^\dagger a_i \delta_{jk} + a_h^\dagger a_j \delta_{ik} \\ =& \delta_{ik}\delta_{jh} + a_k^\dagger a_i \delta_{jh} + \delta_{jk}\delta_{ih} + a_k^\dagger a_j \delta_{ih} + a_h^\dagger a_i \delta_{jk} + a_h^\dagger a_j \delta_{ik} \ ,\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}[][a_i^\dagger a_j, a_h^\dagger a_k] =& [a_i^\dagger a_j, a_h^\dagger]a_k + a_h^\dagger[a_i^\dagger a_j, a_k] \\ =& a_i^\dagger[a_j, a_h^\dagger]a_k + a_h^\dagger[a_i^\dagger, a_k]a_j \\ =& a_i^\dagger a_k \delta_{j,h} - a_h^\dagger a_j \delta_{i,k} \ .\end{aligned} \tag{2.9}$$

Equation (2.3) follows from

$$\begin{aligned}&[aMa, a^\dagger N a^\dagger] = M_{ij}N_{hk}[a_i a_j, a_h^\dagger a_k^\dagger] \\ =&^{(2.8)} M_{ji}N_{hk}\left(a_i a_k^\dagger \delta_{jh} + a_j a_k^\dagger \delta_{ih} + a_h^\dagger a_i \delta_{jk} + a_h^\dagger a_j \delta_{ik}\right) \\ =& M_{ji}N_{hk}\left(\delta_{ik}\delta_{jh} + a_k^\dagger a_i \delta_{jh} + \delta_{jk}\delta_{ih} + a_k^\dagger a_j \delta_{ih} + a_h^\dagger a_i \delta_{jk} + a_h^\dagger a_j \delta_{ik}\right) \\ =& M_{ji}N_{hk}\delta_{ik}\delta_{jh} + M_{ji}N_{hk}a_k^\dagger a_i \delta_{jh} + M_{ji}N_{hk}\delta_{jk}\delta_{ih} + \\ & M_{ji}N_{hk}a_k^\dagger a_j \delta_{ih} + M_{ji}N_{hk}a_h^\dagger a_i \delta_{jk} + M_{ji}N_{hk}a_h^\dagger a_j \delta_{ik} \\ =& M_{ji}N_{ji} + M_{ji}N_{jk}a_k^\dagger a_i + M_{ji}N_{ij} + M_{ji}N_{ik}a_k^\dagger a_j + M_{ji}N_{hj}a_h^\dagger a_i + M_{ji}N_{hi}a_h^\dagger a_j \\ =& 2M_{ji}N_{ij} + N_{kj}M_{ji}a_k^\dagger a_i + M_{ji}N_{ik}a_k^\dagger a_j + N_{hj}M_{ji}a_h^\dagger a_i + M_{ji}N_{ih}a_h^\dagger a_j \\ =& 2\mathrm{Tr}\ (MN) + (NM)_{ki}a_k^\dagger a_i + (MN)_{jk}a_k^\dagger a_j + (NM)_{hi}a_h^\dagger a_i + (MN)_{jh}a_h^\dagger a_j \\ =& 2\mathrm{Tr}\ (MN) + (NM)_{ki}a_k^\dagger a_i + (NM)_{kj}a_k^\dagger a_j + (NM)_{hi}a_h^\dagger a_i + (NM)_{hj}a_h^\dagger a_j \\ =& 2\mathrm{Tr}\ (MN) + (NM)_{ki}a_k^\dagger a_i + (NM)_{ki}a_k^\dagger a_i + (NM)_{ki}a_k^\dagger a_i + (NM)_{kj}a_k^\dagger a_j \\ =& 2\mathrm{Tr}\ (MN) + 4\,(NM)_{ki}\,a_k^\dagger a_i = 2\,\mathrm{Tr}(NM) + 4\,a^\dagger NMa\ .\end{aligned}$$

Equation (2.7) follows from

$$\begin{aligned}&[a^\dagger Ma, a^\dagger N a^\dagger] = M_{ij}N_{hk}[a_i^\dagger a_j, a_h^\dagger a_k^\dagger] \\ =&^{(2.8)} M_{ij}N_{hk}\left(\delta_{ik}\delta_{jh} + a_k^\dagger a_i \delta_{jh} + \delta_{jk}\delta_{ih} + a_k^\dagger a_j \delta_{ih} + a_h^\dagger a_i \delta_{jk} + a_h^\dagger a_j \delta_{ik}\right) \\ =& M_{ij}N_{hk}a_i^\dagger a_k^\dagger \delta_{jh} + M_{ij}N_{hk}a_h^\dagger a_i^\dagger \delta_{jk} + M_{ij}N_{jk}a_i^\dagger a_k^\dagger + M_{ij}N_{hj}a_h^\dagger a_i^\dagger \\ & + (MN)_{ik}a_i^\dagger a_k^\dagger + M_{ij}N_{jh}a_h^\dagger a_i^\dagger \\ =& (MN)_{ik}a_i^\dagger a_k^\dagger + (MN)_{ih}a_h^\dagger a_i^\dagger + (MN)_{ik}a_i^\dagger a_k^\dagger + (MN)_{ki}a_i^\dagger a_k^\dagger \\ & + (MN)_{ik}a_i^\dagger a_k^\dagger + ((MN)^T)_{ik}a_i^\dagger a_k^\dagger \\ =& a_i^\dagger((MN) + (MN)^T)_{ik}a_k^\dagger \\ =& a^\dagger((MN) + (MN)^T)a^\dagger\ .\end{aligned}$$

Equation (2.4) is the adjoint of (2.7). Equation (2.5) follows from

$$\begin{aligned}[a^\dagger Ma, a^\dagger Na] =& M_{ij}N_{hk}[a_i^\dagger a_j, a_h^\dagger a_k] \\ =&^{(2.9)} M_{ij}N_{hk}(a_i^\dagger a_k\delta_{j,h} - a_h^\dagger a_j\delta_{i,k}) \\ =& M_{ij}N_{hk}a_i^\dagger\delta_{jh}a_k - M_{ij}N_{hk}a_h^\dagger\delta_{ki}a_j \\ =& a_i^\dagger M_{ij}N_{jk}a_k - a_h^\dagger N_{hi}M_{ij}a_j \\ =& a_i^\dagger(MN)_{ik}a_k - a_h^\dagger(NM)_{hj}a_j \\ =& a^\dagger(MN)a - a^\dagger(NM)a \\ =& a^\dagger(MN - NM)a \\ =& a^\dagger[M,N]a \ .\end{aligned}$$

The proof of (2.4) follows directly from the commutation relations

$$[a_i^\dagger a_j^\dagger, a_h^\dagger a_k^\dagger] = [a_i a_j, a_h a_k] = 0 \ ,$$

for all indices i, j, h, k. □

The commutation relations (2.5), (2.7) suggest that it is convenient to introduce the composition law discussed in the following Lemma. In section 7 below we will see that, using the ◦–notation, several formulas acquire a shape that strongly reminds the corresponding result in the commutative case.

Lemma 2.2. *For any $X, Y \in M_d(\mathbb{C})$, the binary composition law*

$$X \circ Y := XY + (XY)^T \ , \tag{2.10}$$

is commutative, distributive, complex bi–linear and the following properties hold.

$$X \circ 1 = X + X^T \ , \tag{2.11}$$

$$(X \circ Y)^* = Y^* \circ X^* \ . \tag{2.12}$$

Proof. Commutativity, distributivity, complex bi–linearity and (2.11) are clear. (2.12) follows from

$$\begin{aligned}(X \circ Y)^* :=& (XY + (XY)^T)^* = Y^*X^* + ((XY)^T)^* \\ =& Y^*X^* + ((XY)^*)^T = Y^*X^* + (Y^*X^*)^T = Y^* \circ X^* \ .\end{aligned}$$

□

Remark 2.3. The operation ◦ is neither commutative nor associative. Restricted to the hermitian (or skew–hermitian) elements of $\mathrm{heis}_{2;\mathbb{C}}(n)$ it reduces to the **Jordan product** which is commutative.

With the ◦–notation, Lemma 2.1 becomes the following.

Corollary 2.4. *For $M, N \in M_n(\mathbb{C})$,*

$$[aMa, a^\dagger Na^\dagger] = 2\ Tr(NM) + 4\, a^\dagger NMa\ ,\ \ M, N \in M_{n,sym}(\mathbb{C})\ , \tag{2.13}$$

$$[aMa, a^\dagger Na] = a(M \circ N)a\ ,\ \ M \in M_{n,sym}(\mathbb{C})\ ,\ \ N \in M_n(\mathbb{C})\ , \tag{2.14}$$

$$[a^\dagger Ma, a^\dagger Na] = a^\dagger [M, N]\ ,\ \ M, N \in M_n(\mathbb{C})a\ , \tag{2.15}$$

$$[a^\dagger Ma^\dagger, a^\dagger Na^\dagger] = [aMa, aNa] = 0\ ,\ \ M, N \in M_{n,sym}(\mathbb{C})\ , \tag{2.16}$$

$$[a^\dagger Ma^\dagger, a^\dagger Na] = -\, a^\dagger (N \circ M)a^\dagger\ ,\ \ M \in M_{n,sym}(\mathbb{C})\ ,\ \ N \in M_n(\mathbb{C})\ . \tag{2.17}$$

Proof. (2.17) is, up to change of notations, the adjoint of (2.14). In fact

$$\begin{aligned}(2.14)\ &\Longleftrightarrow ([aMa, a^\dagger Na])^* = (a(M \circ N)a)^* \\ &\Longleftrightarrow [(a^\dagger Na)^*, (aMa)^*] = a^\dagger (M \circ N)^* a^\dagger \\ &\Longleftrightarrow [a^\dagger N^* a, a^\dagger M^* a^\dagger] = a^\dagger (N^* \circ M^*)a^\dagger\ .\end{aligned}$$

Since M, N are arbitrary, one can replace M by M^* and N by N^* obtaining

$$[a^\dagger Ma^\dagger, a^\dagger Na] = -a^\dagger (N \circ M)a^\dagger\ .$$

Exchanging the roles of M and N, (2.17) takes the form (2.7), i.e.

$$[a^\dagger Ma, a^\dagger Na^\dagger] = a^\dagger (M \circ N)a^\dagger\ .$$

□

Definition 2.5. The **complex n–dimensional quadratic Boson algebra**, still denoted $\mathrm{heis}_{2;\mathbb{C}}(n)$, is the $*$–Lie algebra with linearly independent generators

$$\left\{\mathbf{1}\ ,\ B_0^2(A)\ ,\ B_1^1(B)\ ,\ B_2^0(C)\ :\ A, C \in M_{n,sym}(\mathbb{C})\ ,\ \ B \in M_n(\mathbb{C})\right\}\ , \tag{2.18}$$

central element $\mathbf{1}$, involution given by

$$B_0^2(A)^* = B_2^0(A^*)\ ;\ \ B_1^1(B)^* = B_1^1(B^*)\ ;\ \ B_2^0(C)^* = B_0^2(C^*)\ , \tag{2.19}$$

and Lie brackets given by:

$$[B_2^0(M), B_0^2(N)] = 2\,\mathrm{Tr}(NM) + 4\, B_1^1(NM)\ , \tag{2.20}$$

$$[B_1^1(M), B_1^1(N)] = B_1^1([M, N])\ , \tag{2.21}$$

$$[B_2^0(M), B_1^1(N)] = B_2^0\,(M \circ N)\ \ , \tag{2.22}$$

$$[B_0^2(M), B_1^1(N)] = -\, B_0^2\,(N \circ M)\ \ , \tag{2.23}$$

$$[B_0^2(M), B_0^2(N)] = [B_2^0(M), B_2^0(N)] = 0\ . \tag{2.24}$$

The **the skew–adjoint** elements of $\mathrm{heis}_{2;\mathbb{C}}(n)$, are a **real** $*$–Lie sub–algebra denoted $\mathrm{heis}_2(n)$, called the **n–dimensional quadratic Heisenberg algebra**. The local Lie groups associated to $\mathrm{heis}_{2;\mathbb{C}}(n)$ and $\mathrm{heis}_2(n)$ are denoted respectively $\mathrm{Heis}_{2;\mathbb{C}}(n)$ and $\mathrm{Heis}_2(n)$.

Remark 2.6. With the additional prescriptions

$$B_0^2(A) := B_0^2(A^T)\ ;\ \ B_2^0(A) := B_2^0(A^T)\ ;\ \forall A \in M_n(\mathbb{C})\ , \tag{2.25}$$

(automatically satisfied in the Boson realization (see (2.1))) allows to replace the parametrization of $\mathrm{heis}_{2;\mathbb{C}}(n)$ given by (2.18)

$$\left\{\mathbf{1}\ ,\ B_0^2(A)\ ,\ B_1^1(B)\ ,\ B_2^0(C)\ :\ A, B, C \in M_n(\mathbb{C})\right\}\ . \tag{2.26}$$

The advantage of (2.18) is that it is one–to–one.

Remark 2.7. In order to simplify notations we use the **same symbol** $\mathrm{heis}_{2;\mathbb{C}}(n)$ for the abstract $*$–Lie algebra and for its Boson realization and the same for their central element. The identifications

$$B^2_0(A) \equiv a^\dagger A a^\dagger \ ; \ B^1_1(B) \equiv a^\dagger B a \ ; \ B^0_2(C) \equiv aCa \ ,$$

are clear from the commutation relations and in the following we will use them constantly because the Boson notation are more intuitive. These identifications should not create confusion provided they are handled carefully. For example the following identities, where summation on repeated indices is understood, make sense in a Boson context:

$$\begin{aligned} aAa^\dagger =& a_i(A)_{ij}a^\dagger_j = a_iA_{ij}a^\dagger_j = A_{ij}([a_i, a^\dagger_j] + a^\dagger_j a_i) \\ =& A_{ij}(\delta_{ij} + a^\dagger_j a_i) = A_{ij}\delta_{ij} + A_{ij}a^\dagger_j a_i = A_{ii}\delta_{ij} + a^\dagger_j (A^T)_{ji} a_i \\ =& \mathrm{Tr}(A) + a^\dagger_j((A)^T)_{ji}a_i \\ =& \mathrm{Tr}(A) + a^\dagger (A)^T a \ , \end{aligned} \tag{2.27}$$

but, while expressions of the form $aAa^\dagger$ do not make sense in the quadratic Boson algebra, both terms in the sum appearing in the last identity in 2.27 are in the quadratic Boson algebra.

Remark 2.8. In the physics literature, instead of (2.19), one uses the involution

$$B^1_1(B)^* = B^1_1(B^*) \ ; \ B^0_2(C)^* = B^2_0(C) \ ,$$

where annihilators depend **anti–linearly** on their **test matrices**. This has the advantage that annihilators are simply defined as adjoints of creators and one is not obliged to define an involution on the test function space. However, since most test function spaces concretely used have a natural involution, the choice (2.19) seems to be more natural.

Lemma 2.9. *The pairs $(N, n) \in \mathbb{N}^2$ such that $sl(N, \mathbb{R})$ and $heis_2(n)$ are isomorphic as vector spaces if and only if N and n have the form*

$$n = 2n_1 + 1 \ ; \ N = 2(2p_1 + 1) \ ; \ n_1, p_1 \in \mathbb{N} \ ,$$

where the pair $(n_1, p_1) \in \mathbb{N}^2$ is any solution of the quadratic diophantine equation

$$2(2p_1 + 1)^2 = (2n_1 + 1)^2 + 1 \ .$$

Proof. (2.2) implies that the complex dimension of $\mathrm{heis}_{2;\mathbb{C}}(n)$ as a vector space is

$$1 + 2\frac{n(n+1)}{2} + n^2 = 1 + n^2 + n + n^2 = 2n^2 + n + 1 \ ,$$

while the real dimension of $\mathrm{heis}_2(n)$ is

$$1 + n(n+1) + n + (n-1)n = 1 + n^2 + n + n^2 - n = 1 + 2n^2 \ ,$$

The real dimension of $\mathrm{sl}(N, \mathbb{R})$ is $N^2 - 1$. Therefore $\mathrm{heis}_2(n)$ can be isomorphic to $\mathrm{sl}(N, \mathbb{R})$ as vector space only if the pair $(n, N) \in \mathbb{N}^2$ satisfies the equation

$$N^2 - 1 = 2n^2 + 1 \iff N^2 = 2n^2 + 2 \ . \tag{2.28}$$

If this is the case, given n (resp. N) N (resp. n) is uniquely determined. Since odd numbers are a multiplicative semi-group, given n, N **has to be an even number**:

$$N = 2p\ .$$

In this case (2.28) becomes equivalent to

$$(2p)^2 = 2n^2 + 2 \iff 2p^2 = n^2 + 1 \iff 2p^2 - 1 = n^2\ . \tag{2.29}$$

This shows that a necessary condition for (2.29) to have a solution is that n **is odd**:

$$n = 2n_1 + 1\ . \tag{2.30}$$

In this case (2.29) becomes

$$\begin{aligned} 2p^2 - 1 =& (2n_1 + 1)^2 = 4n_1^2 + 4n_1 + 1 \\ &\iff 2p^2 = 4n_1^2 + 4n_1 + 2 \\ &\iff p^2 = 2n_1(n_1 + 1) + 1\ . \end{aligned} \tag{2.31}$$

It follows, for the same reason as above, that a necessary condition for (2.31) to have a solution is that p **is odd**:

$$p = 2p_1 + 1\ . \tag{2.32}$$

In view of (2.30) and (2.32), (2.28) becomes

$$\begin{aligned} (2(2p_1 + 1))^2 =& 2(2n_1 + 1)^2 + 2 \qquad (2.33) \\ &\iff 4(2p_1 + 1)^2 = 2(2n_1 + 1)^2 + 2 \\ &\iff 2(2p_1 + 1)^2 = (2n_1 + 1)^2 + 1\ . \end{aligned}$$

□

Remark 2.10. Equation (2.33) for given n_1 has the non–trivial solution

$$n_1 = 3 \Rightarrow (2n_1 + 1)^2 + 1 = 50 = 2 \cdot 5^2\ ,$$

while for given p_1 it has the non–trivial solution

$$p_1 = 2 \Rightarrow 2(2p_1 + 1)^2 - 1 = 49 = 7^2\ .$$

More generally one can prove, by direct computation, that equation (2.28) has non–trivial solutions in the following cases:

$$\begin{aligned} n = 1 \Rightarrow & N^2 = 2n^2 + 2 = 4 = 2^2 \Rightarrow N = 2\ , \\ n = 2 \Rightarrow & N^2 = 2n^2 + 2 = 10 \ \Rightarrow \ \text{no solutions}\ , \\ n = 3 \Rightarrow & N^2 = 2n^2 + 2 = 20 \ \Rightarrow \ \text{no solutions}\ , \\ n = 4 \Rightarrow & N^2 = 2n^2 + 2 = 34 \ \Rightarrow \ \text{no solutions}\ , \\ n = 5 \Rightarrow & N^2 = 2n^2 + 2 = 52 \ \Rightarrow \ \text{no solutions}\ , \\ n = 6 \Rightarrow & N^2 = 2n^2 + 2 = 74 \ \Rightarrow \ \text{no solutions}\ , \\ n = 7 \Rightarrow & N^2 = 2n^2 + 2 = 100 = 10^2 \ \Rightarrow N = 10\ . \end{aligned}$$

Therefore non–trivial solutions exist.

2.1. Group elements and their 1–st and 2–d kind coordinates. In this section, we identify the first and second kind coordinates in the case of $heis_{\mathbb{C}}(2;n)$. The elements of $heis_{\mathbb{C}}(2;n)$ are parametrized by quadruples

$$(x, A, B, C) \in \mathbb{C} \times M_{n,sym}(\mathbb{C}) \times M_n(\mathbb{C}) \times M_{n,sym}(\mathbb{C}) \ ,$$

and we consider the natural topology induced by this parametrization. From this section on we suppose that, for (z, A, B, C) near the origin the corresponding element of $heis_{\mathbb{C}}(2;n)$ can be exponentiated in the sense that the corresponding exponential series converges on a dense sub–space of the representation space. In section 8 below we prove that this is always the case in the Fock representation.

Following the general theory of Lie groups, we say that the quadruple (x, A, B, C) defines the **second kind coordinates** of

$$G(x, A, B, C) = e^{x\mathbf{1}} e^{a^\dagger A a^\dagger} e^{a^\dagger B a} e^{aCa} = e^{x\mathbf{1}} e^{B_0^2(A)} e^{B_1^1(B)} e^{B_2^0(C)} \in Heis_{\mathbb{C}}(2;n) \ ,$$

and the **first kind coordinates** of

$$W(x, A, B, C) = e^{x\mathbf{1} + a^\dagger A a^\dagger + a^\dagger B a + aCa} = e^{x\mathbf{1} + B_0^2(A) + B_1^1(B) + B_2^0(C)} \in Heis_{\mathbb{C}}(2;n) \ . \tag{2.34}$$

In both representations one can find a **sub–set** of the whole domain of the coordinates, i.e. $\mathbb{C}\cdot 1_{M_n} \times M_{n,sym}(\mathbb{C}) \times M_n(\mathbb{C}) \times M_{n,sym}(\mathbb{C})$, in which the correspondence

$$W(x, A, B, C) \mapsto (x, A, B, C) \qquad ; \qquad G(x, A, B, C) \mapsto (x, A, B, C) \ ,$$

is one–to–one. This domain can be considered as an **embedding** of the group manifold of $\mathrm{Heis}_{2;\mathbb{C}}(n)$ into the vector space $\mathbb{C}\cdot 1_{M_n} \times M_{n,sym}(\mathbb{C}) \times M_n(\mathbb{C}) \times M_{n,sym}(\mathbb{C})$. **On this domain** the group multiplication law induces a group composition law through the identities

$$W(x_1, A_1, B_1, C_1) W(x_2, A_2, B_2, C_2) =: W\left((x_1, A_1, B_1, C_1) \diamond_1 (x_2, A_2, B_2, C_2)\right) \ ,$$
$$G(x_1, A_1, B_1, C_1) G(x_2, A_2, B_2, C_2) =: G\left((x_1, A_1, B_1, C_1) \diamond_2 (x_2, A_2, B_2, C_2)\right) \ .$$

Typically both composition laws $\diamond_1$ and $\diamond_2$ are strongly non–linear functions of the coordinates. In the 1–dimensional case and for the sub–group $\mathrm{Heis}_2(1)$ of $\mathrm{Heis}_{2;\mathbb{C}}(1)$, i.e. up to isomorphism $\mathrm{sl}(2,\mathbb{R})$, both the domain and the explicit form of $\diamond_1$ were determined in the paper [6]. Our goal is to extend this result to the multi–dimensional case.

2.2. $*$–Lie sub–algebras of $\mathbf{heis}_{2;\mathbb{C}}(n)$. For special $*$–Lie sub–algebras of $\mathrm{heis}_{2;\mathbb{C}}(n)$, the formulas of the exponential of their elements are considerably simplified. The following Proposition describes the spaces of **test matrices** that parametrize some natural $*$–Lie sub–algebras of $\mathrm{heis}_{2;\mathbb{C}}(n)$.

Proposition 2.11. *Let $\mathcal{L}$ be a $*$–Lie sub–algebra of $heis_{2;\mathbb{C}}(n)$. Denote*

$$M_d(2,0) := \{A \in M_d(\mathbb{C}) \ : \ B_0^2(A) \in \mathcal{L}\} \ ,$$
$$M_d(1,1) := \{B \in M_d(\mathbb{C}) \ : \ B_1^1(B) \in \mathcal{L}\} \ ,$$
$$M_d(0,2) := \{C \in M_d(\mathbb{C}) \ : \ B_2^0(C) \in \mathcal{L}\} \ .$$

Then $M_d(1,1)$ is a sub–$*$–Lie algebra of $M_d(\mathbb{C})$. If in addition

$$\mathbf{1} \in M_d(2,0) \ ,$$

then $M_d(1,1)$ is a ∗–sub–algebra of $M_d(\mathbb{C})$ ***closed under conjugation*** *(or equivalently under transposition) and*

$$M_d(2,0) = M_d(1,1) = M_d(0,2)\ . \tag{2.35}$$

Proof. The assumption that $\mathcal{L}$ is closed under involution and (2.19) imply that

$$M_d(2,0)^* = M_d(0,2) \qquad ; \qquad M_d(1,1)^* = M_d(1,1)\ . \tag{2.36}$$

The assumption that $\mathcal{L}$ is a Lie algebra and the linearity of the maps $B^j_k(\ \cdot\ )$ imply that $M_d(2,0)$, $M_d(1,1)$, $M_d(0,2)$ are vector spaces. (2.5) implies that, if $M, N \in M_d(1,1)$, then $[M,N] \in M_d(1,1)$. Hence (2.36) implies that $M_d(1,1)$ is a sub–∗–Lie algebra of $M_d(\mathbb{C})$. From (2.4) it follows that, if $M \in M_d(0,2)$ and $N \in M_d(1,1)$ then, (see (2.25)) since $a\left(MN + (MN)^T\right)a = aMNa$, $MN \in M_d(0,2)$. Equivalently

$$M_d(0,2)M_d(1,1) \subseteq M_d(0,2)\ . \tag{2.37}$$

But, because of (2.36), $\mathbf{1} \in M_d(2,0) \iff \mathbf{1} \in M_d(0,2)$. Therefore, if $\mathbf{1} \in M_d(2,0)$, then $M_d(1,1) \subseteq M_d(0,2)$. On the other hand, (2.3) implies that $M_d(2,0)M_d(0,2) \subseteq M_d(1,1)$. By the same argument, $M_d(0,2) \subseteq M_d(1,1)$. Therefore $M_d(0,2) = M_d(1,1)$ and (2.36) implies that also $M_d(2,0) = M_d(1,1)$. But then (2.37) becomes

$$M_d(0,2)M_d(1,1) = M_d(1,1)M_d(1,1) \subseteq M_d(1,1)\ ,$$

i.e. $M_d(1,1)$ is an algebra. Since it is closed under involution, it is a sub–∗–algebra $M_d(\mathbb{C})$. From (2.22), (2.35) and the fact that $M_d(1,1)$ is a ∗–algebra with identity, it follows that $M_d(1,1)$ is closed under transposition and, by the ∗–algebra property, this is equivalent to be closed under conjugation. Conversely, if $M_d(1,1)$ is a ∗–algebra (not necessarily with identity) closed under transposition, then the set

$$\mathcal{L} := \left\{B^2_0(A), B^1_1(B), B^0_2(C)\ :\ A, B, C \in M_d(1,1)\right\}\ ,$$

is closed under the involution (2.19) and under the Lie brackets (2.3), ..., (2.7), i.e. it is a ∗–Lie sub–algebra of $\mathrm{heis}_{2;\mathbb{C}}(n)$. □

Remark 2.12. From (2.5) it follows that, for any real or complex Lie sub–algebra $\mathcal{L}_n$ of $M_n(\mathbb{C})$ the family

$$\Lambda_2(\mathcal{L}_n) := \{a^\dagger Ma\ :\ M \in \mathcal{L}_n\}\ ,$$

is a real or complex Lie sub–algebra (resp. ∗–Lie sub–algebra) of $heis_{\mathbb{C}}(2;n)$ called **the quadratic preservation algebra of order** n. If $\mathcal{L}_n = M_n(\mathbb{C})$, we simply write $\Lambda_{2,n}$.

3. The Splitting Lemma

In this section, we recall, in our notations, Feinsilver–Pap's splitting lemma [9] which will be our basic tool for the calculation of vacuum characteristic functions of homogeneous quadratic fields. Formulas expressing second kind coordinates in terms of first kind ones are called **splitting or disentangling formulas**. In the case of $Heis_{\mathbb{C}}(2;n)$, they are given by the following lemma.

Lemma 3.1. *For $A, C \in Sym\,(M_n(\mathbb{C}))$, $B \in M_n(\mathbb{C})$, define*

$$v := \begin{pmatrix} B & 2A \\ -2C & -B^T \end{pmatrix} \ ,$$

and P, Q, R, S by

$$\begin{pmatrix} P(t) & Q(t) \\ -R(t) & S(t) \end{pmatrix} := e^{tv} \ .$$

Then, for $t \in \mathbb{R}$ sufficiently close to 0:

$$e^{t\left(a^\dagger A a^\dagger + a^\dagger B a + aCa\right)} = e^{-\frac{t}{2}\,Tr(B) + \frac{1}{2}\,Tr(g_t(A,B,C))} e^{\frac{1}{2} a^\dagger \hat{f}_t(A,B,C) a^\dagger} \cdot e^{a^\dagger g_t(A,B,C) a} e^{\frac{1}{2} a \hat{h}_t(A,B,C) a} \ ,$$

or, in the B-notation

$$e^{t\left(B_0^2(A) + B_1^1(B) + B_2^0(C)\right)} = e^{-\frac{t}{2}\,Tr(B) + \frac{1}{2}\,Tr(g_t(A,B,C))} e^{\frac{1}{2} B_0^2(\hat{f}_t(A,B,C))} \cdot e^{B_1^1(g_t(A,B,C))} e^{\frac{1}{2} B_2^0(\hat{h}_t(A,B,C))} \ ,$$

where

$$f_t(A,B,C) = Q(t)S(t)^{-1} \ , \ g_t(A,B,C) = -\log S(t)^T \ , \ h_t(A,B,C) = S(t)^{-1}R(t) \ ,$$

and $\hat{f} = (f + f^T)/2, \hat{h} = (h + h^T)/2$ denote the symmetric parts of f, h.

Remark 3.2. The invertibility of $S(t)$ for each $t \in \mathbb{R}$ is proved in Lemma 3.3 below.

Proof. From Lemma 6 of [9], with the following change of notations with respect to those used there:

$$R_{2A} \to a^\dagger A a^\dagger \ , \ \Delta_{2C} \to aCa \ , \ \rho_B \to a^\dagger B a + \mathrm{Tr}(B/2) \ ,$$

we obtain

$$e^{t\left(a^\dagger A a^\dagger + a^\dagger B a + aCa\right)} = e^{-\frac{t}{2}\,\mathrm{Tr}(B)} e^{t(R_{2A} + \rho_B + \Delta_{2C})} = e^{-\frac{t}{2}\,\mathrm{Tr}(B)} e^{R_{A_4(t)}} e^{\rho_{A_5(t)}} e^{\Delta_{A_6(t)}} \ ,$$

where

$$A_4(t) = f_t(A,B,C) \ , \ A_5(t) = g_t(A,B,C) \ , \ A_6(t) = h_t(A,B,C) \ ,$$

with f_t, g_t, h_t as in the statement of this Lemma. Thus, using

$$e^{aXa} = e^{a\hat{X}a} \ , \ e^{a^\dagger X a^\dagger} = e^{a^\dagger \hat{X} a^\dagger} \ ,$$

one finds

$$\begin{aligned} & e^{t\left(a^\dagger A a^\dagger + a^\dagger B a + aCa\right)} = e^{-\frac{t}{2}\,\mathrm{Tr}(B)} e^{R_{A_4(t)}} e^{\rho_{A_5(t)}} e^{\Delta_{A_6(t)}} \\ =& e^{-\frac{t}{2}\,\mathrm{Tr}(B)} e^{R_{f_t(A,B,C)}} e^{\rho_{g_t(A,B,C)}} e^{\Delta_{h_t(A,B,C)}} \\ =& e^{-\frac{t}{2}\,\mathrm{Tr}(B)} e^{a^\dagger \frac{1}{2} f_t(A,B,C) a^\dagger} e^{a^\dagger g_t(A,B,C) a + \frac{1}{2}\mathrm{Tr}(g_t(A,B,C))} e^{a \frac{1}{2} h_t(A,B,C)} \\ =& e^{-\frac{t}{2}\,\mathrm{Tr}(B) + \frac{1}{2}\mathrm{Tr}(g_t(A,B,C))} e^{\frac{1}{2} a^\dagger f_t(A,B,C) a^\dagger} e^{a^\dagger g_t(A,B,C) a} e^{\frac{1}{2} a h_t(A,B,C) a} \\ =& e^{-\frac{t}{2}\,\mathrm{Tr}(B) + \frac{1}{2}\mathrm{Tr}(g_t(A,B,C))} e^{\frac{1}{2} a^\dagger \hat{f}_t(A,B,C) a^\dagger} e^{a^\dagger g_t(A,B,C) a} e^{\frac{1}{2} a \hat{h}_t(A,B,C) a} \ . \end{aligned}$$

□

Lemma 3.3. *$S(t)$ and $P(t)$ are invertible for $t \in \mathbb{R}$ with $|t|$ sufficiently small.*

Proof. For any $s,t\in\mathbb{R}$, one has

$$e^{sv}e^{tv}=\begin{pmatrix}P_s & Q_s\\ -R_s & S_s\end{pmatrix}\begin{pmatrix}P_t & Q_t\\ -R_t & S_t\end{pmatrix}=\begin{pmatrix}P_sP_t-Q_sR_t & P_sQ_t+Q_sS_t\\ -R_sP_t-S_sR_t & -R_sQ_t+S_sS_t\end{pmatrix}$$
$$=\begin{pmatrix}P_{s+t} & Q_{s+t}\\ -R_{s+t} & S_{s+t}\end{pmatrix}=e^{(s+t)v}\ .$$

Moreover

$$\begin{pmatrix}P_0 & Q_0\\ -R_0 & S_0\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\ .$$

By continuity, for each $\epsilon>0$ there exists t_ϵ such that, for any $t\in\mathbb{R}$ with $|t|<t_\epsilon$,

$$\|P_t-1\|\ ,\ \|S_t-1\|<\epsilon\ ;\ \|Q_t\|\ ,\ \|R_t\|<\epsilon\ . \tag{3.1}$$

Since the set of invertible elements in $M_n(\mathbb{C})$ is open, ϵ can be chosen so that if $X\in M_n(\mathbb{C})$ is such that

$$\|X-1\|<\epsilon \tag{3.2}$$

then X is invertible. In particular, if ϵ is as in (3.2) and P_s and S_s satisfy (3.1), then they are invertible. □

Remark 3.4. In general P_t and S_t **are not invertible for each** $t\in\mathbb{R}$. For example, taking

$$A=\frac{1}{2}i\mathbf{I}\quad ,\quad C=-\frac{1}{2}i\mathbf{I}=A^*\quad ,\quad B=0\ ,$$

$\mathbf{I}$ being the identity matrix, one finds

$$P\,(t)=S\,(t)=\cos{(t)}\,\mathbf{I}\qquad ,\qquad \forall t\ ,$$

which is identically zero for $t=\left(k+\frac{1}{2}\right)\pi$ for any $k\in\mathbb{Z}$.

Remark 3.5. If $A=0$ then v and e^{tv} are both lower triangular, therefore $Q=0$, so $f_t(0,B,C)=0$ as well. Similarly, if $C=0$ then v and e^{tv} are both upper triangular, therefore $R=0$, so $h_t(A,B,0)=0$ as well.

Notation 1. For $t=1$ we will denote the functions f_t,g_t,h_t of Lemma 3.1 by f,g,h respectively.

Notation 2. For $x\in\mathbb{C}$ we denote $e^{x\mathbf{1}}$ by just e^x.

Notation 3. In view of Lemma 3.1,

$$E=W(x,A,B,C)=G(x',A',B',C')$$

where

$$x'=x-\frac{1}{2}\operatorname{Tr}(B)+\frac{1}{2}\operatorname{Tr}\left(g(A,B,C)\right)$$
$$A'=\frac{1}{2}\hat{f}(A,B,C)\ ;\ B'=g(A,B,C)\ ;\ C'=\frac{1}{2}\hat{h}(A,B,C)\ .$$

Remark 3.6. By the Baker- Campbell -Hausdorff formula (see [10]), for all $M,N\in M_n(\mathbb{C})$,

$$e^Me^N=e^{M+\int_0^1 g\left(e^{\mathrm{ad}_M}e^{t\mathrm{ad}_N}\right)(N)\,dt}=e^{M+N+BCH(M,N)}\ ,$$

where

$$g(z)=\frac{\log z}{1-\frac{1}{z}}\ ,$$

and

$$BCH(M,N)=\frac{1}{2}[M,N]+\frac{1}{12}\left([M,[M,N]]+[N,[N,M]]\right)-\frac{1}{24}[N,[M,[M,N]]] + \text{ higher order commutators....}\ ,$$

(see Chapter 5 of [10]).

Lemma 3.7. *For $M,N\in M_n(\mathbb{C})$,*

$$e^{aMa}e^{a^\dagger Na^\dagger}=e^{\frac{1}{2}Tr\left(g\left(N,4NM,2\left(MNM+(MNM)^T\right)\right)\right)} \cdot e^{\frac{1}{2}a^\dagger \hat{f}\left(N,4NM,2\left(MNM+(MNM)^T\right)\right)a^\dagger} \cdot e^{a^\dagger g\left(N,4NM,2\left(MNM+(MNM)^T\right)\right)a} \cdot e^{\frac{1}{2}a\hat{h}\left(N,4NM,2\left(MNM+(MNM)^T\right)\right)a}\ , \tag{3.3}$$

$$e^{aMa}e^{a^\dagger Na}=e^{Tr\left(-\frac{1}{2}N+\frac{1}{2}g\left(0,N,MN+(MN)^T\right)\right)}\ , \cdot e^{a^\dagger g\left(0,N,MN+(MN)^T\right)a}\cdot e^{a\left(M+\frac{1}{2}\hat{h}\left(0,N,MN+(MN)^T\right)\right)a}\ , \tag{3.4}$$

$$e^{a^\dagger Ma}e^{a^\dagger Na^\dagger}=e^{Tr\left(-\frac{1}{2}M+\frac{1}{2}g\left(MN+(MN)^T,M,0\right)\right)} \cdot e^{a^\dagger\left(N+\frac{1}{2}\hat{f}\left(MN+(MN)^T,M,0\right)\right)a^\dagger}\cdot e^{a^\dagger g\left(MN+(MN)^T,M,0\right)a}\ , \tag{3.5}$$

where in (3.3) $M,N\in M_{n,sym}(\mathbb{C})$, in (3.4) $M\in Sym\left(M_n(\mathbb{C})\right)$, and in (3.5) $N\in M_{n,sym}(\mathbb{C})$. In the B-notation, (3.3)-(3.5) take the form

$$e^{B_2^0(M)}e^{B_0^2(N)}=e^{\frac{1}{2}Tr\left(g\left(N,4NM,2\left(MNM+(MNM)^T\right)\right)\right)} \cdot e^{\frac{1}{2}B_0^2\left(\hat{f}\left(N,4NM,2\left(MNM+(MNM)^T\right)\right)\right)} \cdot e^{B_1^1\left(g\left(N,4NM,2\left(MNM+(MNM)^T\right)\right)\right)} \cdot e^{\frac{1}{2}B_2^0\left(\hat{h}\left(N,4NM,2\left(MNM+(MNM)^T\right)\right)\right)}\ , \tag{3.6}$$

$$e^{B_2^0(M)}e^{B_1^1(N)}=e^{Tr\left(-\frac{1}{2}N+\frac{1}{2}g\left(0,N,MN+(MN)^T\right)\right)} \cdot e^{B_1^1\left(g\left(0,N,MN+(MN)^T\right)\right)}\cdot e^{B_2^0\left(M+\frac{1}{2}\hat{h}\left(0,N,MN+(MN)^T\right)\right)}\ , \tag{3.7}$$

$$e^{B_1^1(M)}e^{B_0^2(N)}=e^{Tr\left(-\frac{1}{2}M+\frac{1}{2}g\left(MN+(MN)^T,M,0\right)\right)}\ , \cdot e^{B_0^2\left(N+\frac{1}{2}\hat{f}\left(MN+(MN)^T,M,0\right)\right)}\cdot e^{B_1^1\left(g\left(MN+(MN)^T,M,0\right)\right)}\ , \tag{3.8}$$

where f,g,h are as in Notation 1 and $\hat{f},\hat{h}$ denote the symmetric parts of f,h.

Proof. Using Lemma 7.1 we have

$$e^{aMa}e^{a^\dagger Na^\dagger}=\left(e^{aMa}e^{a^\dagger Na^\dagger}e^{-aMa}\right)e^{aMa} =e^{a^\dagger Na^\dagger+[aMa,a^\dagger Na^\dagger]+\frac{1}{2!}[aMa,[aMa,a^\dagger Na^\dagger]]+\cdots}e^{aMa}\ .$$

By Lemma 2.1,

$$
\begin{aligned}
[aMa, a^\dagger Na^\dagger] =& 2\,\mathrm{Tr}(MN) + 4\,a^\dagger NMa\ ,\\
[aMa, [aMa, a^\dagger Na^\dagger]] =& [aMa, 2\,\mathrm{Tr}(MN) + 4\,a^\dagger NMa]\\
=& 4[aMa, a^\dagger NMa] = 4a\left(MNM + (MNM)^T\right)a\ ,
\end{aligned}
$$

and by (2.6) all higher order commutators in the exponent are zero. Thus

$$
e^{aMa}e^{a^\dagger Na^\dagger} = e^{2\,\mathrm{Tr}(MN)}e^{a^\dagger Na^\dagger + a^\dagger 4NMa + a\,2\left(MNM+(MNM)^T\right)a}e^{aMa}\ ,
$$

which, using Lemma 3.1 to split the middle exponential, yields (3.6). Similarly, to prove (3.7) we notice that, by 2.1,

$$
\begin{aligned}
e^{aMa}e^{a^\dagger Na} =& \left(e^{aMa}e^{a^\dagger Na}e^{-aMa}\right)e^{aMa}\\
=& e^{a^\dagger Na + [aMa, a^\dagger Na] + \frac{1}{2!}[aMa,[aMa,a^\dagger Na]] + \cdots}e^{aMa}\ .
\end{aligned}
$$

By Lemma 2.1,

$$
[aMa, a^\dagger Na] = a\left(MN + (NM)^T\right)a\ ,
$$

so $[aMa, [aMa, a^\dagger Na]]$ and all higher order commutators in the exponent are all equal to zero. Thus

$$
e^{aMa}e^{a^\dagger Na} = e^{a^\dagger Na + a\left(MN+(NM)^T\right)a}e^{aMa}
$$

which, using Lemma 3.1 to split the exponential, yields

$$
\begin{aligned}
e^{aMa}e^{a^\dagger Na} =& e^{\mathrm{Tr}\left(-\frac{1}{2}N + \frac{1}{2}g\left(0,N,MN+(MN)^T\right)\right)} \cdot e^{\frac{1}{2}a^\dagger \hat{f}\left(0,N,MN+(MN)^T\right)a^\dagger}\\
&\cdot e^{a^\dagger g\left(0,N,MN+(MN)^T\right)a} \cdot e^{a\left(M+\frac{1}{2}\hat{h}\left(0,N,MN+(MN)^T\right)\right)a}\ ,
\end{aligned}
$$

from which (3.7) follows with the use of Remark 3.5. Finally, to prove (3.8) we notice that by 2.1,

$$
\begin{aligned}
e^{a^\dagger Ma}e^{a^\dagger Na^\dagger} =& e^{a^\dagger Na^\dagger}\left(e^{-a^\dagger Na^\dagger}e^{a^\dagger Ma}e^{a^\dagger Na^\dagger}\right)\\
=& e^{a^\dagger Na^\dagger}e^{a^\dagger Ma + [-a^\dagger Na^\dagger, a^\dagger Ma] + \frac{1}{2!}[-a^\dagger Na^\dagger,[-a^\dagger Na^\dagger,a^\dagger Ma]] + \cdots}\\
=& e^{a^\dagger Na^\dagger}e^{a^\dagger Ma + [a^\dagger Ma, a^\dagger Na^\dagger] + \frac{1}{2!}[a^\dagger Na^\dagger,[a^\dagger Na^\dagger,a^\dagger Ma]] + \cdots}\ .
\end{aligned}
$$

By Lemma 2.1,

$$
[a^\dagger Ma, a^\dagger Na^\dagger] = a^\dagger\left(MN + (NM)^T\right)a^\dagger\ ,
$$

so $[a^\dagger Na^\dagger, [a^\dagger Na^\dagger, a^\dagger Ma]]$ and all higher order commutators in the exponent are all equal to zero. Thus

$$
e^{a^\dagger Ma}e^{a^\dagger Na^\dagger} = e^{a^\dagger Na^\dagger}e^{a^\dagger\left(MN+(NM)^T\right)a^\dagger + a^\dagger Ma}\ ,
$$

which, using Lemma 3.1 to split the exponential, yields

$$
\begin{aligned}
e^{a^\dagger Ma}e^{a^\dagger Na^\dagger} =& e^{a^\dagger Na^\dagger}e^{\mathrm{Tr}\left(-\frac{1}{2}M + \frac{1}{2}g\left(MN+(MN)^T,M,0\right)\right)} \cdot e^{\frac{1}{2}a^\dagger \hat{f}\left(MN+(MN)^T,M,0\right)a^\dagger}\\
&\cdot e^{a^\dagger g\left(MN+(MN)^T,M,0\right)a} \cdot e^{\frac{1}{2}a\hat{h}\left(MN+(MN)^T,M,0\right)a}\ ,
\end{aligned}
$$

from which (3.7) follows with the use of Remark 3.5. □

Lemma 3.8. *For $M, N \in M_n(\mathbb{C})$,*

$$e^{a^\dagger M a^\dagger} e^{a^\dagger N a^\dagger} = e^{a^\dagger (M+N) a^\dagger} \ , \tag{3.9}$$

$$e^{aMa} e^{aNa} = e^{a(M+N)a} \ , \tag{3.10}$$

$$e^{a^\dagger M a} e^{a^\dagger N a} = e^{a^\dagger (M+N+BCH(M,N))a} \ , \tag{3.11}$$

or in the B-notation

$$e^{B_0^2(M)} e^{B_0^2(N)} = e^{B_0^2(M+N)} \ ,$$

$$e^{B_2^0(M)} e^{B_2^0(N)} = e^{B_2^0(M+N)} \ ,$$

$$e^{B_1^1(M)} e^{B_1^1(N)} = e^{B_1^1(M+N+BCH(M,N))} \ ,$$

where $BCH(M,N)$ is as in Remark 3.6.

Proof. For (3.11), by the Baker-Campbell-Hausdorff formula of Remark 3.6, we have

$$e^{a^\dagger M a} e^{a^\dagger N a} = e^{a^\dagger M a + a^\dagger N a + BCH(a^\dagger M a, a^\dagger N a)} \ .$$

Using (2.5) we see that

$$BCH(a^\dagger M a, a^\dagger N a) = a^\dagger BCH(M,N) a \ .$$

Thus,

$$e^{a^\dagger M a} e^{a^\dagger N a} = e^{a^\dagger (M+N+BCH(M,N))a} \ .$$

The proof of (3.9) and (3.10) is similar, with the BCH term equal to 0 in both cases by (2.6). □

4. The Group Law in Coordinates of the Second Kind

The homogeneous quadratic Weyl operators are group elements of $heis_{\mathbb{C}}(2;n)$ in coordinates of the second kind. In order to calculate their vacuum expectation values, i.e. the characteristic function of the vacuum distribution of the corresponding hermitian elements of $heis_{\mathbb{C}}(2;n)$, we have to find the transition formula, from coordinates of the second kind to coordinates of the first kind (which, in physical language corresponds to find the **normally ordered form** of these expressions). This is done in this and the next section.

Theorem 4.1. *In the notation of Section 1, let $x_i \in \mathbb{C}$, $A_i, C_i \in M_{n,sym}(\mathbb{C})$ and $B_i \in M_n(\mathbb{C})$, $i = 1, 2$. Then*

$$G(x_1, A_1, B_1, C_1) G(x_2, A_2, B_2, C_2) = G(x, A, B, C) \ , \tag{4.1}$$

where the coordinates x, A, B, C are given by,

$$
\begin{aligned}
x =& x_1 + x_2 + \frac{1}{2} Tr\big(-(B_1 + B_2) + g\left(A_2, 4A_2C_1, 2(C_1A_2C_1 + (C_1A_2C_1)^T)\right) \\
&+ g\left(B_1X + (B_1X)^T, B_1, 0\right) + g\left(0, B_2, ZB_2 + (ZB_2)^T\right)\big) \ ,
\end{aligned}
$$

$$
A = X + A_1 + \frac{1}{2}\hat{f}\left(B_1X + (B_1X)^T, B_1, 0\right) \ ,
$$

$$
\begin{aligned}
B =& Y + g\left(B_1X + (B_1X)^T, B_1, 0\right) + g\left(0, B_2, ZB_2 + (ZB_2)^T\right) \\
&+ BCH\left(g\left(B_1X + (B_1X)^T, B_1, 0\right), Y\right) \\
&+ BCH\left(E, g\left(0, B_2, ZB_2 + (ZB_2)^T\right)\right) \ ,
\end{aligned}
$$

$$
C = C_2 + Z + \frac{1}{2}\hat{h}\left(0, B_2, ZB_2 + (ZB_2)^T\right) \ ,
$$

$$
E = Y + g\left(B_1X + (B_1X)^T, B_1, 0\right) + BCH\left(g\left(B_1X + (B_1X)^T, B_1, 0\right), Y\right) \ ,
$$

and

$$
X = \frac{1}{2}\hat{f}\left(A_2, 4A_2C_1, 2\left(C_1A_2C_1 + (C_1A_2C_1)^T\right)\right) \ , \tag{4.2}
$$

$$
Y = g\left(A_2, 4A_2C_1, 2\left(C_1A_2C_1 + (C_1A_2C_1)^T\right)\right) \ , \tag{4.3}
$$

$$
Z = \frac{1}{2}\hat{h}\left(A_2, 4A_2C_1, 2\left(C_1A_2C_1 + (C_1A_2C_1)^T\right)\right) \ . \tag{4.4}
$$

Proof. Computing $e^{aC_1a}e^{a^\dagger A_2 a^\dagger}$ with the use of Lemma 3.7, we obtain

$$
\begin{aligned}
&G(x_1, A_1, B_1, C_1)G(x_2, A_2, B_2, C_2) \\
&= e^{x_1+x_2} e^{a^\dagger A_1 a^\dagger} e^{a^\dagger B_1 a} e^{aC_1a} e^{a^\dagger A_2 a^\dagger} e^{a^\dagger B_2 a} e^{aC_2a} \\
&= e^{x_1+x_2} e^{a^\dagger A_1 a^\dagger} e^{a^\dagger B_1 a} \cdot e^{\mathrm{Tr}\left(\frac{1}{2} g\left(A_2, 4A_2C_1, 2\left(C_1A_2C_1 + (C_1A_2C_1)^T\right)\right)\right)} \\
&\quad \cdot e^{\frac{1}{2} a^\dagger \hat{f}\left(A_2, 4A_2C_1, 2\left(C_1A_2C_1 + (C_1A_2C_1)^T\right)\right) a^\dagger} \\
&\quad \cdot e^{a^\dagger g\left(A_2, 4A_2C_1, 2\left(C_1A_2C_1 + (C_1A_2C_1)^T\right)\right) a} \\
&\quad \cdot e^{\frac{1}{2} a \hat{h}\left(A_2, 4A_2C_1, 2\left(C_1A_2C_1 + (C_1A_2C_1)^T\right)\right) a} e^{a^\dagger B_2 a} e^{aC_2a} \\
&= e^{x_1+x_2+\mathrm{Tr}\left(\frac{1}{2} g\left(A_2, 4A_2C_1, 2\left(C_1A_2C_1 + (C_1A_2C_1)^T\right)\right)\right)} \\
&\quad \cdot e^{a^\dagger A_1 a^\dagger} e^{a^\dagger B_1 a} e^{a^\dagger X a^\dagger} e^{a^\dagger Y a} e^{aZa} e^{a^\dagger B_2 a} e^{aC_2a} \ ,
\end{aligned}
$$

where X, Y, Z are as in (4.2)-(4.4). Computing $e^{a^\dagger B_1 a} e^{a^\dagger X a^\dagger}$ and $e^{aZa} e^{a^\dagger B_2 a}$ with the use of Lemma 3.7, we obtain

$$
\begin{aligned}
G(x_1,&A_1, B_1, C_1)G(x_2, A_2, B_2, C_2)\\
&= e^{x_1+x_2+\ \mathrm{Tr}\left(\frac{1}{2}g\left(A_2,4A_2C_1,2\left(C_1A_2C_1+(C_1A_2C_1)^T\right)\right)\right)} e^{a^\dagger A_1 a^\dagger}\\
&\quad \cdot e^{\mathrm{Tr}\left(-\frac{1}{2}B_1+\frac{1}{2}g\left(B_1X+(B_1X)^T,B_1,0\right)\right)} e^{a^\dagger\left(\frac{1}{2}\hat{f}\left(B_1X+(B_1X)^T,B_1,0\right)+X\right)a^\dagger}\\
&\quad \cdot e^{a^\dagger g\left(B_1X+(B_1X)^T,B_1,0\right)a} e^{a^\dagger Y a} e^{\mathrm{Tr}\left(-\frac{1}{2}B_2+\frac{1}{2}g\left(0,B_2,ZB_2+(ZB_2)^T\right)\right)}\\
&\quad \cdot e^{a^\dagger g\left(0,B_2,ZB_2+(ZB_2)^T\right)a} e^{a\left(Z+\frac{1}{2}\hat{h}\left(0,B_2,ZB_2+(ZB_2)^T\right)\right)a} e^{aC_2a}\\
&= e^{x_1+x_2+\mathrm{Tr}\left(\frac{1}{2}g\left(A_2,4A_2C_1,2\left(C_1A_2C_1+(C_1A_2C_1)^T\right)\right)\right)}\\
&\quad \cdot e^{\mathrm{Tr}\left(-\frac{1}{2}B_1+\frac{1}{2}g\left(B_1X+(B_1X)^T,B_1,0\right)\right)} e^{\mathrm{Tr}\left(-\frac{1}{2}B_2+\frac{1}{2}g\left(0,B_2,ZB_2+(ZB_2)^T\right)\right)}\\
&\quad \cdot e^{a^\dagger A_1 a^\dagger} e^{a^\dagger\left(\frac{1}{2}\hat{f}\left(B_1X+(B_1X)^T,B_1,0\right)+X\right)a^\dagger}\\
&\quad \cdot e^{a^\dagger g\left(B_1X+(B_1X)^T,B_1,0\right)a} e^{a^\dagger Y a} e^{a^\dagger g\left(0,B_2,ZB_2+(ZB_2)^T\right)a}\\
&\quad \cdot e^{a\left(Z+\frac{1}{2}\hat{h}\left(0,B_2,ZB_2+(ZB_2)^T\right)\right)a} e^{aC_2a}\ ,
\end{aligned}
$$

from which (4.1) follows by combining the exponentials with the use of Lemma 3.8. □

5. The Group Law in Coordinates of the First Kind

Theorem 5.1. *In the notation of Section 1, let $x_i \in \mathbb{C}$, $A_i, C_i \in M_{n,sym}(\mathbb{C})$ and $B_i \in M_n(\mathbb{C})$, $i = 1, 2$. Then*

$$W(x_1, A_1, B_1, C_1)W(x_2, A_2, B_2, C_2) = W(x, A, B, C)\ ,$$

where x, A, B, C are determined by the system

$$
\begin{aligned}
x' &= x - \frac{1}{2}\, Tr(B) + \frac{1}{2}\, Tr\left(g(A, B, C)\right)\ ,\\
A' &= \frac{1}{2}\hat{f}(A, B, C)\ ,\\
B' &= g(A, B, C)\ ,\\
C' &= \frac{1}{2}\hat{h}(A, B, C)\ ,
\end{aligned}
$$

where

$$
\begin{aligned}
x' =& x_1 + x_2 + \frac{1}{2} Tr\big(- (L_1 + L_2) + g\left(K_2, 4K_2M_1, 2(M_1K_2M_1 + (M_1K_2M_1)^T)\right) \\
&+ g\left(L_1X + (L_1X)^T, L_1, 0\right) + g\left(0, L_2, ZL_2 + (ZL_2)^T)\right) \\
&+ \frac{1}{2} Tr(-(B_1 + B_2) + g(A_1, B_1, C_1) + g(A_2, B_2, C_2)) \ ,
\end{aligned}
$$

$$
A' = X + K_1 + \frac{1}{2}\hat{f}\left(L_1X + (L_1X)^T, L_1, 0\right) \ ,
$$

$$
\begin{aligned}
B' =& Y + g\left(L_1X + (L_1X)^T, L_1, 0\right) + g\left(0, L_2, ZL_2 + (ZL_2)^T\right) \\
&+ BCH\left(g\left(L_1X + (L_1X)^T, L_1, 0\right), Y\right) \\
&+ BCH\left(E, g\left(0, L_2, ZL_2 + (ZL_2)^T\right)\right) \ ,
\end{aligned}
$$

$$
C' = M_2 + Z + \frac{1}{2}\hat{h}\left(0, L_2, ZL_2 + (ZL_2)^T\right) \ ,
$$

and

$$
E = Y + g\left(L_1X + (L_1X)^T, L_1, 0\right) + BCH\left(g\left(L_1X + (L_1X)^T, L_1, 0\right), Y\right) \ ,
$$

$$
X = \frac{1}{2}\hat{f}\left(K_2, 4K_2M_1, 2\left(M_1K_2M_1 + (M_1K_2M_1)^T\right)\right) \ ,
$$

$$
Y = g\left(K_2, 4K_2M_1, 2\left(M_1K_2M_1 + (M_1K_2M_1)^T\right)\right) \ ,
$$

$$
Z = \frac{1}{2}\hat{h}\left(K_2, 4K_2M_1, 2\left(M_1K_2M_1 + (M_1K_2M_1)^T\right)\right) \ ,
$$

where, for $i = 1, 2$,

$$
K_i = \frac{1}{2}\hat{f}(A_i, B_i, C_i) \ ; \ L_i = g(A_i, B_i, C_i) \ ; \ M_i = \frac{1}{2}\hat{h}(A_i, B_i, C_i) \ .
$$

Proof. We have

$$
\begin{aligned}
W(x_1, A_1, B_1, C_1)&W(x_2, A_2, B_2, C_2) \\
&= e^{x_1\mathbf{1} + a^\dagger A_1 a^\dagger + a^\dagger B_1 a + aC_1 a} e^{x_2\mathbf{1} + a^\dagger A_2 a^\dagger + a^\dagger B_2 a + aC_2 a} \\
&= e^{x_1 + x_2} e^{a^\dagger A_1 a^\dagger + a^\dagger B_1 a + aC_1 a} e^{a^\dagger A_2 a^\dagger + a^\dagger B_2 a + aC_2 a} \ .
\end{aligned}
$$

Splitting the exponentials with the use of Lemma 3.1, we obtain

$$\begin{aligned}
&W(x_1,A_1,B_1,C_1)W(x_2,A_2,B_2,C_2)\\
&\quad=e^{x_1+x_2+\mathrm{Tr}\left(-\frac{1}{2}(B_1+B_2)+\frac{1}{2}g(A_1,B_1,C_1)+\frac{1}{2}g(A_2,B_2,C_2)\right)}\\
&\qquad\cdot e^{\frac{1}{2}a^\dagger\hat{f}(A_1,B_1,C_1)a^\dagger}e^{a^\dagger g(A_1,B_1,C_1)a}e^{\frac{1}{2}a\hat{h}(A_1,B_1,C_1)a}\\
&\qquad\cdot e^{\frac{1}{2}a^\dagger\hat{f}(A_2,B_2,C_2)a^\dagger}e^{a^\dagger g(A_2,B_2,C_2)a}e^{\frac{1}{2}a\hat{h}(A_2,B_2,C_2)a}\\
&\quad=e^{\frac{1}{2}\mathrm{Tr}(-(B_1+B_2)+g(A_1,B_1,C_1)+g(A_2,B_2,C_2))}G(x_1,K_1,L_1,M_1)G(x_2,K_2,L_2,M_2)\ ,
\end{aligned}$$

$$K_i=\frac{1}{2}\hat{f}(A_i,B_i,C_i)\ ;\ L_i=g(A_i,B_i,C_i)\ ;\ M_i=\frac{1}{2}\hat{h}(A_i,B_i,C_i)\,,\ i=1,2\ .$$

Thus, by Theorem 4.1,

$$W(x_1,A_1,B_1,C_1)W(x_2,A_2,B_2,C_2)=G(x',A',B',C')$$

where

$$\begin{aligned}
x'=&x_1+x_2+\frac{1}{2}\mathrm{Tr}\left(-(L_1+L_2)+g\left(K_2,4K_2M_1,2(M_1K_2M_1+(M_1K_2M_1)^T)\right)\right.\\
&\left.+g\left(L_1X+(L_1X)^T,L_1,0\right)+g\left(0,L_2,ZL_2+(ZL_2)^T\right)\right)\\
&+\frac{1}{2}\mathrm{Tr}\left(-(B_1+B_2)+g(A_1,B_1,C_1)+g(A_2,B_2,C_2)\right)\ ,
\end{aligned}$$

$$A'=X+K_1+\frac{1}{2}\hat{f}\left(L_1X+(L_1X)^T,L_1,0\right)\ ,$$

$$\begin{aligned}
B'=&Y+g\left(L_1X+(L_1X)^T,L_1,0\right)+g\left(0,L_2,ZL_2+(ZL_2)^T\right)\\
&+BCH\left(g\left(L_1X+(L_1X)^T,L_1,0\right),Y\right)\\
&+BCH\left(E,g\left(0,L_2,ZL_2+(ZL_2)^T\right)\right)\ ,
\end{aligned}$$

$$C'=M_2+Z+\frac{1}{2}\hat{h}\left(0,L_2,ZL_2+(ZL_2)^T\right)\ ,$$

and

$$\begin{aligned}
E=&Y+g\left(L_1X+(L_1X)^T,L_1,0\right)+BCH\left(g\left(L_1X+(L_1X)^T,L_1,0\right),Y\right)\ ,\\
X=&\frac{1}{2}\hat{f}\left(K_2,4K_2M_1,2\left(M_1K_2M_1+(M_1K_2M_1)^T\right)\right)\ ,\\
Y=&g\left(K_2,4K_2M_1,2\left(M_1K_2M_1+(M_1K_2M_1)^T\right)\right)\ ,\\
Z=&\frac{1}{2}\hat{h}\left(K_2,4K_2M_1,2\left(M_1K_2M_1+(M_1K_2M_1)^T\right)\right)\ .
\end{aligned}$$

Therefore, in Notation 3,

$$W(x_1, A_1, B_1, C_1)W(x_2, A_2, B_2, C_2) = W(x, A, B, C) \ ,$$

where x, A, B, C are defined by

$$\begin{aligned} x' =& x - \frac{1}{2}\operatorname{Tr}(B) + \frac{1}{2}\operatorname{Tr}\left(g(A, B, C)\right) \\ A' =& \frac{1}{2}\hat{f}(A, B, C) \ ; \ B' = g(A, B, C) \ ; \ C' = \frac{1}{2}\hat{h}(A, B, C) \ . \end{aligned}$$

□

Example 5.2. For $j = 1, 2$, assuming $x_j\mathbf{1} = A_j = C_j = 0$, by Lemma 3.8 we know that

$$\begin{aligned} W(0, 0, B_1, 0)W(0, 0, B_2, 0) =& e^{a^\dagger B_1 a} e^{a^\dagger B_2 a} = e^{a^\dagger (B_1 + B_2 + BCH(B_1, B_2))a} \\ =& W(0, 0, B_1 + B_2 + BCH(B_1, B_2), 0) \ . \end{aligned}$$

To verify this using the group law, we notice that in the notation of Theorem 5.1 we have

$$K_j = \frac{1}{2}\hat{f}\left(0, B_j, 0\right) \ , \ L_j = g\left(0, B_j, 0\right) \ , \ M_j = \frac{1}{2}\hat{h}\left(0, B_j, 0\right) \ .$$

In the notation of Lemma 3.1,

$$v = \begin{pmatrix} B_j & 0 \\ 0 & -B_j^T \end{pmatrix} \ , \ e^v = \begin{pmatrix} e^{B_j} & 0 \\ 0 & e^{-B_j^T} \end{pmatrix} \ ,$$

i.e.,

$$P = e^{B_j} \, , \, Q = R = 0 \, , \, S = e^{-B_j^T} \ ,$$

so

$$\hat{f}\left(0, B_j, 0\right) = \hat{h}\left(0, B_j, 0\right) = 0 \ ,$$

and

$$g\left(0, B_j, 0\right) = -\log\left(e^{-B_j^T}\right)^T = B_j \ .$$

Thus

$$K_j = M_j = 0 \, , \, L_j = B_j \ ,$$

and, by Remark 3.5,

$$X = \frac{1}{2}\hat{f}\left(0, 0, 0\right) = 0 \, , \, Y = g(0, 0, 0) = 0 \, , \, Z = \frac{1}{2}\hat{h}\left(0, 0, 0\right) = 0 \ ,$$

which imply that $A' = C' = 0$,

$$\begin{aligned} B' =& g\left(0, B_1, 0\right) + g\left(0, B_2, 0\right) + BCH\left(g\left(0, B_1, 0\right), 0\right) \\ & + BCH\left(g\left(0, B_1, 0\right), g\left(0, B_2, 0\right)\right) \\ =& B_1 + B_2 + 0 + BCH\left(B_1, B_2\right) = B_1 + B_2 + BCH\left(B_1, B_2\right) \ , \end{aligned}$$

and

$$\begin{aligned} x' =&\frac{1}{2}\mathrm{Tr}\left(-\left(B_1+B_2\right)+g\left(0,B_1,0\right)+g\left(0,B_2,0\right)\right)\\ &+\frac{1}{2}\mathrm{Tr}\left(-(B_1+B_2)+g(0,B_1,0)+g(0,B_2,0)\right)\\ =&\mathrm{Tr}\left(-\left(B_1+B_2\right)+B_1+B_2\right)=0\ . \end{aligned}$$

Therefore x,A,B,C are determined by the system

$$\begin{aligned} 0 =&x-\frac{1}{2}\,\mathrm{Tr}(B)+\frac{1}{2}\mathrm{Tr}\left(g(A,B,C)\right)\\ 0 =&\frac{1}{2}\hat{f}(A,B,C)=\frac{1}{2}QS^{-1}\\ B_1+B_2+BCH\left(B_1,B_2\right) =&g(A,B,C)=-\log S^T\\ 0 =&\frac{1}{2}\hat{h}(A,B,C)=\frac{1}{2}S^{-1}R\ . \end{aligned}$$

We see that

$$Q=R=0\,,\; S=e^{-(B_1+B_2+BCH(B_1,B_2))^T}\ ,$$

where these new P,Q,R,S are again as in Lemma 3.1. Thus, in the notation of Theorem 5.1,

$$e^v=\begin{pmatrix} P & Q\\ -R & S\end{pmatrix}=\begin{pmatrix} P & 0\\ 0 & e^{-(B_1+B_2+BCH(B_1,B_2))^T}\end{pmatrix}=e^{\begin{pmatrix} B & 2A\\ -2C & -B^T\end{pmatrix}}\ ,$$

which implies

$$A=C=0\,,\; B=B_1+B_2+BCH\left(B_1,B_2\right)\ ,$$

and the equation

$$0=x-\frac{1}{2}\,\mathrm{Tr}(B)+\frac{1}{2}\mathrm{Tr}\left(g(A,B,C)\right)\ ,$$

becomes

$$0=x-\frac{1}{2}\,\mathrm{Tr}\left(B_1+B_2+BCH\left(B_1,B_2\right)\right)+\frac{1}{2}\mathrm{Tr}\left(g(0,B_1+B_2+BCH\left(B_1,B_2\right),0)\right)$$

i.e.,

$$0=x-\frac{1}{2}\,\mathrm{Tr}\left(B_1+B_2+BCH\left(B_1,B_2\right)\right)+\frac{1}{2}\mathrm{Tr}\left(B_1+B_2+BCH\left(B_1,B_2\right)\right)\ ,$$

so

$$x=0\ .$$

Therefore

$$\begin{aligned} W(0,0,B_1,0)W(0,0,B_2,0) &= W(x,A,B,C)\\ &= W(0,0,B_1+B_2+BCH\left(B_1,B_2\right),0)\ . \end{aligned}$$

6. Quadratic Weyl Operators and Vacuum Characteristic Function

In this section we consider the Fock representation of $heis_{\mathbb{C}}(2;n)$ and we apply all the tools developed in the first part of the paper to calculate the vacuum characteristic functions of its hermitian elements, considered as real valued classical random variables, as well as the explicit form of the scalar product of the vacuum cyclic space of $heis_{\mathbb{C}}(2;n)$. Let $\lambda \in \mathbb{R}$, A a symmetric matrix, and B a hermitian matrix. Then

$$H(\lambda, A, B) = \lambda \mathbf{1} + a^\dagger A a^\dagger + a^\dagger B a + a\bar{A}a = \lambda \mathbf{1} + B_0^2(A) + B_1^1(B) + B_2^0(\bar{A}) \ ,$$

is a hermitian operator (in Section (8) we will prove that it is self-adjoint). For A, B as above, we define the unitary *Weyl operator*

$$U(\lambda, A, B) = e^{iH(\lambda,A,B)} = e^{i\left(\lambda\mathbf{1}+a^\dagger A a^\dagger + a^\dagger B a + a\bar{A}a\right)} = e^{i\left(\lambda\mathbf{1}+B_0^2(A)+B_1^1(B)+B_2^0(\bar{A})\right)} \ .$$

Notice that, in the notation of section 1,

$$U(\lambda, A, B) = W(i\lambda, iA, iB, i\bar{A}) \ .$$

The Weyl group multiplication law is

$$\begin{aligned} U(\lambda_1, A_1, B_1)U(\lambda_2, A_2, B_2) =& W(i\lambda_1, iA_1, iB_1, i\bar{A}_1)W(i\lambda_2, iA_2, iB_2, i\bar{A}_2) \\ =& W(i\lambda, iA, iB, i\hat{\bar{A}}) = U(\lambda, A, B) \ , \end{aligned}$$

where λ, A, B are determined by Theorem 5.1.

Theorem 6.1. *If Φ is a normalized Fock vacuum vector, i.e. $a\Phi = 0$ and $\|\Phi\| = 1$, then the vacuum characteristic function of the quantum observable $H(\lambda, A, B)$ is given by*

$$\langle \Phi, U(\lambda, A, B)\Phi\rangle = e^{i\lambda - \frac{1}{2}\, Tr(iB) + \frac{1}{2}\, Tr\left(g(iA,iB,i\bar{A})\right)} \ .$$

Proof. In the context of Notation 3, we have

$$\begin{aligned} \langle \Phi, U(\lambda, A, B)\Phi\rangle =& \langle \Phi, W(i\lambda, iA, iB, i\bar{A})\Phi\rangle \\ =& \langle \Phi, G(x', A', B', C')\Phi\rangle \\ =& \langle \Phi, e^{x'} e^{a^\dagger A' a^\dagger} e^{a^\dagger B' a} e^{aC'a}\Phi\rangle \\ =& e^{x'}\langle \Phi, \Phi\rangle = e^{x'} \ , \end{aligned}$$

where

$$\begin{aligned} x' =& i\lambda - \frac{1}{2}\,\mathrm{Tr}(iB) + \frac{1}{2}\mathrm{Tr}\left(g(iA, iB, i\bar{A})\right) \\ A' =& \frac{1}{2}\hat{f}(iA, iB, i\bar{A}) \ ; \ B' = g(iA, iB, i\bar{A}) \ ; \ C' = \frac{1}{2}\hat{h}(iA, iB, i\bar{A}) \ . \end{aligned}$$

□

Proposition 6.2. *Let $A \in M_{n,sym}(\mathbb{C})$ and let Φ be a normalized Fock vacuum vector. Then*

$$\|e^{a^\dagger A a^\dagger}\Phi\| = e^{\frac{1}{4}\, Tr\left(g\left(A, 4A\bar{A}, 4\bar{A}A\bar{A}\right)\right)} \ .$$

Proof. By (3.3) of Lemma 3.7

$$\begin{aligned}\|e^{a^\dagger A a^\dagger}\Phi\|^2 &= \langle e^{a^\dagger A a^\dagger}\Phi, e^{a^\dagger A a^\dagger}\Phi\rangle = \langle \left(e^{a^\dagger A a^\dagger}\right)^* e^{a^\dagger A a^\dagger}\Phi, \Phi\rangle \\ &= e^{\frac{1}{2}\mathrm{Tr}\left(g\left(A,4A\bar{A},4\bar{A}A\bar{A}\right)\right)} \\ &\quad\cdot \langle e^{\frac{1}{2}a^\dagger \hat{f}\left(A,4A\bar{A},4\bar{A}A\bar{A}\right)a^\dagger} e^{a^\dagger g\left(A,4A\bar{A},4\bar{A}A\bar{A}\right)a} e^{\frac{1}{2}a\hat{h}\left(A,4A\bar{A},4\bar{A}A\bar{A}\right)a}\Phi, \Phi\rangle \\ &= e^{\frac{1}{2}\mathrm{Tr}\left(g\left(A,4A\bar{A},4\bar{A}A\bar{A}\right)\right)} \langle e^{\frac{1}{2}a^\dagger \hat{f}\left(A,4A\bar{A},4\bar{A}A\bar{A}\right)a^\dagger}\Phi, \Phi\rangle \\ &= e^{\frac{1}{2}\mathrm{Tr}\left(g\left(A,4A\bar{A},4\bar{A}A\bar{A}\right)\right)} \langle \Phi, e^{\frac{1}{2}a\left(\hat{f}\left(A,4A\bar{A},4\bar{A}A\bar{A}\right)\right)^* a}\Phi\rangle \\ &= e^{\frac{1}{2}\mathrm{Tr}\left(g\left(A,4A\bar{A},4\bar{A}A\bar{A}\right)\right)} \langle \Phi, \Phi\rangle \\ &= e^{\frac{1}{2}\mathrm{Tr}\left(g\left(A,4A\bar{A},4\bar{A}A\bar{A}\right)\right)} .\end{aligned}$$

□

Proposition 6.3. *Let $A, B \in M_{n,sym}(\mathbb{C})$ and let Φ be a normalized Fock vacuum vector. Then*

$$\langle e^{a^\dagger A a^\dagger}\Phi, e^{a^\dagger B a^\dagger}\Phi\rangle = e^{\frac{1}{2}Tr\left(g\left(A,4A\bar{B},4\bar{B}A\bar{B}\right)\right)} .$$

Proof. By (3.3) of Lemma 3.7

$$\begin{aligned}\langle e^{a^\dagger A a^\dagger}\Phi, e^{a^\dagger B a^\dagger}\Phi\rangle &= \langle \left(e^{a^\dagger B a^\dagger}\right)^* e^{a^\dagger A a^\dagger}\Phi, \Phi\rangle = \langle e^{a\bar{B}a} e^{a^\dagger A a^\dagger}\Phi, \Phi\rangle \\ &= e^{\frac{1}{2}\mathrm{Tr}\left(g\left(A,4A\bar{B},4\bar{B}A\bar{B}\right)\right)} \\ &\quad\cdot \langle e^{\frac{1}{2}a^\dagger \hat{f}\left(A,4A\bar{B},4\bar{B}A\bar{B}\right)a^\dagger} e^{a^\dagger g\left(A,4A\bar{B},4\bar{B}A\bar{B}\right)a} e^{\frac{1}{2}a\hat{h}\left(A,4A\bar{B},4\bar{B}A\bar{B}\right)a}\Phi, \Phi\rangle \\ &= e^{\frac{1}{2}\mathrm{Tr}\left(g\left(A,4A\bar{B},4\bar{B}A\bar{B}\right)\right)} \langle e^{\frac{1}{2}a^\dagger \hat{f}\left(A,4A\bar{B},4\bar{B}A\bar{B}\right)a^\dagger}\Phi, \Phi\rangle \\ &= e^{\frac{1}{2}\mathrm{Tr}\left(g\left(A,4A\bar{B},4\bar{B}A\bar{B}\right)\right)} \langle \Phi, e^{\frac{1}{2}a\left(\hat{f}\left(A,4A\bar{B},4\bar{B}A\bar{B}\right)\right)^* a}\Phi\rangle \\ &= e^{\frac{1}{2}\mathrm{Tr}\left(g\left(A,4A\bar{B},4\bar{B}A\bar{B}\right)\right)} \langle \Phi, \Phi\rangle \\ &= e^{\frac{1}{2}\mathrm{Tr}\left(g\left(A,4A\bar{B},4\bar{B}A\bar{B}\right)\right)} .\end{aligned}$$

□

Proposition 6.4. *Let E be a real projection matrix and let Φ be a normalized Fock vacuum vector. Then for $n = 1, 2, 3, ...$*

$$\| \left(a^\dagger E a^\dagger\right)^n \Phi\|^2 = 4^n n! \left(\frac{TrE}{2}\right)^{(n)} ,$$

where for $x \in \mathbb{R}$,

$$x^{(n)} = x(x+1)(x+2)\cdots(x+n-1) ,$$

is the rising factorial of x, and

$$(x)_n = x(x-1)(x-2)\cdots(x-n+1) ,$$

is the falling factorial of x.

Proof. Using the Lie algebraic identity

$$[X,Y^n]=\sum_{j=0}^{n-1}Y^{n-1-j}[X,Y]Y^j\ ,$$

and Lemma 2.1 we have

$$\begin{aligned}
[a^\dagger Ea,\left(a^\dagger Ea^\dagger\right)^n] &= \sum_{j=0}^{n-1}\left(a^\dagger Ea^\dagger\right)^{n-1-j}[a^\dagger Ea,a^\dagger Ea^\dagger]\left(a^\dagger Ea^\dagger\right)^j\\
&=\sum_{j=0}^{n-1}\left(a^\dagger Ea^\dagger\right)^{n-1-j}2(a^\dagger Ea^\dagger)\left(a^\dagger Ea^\dagger\right)^j\\
&=2(a^\dagger Ea^\dagger)^n\sum_{j=0}^{n-1}1=2n(a^\dagger Ea^\dagger)^n\ ,
\end{aligned}$$

and

$$\begin{aligned}
[aEa,\left(a^\dagger Ea^\dagger\right)^n]\Phi &= \sum_{j=0}^{n-1}\left(a^\dagger Ea^\dagger\right)^{n-1-j}[aEa,a^\dagger Ea^\dagger]\left(a^\dagger Ea^\dagger\right)^j\Phi\\
&=\sum_{j=0}^{n-1}\left(a^\dagger Ea^\dagger\right)^{n-1-j}(2\,\mathrm{Tr}E+4a^\dagger Ea)\left(a^\dagger Ea^\dagger\right)^j\Phi\\
&=2n\,\mathrm{Tr}E\ \left(a^\dagger Ea^\dagger\right)^{n-1}\Phi+4\sum_{j=0}^{n-1}\left(a^\dagger Ea^\dagger\right)^{n-1-j}(a^\dagger Ea)\left(a^\dagger Ea^\dagger\right)^j\Phi\\
&=2n\,\mathrm{Tr}E\ \left(a^\dagger Ea^\dagger\right)^{n-1}\Phi+4\sum_{j=0}^{n-1}\left(a^\dagger Ea^\dagger\right)^{n-1-j}([a^\dagger Ea,\left(a^\dagger Ea^\dagger\right)^j]\\
&\quad+\left(a^\dagger Ea^\dagger\right)^j a^\dagger Ea)\Phi\\
&=2n\,\mathrm{Tr}E\ \left(a^\dagger Ea^\dagger\right)^{n-1}\Phi+4\sum_{j=0}^{n-1}\left(a^\dagger Ea^\dagger\right)^{n-1-j}[a^\dagger Ea,\left(a^\dagger Ea^\dagger\right)^j]\Phi\\
&=2n\,\mathrm{Tr}E\ \left(a^\dagger Ea^\dagger\right)^{n-1}\Phi+4\sum_{j=0}^{n-1}\left(a^\dagger Ea^\dagger\right)^{n-1-j}2j\left(a^\dagger Ea^\dagger\right)^j\Phi\\
&=2n\,\mathrm{Tr}E\ \left(a^\dagger Ea^\dagger\right)^{n-1}\Phi+4n(n-1)\left(a^\dagger Ea^\dagger\right)^{n-1}\Phi\\
&=(2n\,\mathrm{Tr}E+4n(n-1))\left(a^\dagger Ea^\dagger\right)^{n-1}\Phi\ .
\end{aligned}$$

Thus

$$\begin{aligned}
m_n :=& \| \left(a^\dagger E a^\dagger\right)^n \Phi\|^2 \\
=& \langle \left(a^\dagger E a^\dagger\right)^n \Phi, \left(a^\dagger E a^\dagger\right)^n \Phi\rangle \\
=& \langle (aEa) \left(a^\dagger E a^\dagger\right)^n \Phi, \left(a^\dagger E a^\dagger\right)^{n-1} \Phi\rangle \\
=& \langle ([aEa, \left(a^\dagger E a^\dagger\right)^n] + \left(a^\dagger E a^\dagger\right)^n aEa)\Phi, \left(a^\dagger E a^\dagger\right)^{n-1} \Phi\rangle \\
=& \langle [aEa, \left(a^\dagger E a^\dagger\right)^n]\Phi, \left(a^\dagger E a^\dagger\right)^{n-1} \Phi\rangle \\
=& \langle (2n\,\mathrm{Tr}E + 4n(n-1)) \left(a^\dagger E a^\dagger\right)^{n-1} \Phi, \left(a^\dagger E a^\dagger\right)^{n-1} \Phi\rangle \\
=& (2n\,\mathrm{Tr}E + 4n(n-1)) \langle \left(a^\dagger E a^\dagger\right)^{n-1} \Phi, \left(a^\dagger E a^\dagger\right)^{n-1} \Phi\rangle \\
=& (2n\,\mathrm{Tr}E + 4n(n-1)) m_{n-1} \\
& \vdots \\
=& \Pi_{k=1}^n (2k\,\mathrm{Tr}E + 4k(k-1)) m_0 \\
=& \Pi_{k=1}^n (2k\,\mathrm{Tr}E + 4k(k-1)) \\
=& 4^n n! \left(\frac{\mathrm{Tr}E}{2}\right)^{(n)} .
\end{aligned}$$

□

Lemma 6.5. *Let Φ be a normalized Fock vacuum vector and let $n \in \mathbb{N}$. Then, for all $M, N \in Sym\left(\mathbb{R}^{n\times n}\right)$ with $[M,N]=0$,*

$$[a^\dagger N a, \left(a^\dagger M a^\dagger\right)^n] = 2n\, a^\dagger M N a^\dagger \, \left(a^\dagger M a^\dagger\right)^{n-1} ,$$

and

$$\begin{aligned}
[aNa, \left(a^\dagger M a^\dagger\right)^n]\Phi =& 2n\; Tr(NM)\, \left(a^\dagger M a^\dagger\right)^{n-1} \Phi \\
&+ 4n(n-1) \left(a^\dagger M^2 N a^\dagger\right) \left(a^\dagger M a^\dagger\right)^{n-2} \Phi .
\end{aligned}$$

Proof. As in the proof of Proposition 6.4,

$$\begin{aligned}
[a^\dagger N a, \left(a^\dagger M a^\dagger\right)^n] =& \sum_{j=0}^{n-1} \left(a^\dagger M a^\dagger\right)^{n-1-j} [a^\dagger N a, a^\dagger M a^\dagger] \left(a^\dagger M a^\dagger\right)^j \\
=& \sum_{j=0}^{n-1} \left(a^\dagger M a^\dagger\right)^{n-1-j} 2(a^\dagger M N a^\dagger) \left(a^\dagger M a^\dagger\right)^j \\
=& 2\, a^\dagger M N a^\dagger \, \left(a^\dagger M a^\dagger\right)^{n-1} \sum_{j=0}^{n-1} 1 \\
=& 2n\, a^\dagger M N a^\dagger \, \left(a^\dagger M a^\dagger\right)^{n-1} ,
\end{aligned}$$

and

$$
\begin{aligned}
[aNa, \left(a^\dagger Ma^\dagger\right)^n]\Phi &= \sum_{j=0}^{n-1} \left(a^\dagger Ma^\dagger\right)^{n-1-j} [aNa, a^\dagger Ma^\dagger] \left(a^\dagger Ma^\dagger\right)^j \Phi \\
&= \sum_{j=0}^{n-1} \left(a^\dagger Ma^\dagger\right)^{n-1-j} (2\,\mathrm{Tr}(NM) \\
&\quad + 4a^\dagger NMa) \left(a^\dagger Ma^\dagger\right)^j \Phi \\
&=2n\,\mathrm{Tr}(NM)\, \left(a^\dagger Ma^\dagger\right)^{n-1} \Phi \\
&\quad + 4\sum_{j=0}^{n-1} \left(a^\dagger Ma^\dagger\right)^{n-1-j} (a^\dagger NMa) \left(a^\dagger Ma^\dagger\right)^j \Phi \\
&=2n\,\mathrm{Tr}(NM)\, \left(a^\dagger Ma^\dagger\right)^{n-1} \Phi \\
&\quad + 4\sum_{j=0}^{n-1} \left(a^\dagger Ma^\dagger\right)^{n-1-j} [a^\dagger NMa, \left(a^\dagger Ma^\dagger\right)^j]\Phi \\
&=2n\,\mathrm{Tr}(NM)\, \left(a^\dagger Ma^\dagger\right)^{n-1} \Phi \\
&\quad + 4\sum_{j=0}^{n-1} \left(a^\dagger Ma^\dagger\right)^{n-1-j}\ 2j\ \left(a^\dagger M^2Na^\dagger\right) \left(a^\dagger Ma^\dagger\right)^{j-1} \Phi \\
&=2n\,\mathrm{Tr}(NM)\, \left(a^\dagger Ma^\dagger\right)^{n-1} \Phi \\
&\quad + 8\left(a^\dagger M^2Na^\dagger\right) \left(a^\dagger Ma^\dagger\right)^{n-2} \sum_{j=0}^{n-1} j\Phi \\
&=2n\,\mathrm{Tr}(NM)\, \left(a^\dagger Ma^\dagger\right)^{n-1} \Phi \\
&\quad + 4n(n-1)\left(a^\dagger M^2Na^\dagger\right) \left(a^\dagger Ma^\dagger\right)^{n-2} \Phi\ .
\end{aligned}
$$

□

The vacuum cyclic space of $heis_{\mathbb{C}}(2;n)$ is by definition the sub–space of the Fock space obtained by applying to the vacuum vectors all the elements of the polynomial algebra generated by all possible $a^\dagger Ma^\dagger$, $a^\dagger Ha$, aNa. Using the commutation relations and the Fock property, one verifies that this space is the linear span of vectors of the form

$$(a^\dagger M_n a^\dagger)\cdots(a^\dagger M_1 a^\dagger)\Phi\ . \tag{6.1}$$

Moreover, since each M_j has the form $M_j = M_{j;R} + iM_{j;I}$ with $M_{j;R}$ and $M_{j;I}$ having real entries, one can suppose that, in (6.1), all matrices have real entries. Using polarization and the commutativity of the creators, one concludes that the vacuum cyclic space of $heis_{\mathbb{C}}(2;n)$ is the linear span of the vectors of the form

$$\left(a^\dagger Na^\dagger\right)^n \Phi\ , \tag{6.2}$$

when n varies in $\mathbb{N}$ and N varies in $M_{n,sym}(\mathbb{R})$. Therefore the scalar product on the vacuum cyclic space is uniquely determined by the scalar product of vectors of the form (6.2) with N real symmetric.

Proposition 6.6. *Let Φ be a normalized Fock vacuum vector and let $l \in \{1, 2, ..\}$. Then, for all $M, N \in Sym\left(\mathbb{R}^{l\times l}\right)$ with $[M, N] = 0$,*

$$m_n := \langle \left(a^\dagger M a^\dagger\right)^n \Phi, \left(a^\dagger N a^\dagger\right)^n \Phi\rangle \ ,$$

satisfies, for $n = 1, 2, ...$, the recursion

$$m_n = \frac{1}{n}\sum_{k=1}^{n} 2^{2k-1} \left((n)_k\right)^2 \mathrm{Tr}\left((MN)^k\right) m_{n-k} \ ,$$

where

$$m_0 = 1 \ ,$$

and

$$(n)_k = n(n-1)(n-2)\cdots(n-k+1) \ ,$$

is the lowering factorial of n.

Proof. By Lemma 6.5

$$\begin{aligned}
m_n :=&\langle \left(a^\dagger M a^\dagger\right)^n \Phi, \left(a^\dagger N a^\dagger\right)^n \Phi\rangle \\
=&\langle (aNa) \left(a^\dagger M a^\dagger\right)^n \Phi, \left(a^\dagger N a^\dagger\right)^{n-1} \Phi\rangle \\
=&\langle [aNa, \left(a^\dagger M a^\dagger\right)^n]\Phi, \left(a^\dagger N a^\dagger\right)^{n-1} \Phi\rangle \\
=&\langle 2n\,\mathrm{Tr}(MN) \left(a^\dagger M a^\dagger\right)^{n-1} \Phi, \left(a^\dagger N a^\dagger\right)^{n-1} \Phi\rangle \\
&+ \langle 4n(n-1)\, a^\dagger M^2 N a^\dagger \ \left(a^\dagger M a^\dagger\right)^{n-2} \Phi, \left(a^\dagger N a^\dagger\right)^{n-1} \Phi\rangle \\
=&2n\,\mathrm{Tr}(MN) m_{n-1} + 4n(n-1)\langle \left(a^\dagger M a^\dagger\right)^{n-2} \Phi, aM^2Na \ \left(a^\dagger N a^\dagger\right)^{n-1} \Phi\rangle \\
=&2n\,\mathrm{Tr}(MN) m_{n-1} + 4n(n-1)\langle \left(a^\dagger M a^\dagger\right)^{n-2} \Phi, [aM^2Na, \left(a^\dagger N a^\dagger\right)^{n-1}]\Phi\rangle \ .
\end{aligned}$$

By Lemma 6.5,

$$\begin{aligned}
[aM^2Na, \left(a^\dagger N a^\dagger\right)^{n-1}]\Phi =&2(n-1)\mathrm{Tr}(M^2N^2) \ \left(a^\dagger N a^\dagger\right)^{n-2} \Phi \\
&+ 4(n-2)(n-1)\, a^\dagger M^2 N^3 a^\dagger \ \left(a^\dagger N a^\dagger\right)^{n-3} \Phi \ .
\end{aligned}$$

Thus

$$\begin{aligned} m_n =&2n\,\mathrm{Tr}(MN)m_{n-1} + 4n(n-1)\langle \left(a^\dagger M a^\dagger\right)^{n-2}\Phi, \\ &2(n-1)\mathrm{Tr}(M^2N^2)\ \left(a^\dagger N a^\dagger\right)^{n-2}\Phi \\ &+ 4(n-2)(n-1)\, a^\dagger M^2 N^3 a^\dagger\ \left(a^\dagger N a^\dagger\right)^{n-3}\Phi\rangle \\ =&2n\,\mathrm{Tr}(MN)m_{n-1} + 8n(n-1)^2\mathrm{Tr}(M^2N^2)\, m_{n-2} \\ &+ 16n(n-1)^2(n-2)\langle \left(a^\dagger M a^\dagger\right)^{n-2}\Phi, a^\dagger M^2 N^3 a^\dagger\ \left(a^\dagger N a^\dagger\right)^{n-3}\Phi\rangle \\ =&2n\,\mathrm{Tr}(MN)m_{n-1} + 8n(n-1)^2\mathrm{Tr}(M^2N^2)\, m_{n-2} \\ &+ 16n(n-1)^2(n-2)\langle aM^2N^3a\ \left(a^\dagger M a^\dagger\right)^{n-2}\Phi, \left(a^\dagger N a^\dagger\right)^{n-3}\Phi\rangle \\ =&2n\,\mathrm{Tr}(MN)m_{n-1} + 8n(n-1)^2\mathrm{Tr}(M^2N^2)\, m_{n-2} \\ &+ 16n(n-1)^2(n-2)\langle [aM^2N^3a, \left(a^\dagger M a^\dagger\right)^{n-2}]\Phi, \left(a^\dagger N a^\dagger\right)^{n-3}\Phi\rangle\ . \end{aligned}$$

By Lemma 6.5,

$$\begin{aligned} [aM^2N^3a, \left(a^\dagger M a^\dagger\right)^{n-2}]\Phi =&2(n-2)\mathrm{Tr}(M^3N^3)\ \left(a^\dagger M a^\dagger\right)^{n-3}\Phi \\ &+ 4(n-3)(n-2)\, a^\dagger M^4 N^3 a^\dagger\ \left(a^\dagger M a^\dagger\right)^{n-4}\Phi\ . \end{aligned}$$

Thus, since M, N are real,

$$\begin{aligned} m_n =&2n\,\mathrm{Tr}(MN)m_{n-1} + 8n(n-1)^2\mathrm{Tr}(M^2N^2)\, m_{n-2} \\ &+ 16n(n-1)^2(n-2)\langle 2(n-2)\mathrm{Tr}(M^3N^3)\ \left(a^\dagger M a^\dagger\right)^{n-3}\Phi \\ &+ 4(n-3)(n-2)\, a^\dagger M^4 N^3 a^\dagger\ \left(a^\dagger M a^\dagger\right)^{n-4}\Phi, \left(a^\dagger N a^\dagger\right)^{n-3}\Phi\rangle \\ =&2n\,\mathrm{Tr}(MN)m_{n-1} + 8n(n-1)^2\mathrm{Tr}(M^2N^2)\, m_{n-2} \\ &+ 32n(n-1)^2(n-2)^2\mathrm{Tr}(M^3N^3)\, m_{n-3} \\ &+ 64n(n-1)^2(n-2)^2(n-3)\langle a^\dagger M^4 N^3 a^\dagger\ \left(a^\dagger M a^\dagger\right)^{n-4}\Phi, \left(a^\dagger N a^\dagger\right)^{n-3}\Phi\rangle \\ =&2n\,\mathrm{Tr}(MN)m_{n-1} + 8n(n-1)^2\mathrm{Tr}(M^2N^2)\, m_{n-2} \\ &+ 32n(n-1)^2(n-2)^2\mathrm{Tr}(M^3N^3)\, m_{n-3} \\ &+ 64n(n-1)^2(n-2)^2(n-3)\langle \left(a^\dagger M a^\dagger\right)^{n-4}\Phi, [aM^4N^3a, \left(a^\dagger N a^\dagger\right)^{n-3}]\Phi\rangle\ . \end{aligned}$$

Proceeding in this way, until the $n-4$ in $\left(a^\dagger M a^\dagger\right)^{n-4}\Phi$ is reduced to zero, using the fact that

$$m_0 = \|\Phi\|^2 = 1\ ,$$

we end up with

$$\begin{aligned} m_n =&\frac{1}{n}\sum_{k=1}^{n} 2^{2k-1}\left(n(n-1)(n-2)\cdots(n-k+1)\right)^2\mathrm{Tr}\left((MN)^k\right) m_{n-k} \\ =&\frac{1}{n}\sum_{k=1}^{n} 2^{2k-1}\left((n)_k\right)^2\mathrm{Tr}\left((MN)^k\right) m_{n-k}\ . \end{aligned}$$

□

Proposition 6.7. *Let Φ be a normalized Fock vacuum vector and let $l = 1, 2, ...$ Then, for all $M, N \in Sym\left(\mathbb{R}^{l\times l}\right)$ with $[M, N] = 0$,*

$$\sigma_{n,m} := \langle \left(a^\dagger M a^\dagger\right)^n \Phi, \left(a^\dagger N a^\dagger\right)^m \Phi\rangle \ ,$$

satisfies, for $n, m \in \{1, 2, ...\}$ with $n > m$, the recursion

$$\sigma_{n,m} = \frac{1}{m}\sum_{k=1}^{m-1} 2^{2k-1}(n)_k(m)_k \operatorname{Tr}\left((MN)^k\right)\sigma_{n-k,m-k} \ ,$$

where

$$(x)_k = x(x-1)(x-2)\cdots(x-k+1) \ .$$

Proof. By Lemma 6.5

$$\begin{aligned}
\sigma_{n,m} =&\langle \left(a^\dagger M a^\dagger\right)^n \Phi, \left(a^\dagger N a^\dagger\right)^m \Phi\rangle\\
=&\langle (aNa)\left(a^\dagger M a^\dagger\right)^n \Phi, \left(a^\dagger N a^\dagger\right)^{m-1} \Phi\rangle\\
=&\langle [aNa, \left(a^\dagger M a^\dagger\right)^n]\Phi, \left(a^\dagger N a^\dagger\right)^{m-1} \Phi\rangle\\
=&\langle 2n\operatorname{Tr}(MN)\left(a^\dagger M a^\dagger\right)^{n-1} \Phi, \left(a^\dagger N a^\dagger\right)^{m-1} \Phi\rangle\\
&+\langle 4n(n-1)\, a^\dagger M^2 N a^\dagger\ \left(a^\dagger M a^\dagger\right)^{n-2} \Phi, \left(a^\dagger N a^\dagger\right)^{m-1} \Phi\rangle\\
=&2n\operatorname{Tr}(MN)\sigma_{n-1,m-1} + 4n(n-1)\\
&\cdot\langle \left(a^\dagger M a^\dagger\right)^{n-2} \Phi, aM^2Na\ \left(a^\dagger N a^\dagger\right)^{m-1} \Phi\rangle\\
=&2n\operatorname{Tr}(MN)\sigma_{n-1,m-1} + 4n(n-1)\\
&\cdot\langle \left(a^\dagger M a^\dagger\right)^{n-2} \Phi, [aM^2Na, \left(a^\dagger N a^\dagger\right)^{m-1}]\Phi\rangle \ .
\end{aligned}$$

By Lemma 6.5,

$$\begin{aligned}
[aM^2Na, \left(a^\dagger N a^\dagger\right)^{m-1}]\Phi =&2(m-1)\operatorname{Tr}(M^2N^2)\ \left(a^\dagger N a^\dagger\right)^{m-2} \Phi\\
&+4(m-2)(m-1)\, a^\dagger M^2 N^3 a^\dagger\ \left(a^\dagger N a^\dagger\right)^{m-3} \Phi \ .
\end{aligned}$$

Thus

$$\begin{aligned}
\sigma_{n,m} =&2n\operatorname{Tr}(MN)\sigma_{n-1,m-1} + 4n(n-1)\langle \left(a^\dagger M a^\dagger\right)^{n-2} \Phi,\\
&2(m-1)\operatorname{Tr}(M^2N^2)\ \left(a^\dagger N a^\dagger\right)^{m-2} \Phi\\
&+4(m-2)(m-1)\, a^\dagger M^2 N^3 a^\dagger\ \left(a^\dagger N a^\dagger\right)^{m-3} \Phi\rangle\\
=&2n\operatorname{Tr}(MN)\sigma_{n-1,m-1} + 8n(n-1)(m-1)\operatorname{Tr}(M^2N^2)\,\sigma_{n-2,m-2}\\
&+16n(n-1)(m-1)(m-2)\langle \left(a^\dagger M a^\dagger\right)^{n-2} \Phi, a^\dagger M^2 N^3 a^\dagger\ \left(a^\dagger N a^\dagger\right)^{m-3} \Phi\rangle\\
=&2n\operatorname{Tr}(MN)\sigma_{n-1,m-1} + 8n(n-1)(m-1)\operatorname{Tr}(M^2N^2)\,\sigma_{n-2,m-2}\\
&+16n(n-1)(m-1)(m-2)\langle aM^2N^3a\ \left(a^\dagger M a^\dagger\right)^{n-2} \Phi, \left(a^\dagger N a^\dagger\right)^{m-3} \Phi\rangle\\
=&2n\operatorname{Tr}(MN)\sigma_{n-1,m-1} + 8n(n-1)(m-1)\operatorname{Tr}(M^2N^2)\,\sigma_{n-2,m-2}\\
&+16n(n-1)(m-1)(m-2)\langle [aM^2N^3a, \left(a^\dagger M a^\dagger\right)^{n-2}]\Phi, \left(a^\dagger N a^\dagger\right)^{m-3} \Phi\rangle \ .
\end{aligned}$$

By Lemma 6.5,

$$\begin{aligned}[aM^2N^3a, \left(a^\dagger Ma^\dagger\right)^{n-2}]\Phi =&2(n-2)\mathrm{Tr}(M^3N^3)\ \left(a^\dagger Ma^\dagger\right)^{n-3}\Phi\\ &+4(n-3)(n-2)\,a^\dagger M^4N^3a^\dagger\ \left(a^\dagger Ma^\dagger\right)^{n-4}\Phi\ .\end{aligned}$$

Thus, since M, N are real

$$\begin{aligned}\sigma_{n,m} =&2n\,\mathrm{Tr}(MN)\sigma_{n-1,m-1}+8n(n-1)(m-1)\mathrm{Tr}(M^2N^2)\,\sigma_{n-2,m-2}\\ &+16n(n-1)(m-1)(m-2)\langle 2(n-2)\mathrm{Tr}(M^3N^3)\ \left(a^\dagger Ma^\dagger\right)^{n-3}\Phi\\ &+4(n-3)(n-2)\,a^\dagger M^4N^3a^\dagger\ \left(a^\dagger Ma^\dagger\right)^{n-4}\Phi, \left(a^\dagger Na^\dagger\right)^{m-3}\Phi\rangle\\ =&2n\,\mathrm{Tr}(MN)\sigma_{n-1,m-1}+8n(n-1)(m-1)\mathrm{Tr}(M^2N^2)\,\sigma_{n-2,m-2}\\ &+32n(n-1)(n-2)(m-1)(m-2)\mathrm{Tr}(M^3N^3)\,\sigma_{n-3,m-3}\\ &+64n(n-1)(n-2)(n-3)(m-1)(m-2)\\ &\cdot\langle a^\dagger M^4N^3a^\dagger\ \left(a^\dagger Ma^\dagger\right)^{n-4}\Phi, \left(a^\dagger Na^\dagger\right)^{m-3}\Phi\rangle\\ =&2n\,\mathrm{Tr}(MN)\sigma_{n-1,m-1}+8n(n-1)(m-1)\mathrm{Tr}(M^2N^2)\,\sigma_{n-2,m-2}\\ &+32n(n-1)(n-2)(m-1)(m-2)\mathrm{Tr}(M^3N^3)\,\sigma_{n-3,m-3}\\ &+64n(n-1)(n-2)(n-3)(m-1)(m-2)\\ &\cdot\langle\left(a^\dagger Ma^\dagger\right)^{n-4}\Phi, [aM^4N^3a, \left(a^\dagger Na^\dagger\right)^{m-3}]\Phi\rangle\\ =&\frac{1}{m}\Big(2nm\,\mathrm{Tr}(MN)\sigma_{n-1,m-1}+8n(n-1)m(m-1)\mathrm{Tr}(M^2N^2)\,\sigma_{n-2,m-2}\\ &+32n(n-1)(n-2)m(m-1)(m-2)\mathrm{Tr}(M^3N^3)\,\sigma_{n-3,m-3}\\ &+64n(n-1)(n-2)(n-3)m(m-1)(m-2)\\ &\cdot\langle\left(a^\dagger Ma^\dagger\right)^{n-4}\Phi, [aM^4N^3a, \left(a^\dagger Na^\dagger\right)^{m-3}]\Phi\rangle\Big)\ .\end{aligned}$$

Proceeding in this way, until the $\left(a^\dagger Ma^\dagger\right)^{n-4}\Phi$ term is reduced to $\left(a^\dagger Ma^\dagger\right)^{n-m}\Phi$ and the $\left(a^\dagger Na^\dagger\right)^{m-3}$ term is reduced to $a^\dagger Na^\dagger\Phi$, using the fact that by Lemma 2.1 and the fact that $a\Phi = 0$ and $n > m$,

$$\langle\left(a^\dagger Ma^\dagger\right)^{n-m}\Phi, [aM^mN^{m-1}a, a^\dagger Na^\dagger]\Phi\rangle = 0\ ,$$

we arrive at

$$\sigma_{n,m} = \frac{1}{m}\sum_{k=1}^{m-1} 2^{2k-1}(n)_k(m)_k\mathrm{Tr}\left((MN)^k\right)\sigma_{n-k,m-k}\ .$$

□

7. The Adjoint Action of the Quadratic Lie Group on the Quadratic ∗–Lie Algebra

In this section, we show that, expressing the adjoint representation of $heis_{\mathbb{C}}(2;n)$ in terms of the ∘–operation, in several cases one obtains rather explicit formulae for this action. The following Lemma collects some known formulas that we will need.

Lemma 7.1.

$$e^Y X e^{-Y} = e^{[Y,\ \cdot]} X \ , \tag{7.1}$$

$$e^{[Y^*,\ \cdot]} X = \left(e^{[-Y,\ \cdot]} X^* \right)^* \ . \tag{7.2}$$

Moreover, for any holomorphic function f,

$$e^{[Y,\ \cdot]} f(X) = f\left(e^{[Y,\ \cdot]} X \right) \ . \tag{7.3}$$

Proof. (7.1) is a well known identity. Taking adjoint of both sides of (7.1) one finds

$$e^{-Y^*} X^* e^{Y^*} = e^{[-Y^*,\ \cdot]} X^* \ .$$

Therefore

$$e^{[-Y^*,\ \cdot]} X^* = \left(e^{[Y,\ \cdot]} X \right)^* \ .$$

Exchanging Y to $-Y$ and X to X^* one finds (7.2). (7.3) follows expanding f in power series and using the fact that $e^{[Y,\ \cdot]}$ is an homomorphism. □

Remark 7.2. We will mainly use (7.3) in the form

$$e^Y e^X e^{-Y} = e^{e^{[Y,\ \cdot]} X} \ .$$

Lemma 7.3.

$$e^{[aCa,\ \cdot]} aC'a = aC'a \ , \tag{7.4}$$

$$e^{[aCa,\ \cdot]} a^\dagger Ba = a^\dagger Ba + a(C \circ B)a \ , \tag{7.5}$$

$$e^{[aCa,\ \cdot]} a^\dagger A a^\dagger = a^\dagger A a^\dagger + 4\, a^\dagger ACa + 4a(C \circ (AC))a \ , \tag{7.6}$$

$$e^{[a^\dagger A a^\dagger,\ \cdot]} a^\dagger A' a^\dagger = a^\dagger A' a^\dagger \ , \tag{7.7}$$

$$e^{[a^\dagger A a^\dagger,\ \cdot]} a^\dagger Ba = a^\dagger Ba - a(B \circ A)a \ , \tag{7.8}$$

$$e^{[a^\dagger A a^\dagger,\ \cdot]} aCa = aCa - 4\, Tr(AC) - 4\, a^\dagger (AC)^T a + 4a^\dagger ((AC) \circ A) a^\dagger \ . \tag{7.9}$$

Proof. (7.4) is clear.

$$\begin{aligned} e^{aCa} a^\dagger B a e^{-aCa} &= e^{[aCa,\ \cdot]} a^\dagger Ba \\ &= \sum_{n \geq 0} \frac{1}{n!} [aCa,\ \cdot]^n a^\dagger Ba \\ &= a^\dagger Ba + \sum_{n \geq 1} \frac{1}{n!} [aCa,\ \cdot]^n a^\dagger Ba \\ &= a^\dagger Ba + \sum_{n \geq 1} \frac{1}{n!} [aCa,\ \cdot]^{n-1} [aCa, a^\dagger Ba] \\ &=^{(2.14)} a^\dagger Ba + \sum_{n \geq 1} \frac{1}{n!} [aCa,\ \cdot]^{n-1} a(C \circ B)a \\ &= a^\dagger Ba + a(C \circ B)a + \sum_{n \geq 2} \frac{1}{n!} [aCa,\ \cdot]^{n-1} a(C \circ B)a \\ &= a^\dagger Ba + a(C \circ B)a \ , \end{aligned}$$

which is (7.5). Similarly

$$\begin{aligned}
e^{[aCa,\ \cdot]}a^\dagger A a^\dagger =& \sum_{n\geq 0}\frac{1}{n!}[aCa,\ \cdot]^n a^\dagger A a^\dagger \\
=& a^\dagger A a^\dagger + \sum_{n\geq 1}\frac{1}{n!}[aCa,\ \cdot]^n a^\dagger A a^\dagger \\
=& a^\dagger A a^\dagger + \sum_{n\geq 1}\frac{1}{n!}[aCa,\ \cdot]^{n-1}[aCa, a^\dagger A a^\dagger] \\
=^{(2.13)}& a^\dagger A a^\dagger + \sum_{n\geq 1}\frac{1}{n!}[aCa,\ \cdot]^{n-1}(2\,\mathrm{Tr}(CA) + 4\,a^\dagger ACa) \\
=& a^\dagger A a^\dagger + \sum_{n\geq 1}\frac{1}{n!}[aCa,\ \cdot]^{n-1}(4\,a^\dagger ACa) \\
=& a^\dagger A a^\dagger + 4\,a^\dagger ACa + \sum_{n\geq 2}\frac{1}{n!}[aCa,\ \cdot]^{n-1}(4\,a^\dagger ACa) \\
=& a^\dagger A a^\dagger + 4\,a^\dagger ACa + 4\sum_{n\geq 2}\frac{1}{n!}[aCa,\ \cdot]^{n-2}[aCa, a^\dagger ACa] \\
=^{(2.14)}& a^\dagger A a^\dagger + 4\,a^\dagger ACa + 4\sum_{n\geq 2}\frac{1}{n!}[aCa,\ \cdot]^{n-2}[aCa, a^\dagger ACa] \\
=& a^\dagger A a^\dagger + 4\,a^\dagger ACa + 4\sum_{n\geq 2}\frac{1}{n!}[aCa,\ \cdot]^{n-2}a\left(CAC + (CAC)^T\right)a \\
=^{(2.10)}& a^\dagger A a^\dagger + 4\,a^\dagger ACa + 4\sum_{n\geq 2}\frac{1}{n!}[aCa, \cdot]^{n-2}a(C\circ(AC))a \\
=^{(2.10)}& a^\dagger A a^\dagger + 4\,a^\dagger ACa + 4a(C\circ(AC))a \\
&+ 4\sum_{n\geq 3}\frac{1}{n!}[aCa, \cdot]^{n-2}a(C\circ(AC))a \\
=& a^\dagger A a^\dagger + 4\,a^\dagger ACa + 2a(C\circ(AC))a\ ,
\end{aligned}$$

which is (7.6). Applying (7.2) with $Y = aA^*a$, $X = a^\dagger Ba$ and (7.5) with $C \to -A^*$, $B \to B^*$, one finds

$$\begin{aligned}
e^{[a^\dagger A a^\dagger,\ \cdot]}a^\dagger Ba =& \left(e^{[-aA^*a,\ \cdot]}a^\dagger B^* a\right)^* \\
=& \left(a^\dagger B^* a + a(-A^*)\circ(B^*)a\right)^* \\
=& a^\dagger Ba - a(A^*\circ B^*)^* a \\
=& a^\dagger Ba - a(B\circ A)a\ ,
\end{aligned}$$

which is (7.8). Applying (7.2) with $Y = aA^*a$ and $X = aCa$, one obtains

$$e^{[a^\dagger A a^\dagger,\ \cdot]}aCa = \left(e^{[-aA^*a,\ \cdot]}a^\dagger C^* a^\dagger\right)^*\ . \tag{7.10}$$

Using (7.6) with $C \to -A^*$ and $A \to C^*$, one has

$$\begin{aligned} e^{[-aA^*a,\ \cdot]}a^\dagger C^* a^\dagger =& a^\dagger C^* a^\dagger + 4\, a^\dagger C^*(-A^*)a + 4a((-A^*)\circ(C^*(-A^*)))a \\ =& a^\dagger C^* a^\dagger - 4\, a^\dagger C^* A^* a + 4a(A^*\circ(C^*A^*))a\ , \end{aligned}$$

and, replacing this in the right hand side of (7.10) one finds

$$\begin{aligned} e^{[a^\dagger A a^\dagger,\ \cdot]}aCa =& \left(a^\dagger C^* a^\dagger - 4\, a^\dagger C^* A^* a + 4a(A^*\circ(C^*A^*))a\right)^* \\ =& aCa - 4\, aACa^\dagger + 4a^\dagger((C^*A^*)^*\circ A)a^\dagger \\ =& aCa - 4\, aACa^\dagger + 4a^\dagger((AC)\circ A)a^\dagger \\ =& aCa - 4\,(\mathrm{Tr}(AC) + a^\dagger(AC)^T a) + 4a^\dagger((AC)\circ A)a^\dagger \\ =& aCa - 4\,\mathrm{Tr}(AC) - 4\, a^\dagger(AC)^T a + 4a^\dagger((AC)\circ A)a^\dagger\ , \end{aligned}$$

which is (7.9). □

Lemma 7.4. *Introducing, for $n \in \mathbb{N}$ and $B, C \in M_n(\mathbb{C})$, the inductively defined notations (see* (2.10)*)*

$$C \,\widehat{\circ}\, B^{\widehat{\circ}\,(n+1)} := (C \,\widehat{\circ}\, B^{\widehat{\circ}\,n})\circ B\,,\ B^{\widehat{\circ}\,0}\,\widehat{\circ}\, B := B\ , \tag{7.11}$$

$$C \,\widehat{\circ}\, e^{\widehat{\circ}\,(-B)} := \sum_{n\ge 0}\frac{1}{n!}(-1)^n C\,\widehat{\circ}\, B^{\widehat{\circ}\,n}\ , \tag{7.12}$$

one has

$$e^{[a^\dagger Ba,\ \cdot]}(aCa) = a\left(C\,\widehat{\circ}\, e^{\widehat{\circ}\,(-B)}\right)a\ . \tag{7.13}$$

Proof.

$$\begin{aligned} e^{[a^\dagger Ba,\ \cdot]}aCa =& \sum_{n\ge 0}\frac{1}{n!}[a^\dagger Ba,\ \cdot]^n aCa \\ =& a^\dagger Ba + \sum_{n\ge 1}\frac{1}{n!}[a^\dagger Ba,\ \cdot]^n aCa \\ =& a^\dagger Ba + \sum_{n\ge 1}\frac{1}{n!}[a^\dagger Ba,\ \cdot]^{n-1}[a^\dagger Ba, aCa] \\ =&^{(2.14)} a^\dagger Ba + \sum_{n\ge 1}\frac{1}{n!}[a^\dagger Ba,\ \cdot]^{n-1}(-1)a(C\circ B)a\ . \end{aligned}$$

In order to calculate $[a^\dagger Ba,\ \cdot]^{n-1}(-1)a(C\circ B)a$, note that using the notation (7.11),

$$\begin{aligned} [a^\dagger Ba,\ \cdot]^2 aCa =& (-1)[a^\dagger Ba, a(C\circ B)a] \\ =& (-1)^2[a^\dagger Ba, a((C\circ B)\circ B)a] \\ =& a((-1)^2 C\circ B^{\circ 2})a\ . \end{aligned}$$

Suppose by induction that

$$[aBa,\ \cdot]^n(a^\dagger Ca) = a((-1)^n C\,\widehat{\circ}\, B^{\widehat{\circ}\,n})a\ . \tag{7.14}$$

Then

$$\begin{aligned}[aBa,\ \cdot]^{n+1}(a^\dagger Ca) =&[aBa,[aBa,\ \cdot]^n(a^\dagger Ca)]\\ =&[aBa,a((-1)^n C\,\widehat{\circ}\,B^{\widehat{\circ}\,n})a]\\ =&^{(2.14)}a((-1)((-1)^n C\,\widehat{\circ}\,B^{\widehat{\circ}\,n})\circ B)a\\ =&a((-1)^{n+1}C\circ B^{\circ(n+1)})a\ .\end{aligned}$$

Therefore by induction (7.14) holds for each $n\in\mathbb{N}$. This implies, in the notation (7.12),

$$\begin{aligned}e^{[a^\dagger Ba,\ \cdot]}(aCa) =&\sum_{n\geq 0}\frac{1}{n!}[a^\dagger Ba,\ \cdot]^n(aCa)\\ =&\sum_{n\geq 0}\frac{1}{n!}a((-1)^n C\,\widehat{\circ}\,B^{\widehat{\circ}\,n})a\\ =&a\left(\sum_{n\geq 0}\frac{1}{n!}(-1)^n C\,\widehat{\circ}\,B^{\widehat{\circ}\,n}\right)a\\ =&a\left(C\,\widehat{\circ}\,e^{\widehat{\circ}\,(-B)}\right)a\ ,\end{aligned}$$

which is (7.13). □

Remark 7.5. Note the big difference between the symbols $\circ$ and $\widehat{\circ}$. The former is a binary operation, non–commutative and non–associative, but bi–linear and distributive in the two factors and well behaved with respect to the adjoint (see (2.12)). The latter is **a purely symbolic notation** that has only a global meaning. In particular **it is not distributive**. The following Lemma shows however that the notation $\widehat{\circ}$ is well behaved with respect to the adjoint.

Lemma 7.6. *Introducing, for $n\in\mathbb{N}$ and $B,C,G,H\in M_n(\mathbb{C})$, the inductively defined notations (see* (2.10)*)*

$$\begin{aligned}B^{\widehat{\circ}\,(n+1)}\,\widehat{\circ}\,C :=&B\circ(B^{\widehat{\circ}\,n}\,\widehat{\circ}\,C)\,,\ B^{\widehat{\circ}\,0}\,\widehat{\circ}\,C:=C\ ,\\ e^{\widehat{\circ}\,(-B)}\,\widehat{\circ}\,C :=&\sum_{n\geq 0}\frac{1}{n!}(-1)^n B^{\widehat{\circ}\,n}\,\widehat{\circ}\,C\ ,\end{aligned}$$

the following identities hold:

$$\left(G\,\widehat{\circ}\,e^{\widehat{\circ}\,(-H)}\right)^* = e^{\widehat{\circ}\,(-H^*)}\,\widehat{\circ}\,G^*\ , \tag{7.15}$$

$$e^{[a^\dagger Ba,\ \cdot]}a^\dagger A a^\dagger = \left(e^{[-a^\dagger B^* a,\ \cdot]}aA^*a\right)^* = a^\dagger\left(e^{\circ(-B)}\,\widehat{\circ}\,A\right)a^\dagger\ , \tag{7.16}$$

$$e^{[a^\dagger Ba,\ \cdot]}a^\dagger B' a = a^\dagger\left(e^{[B,\ \cdot]}B'\right)a = a^\dagger\left(e^B B' e^{-B}\right)a\ . \tag{7.17}$$

Proof. Recalling (7.12), one has for general matrices G and H

$$G\,\widehat{\circ}\,e^{\widehat{\circ}\,(-H)} := \sum_{n\geq 0}\frac{1}{n!}(-1)^n G\,\widehat{\circ}\,H^{\widehat{\circ}\,n}\ , \tag{7.18}$$

and

$$G\,\widehat{\circ}\,H^{\widehat{\circ}\,(n+1)} := (G\,\widehat{\circ}\,H^{\widehat{\circ}\,n}) \circ H\ ,\ H^{\widehat{\circ}\,0} \circ B := B\ .$$

From (2.12) one deduces that

$$(G\,\widehat{\circ}\,H^{\widehat{\circ}\,1})^* = (G \circ H)^* = H^* \circ G^* = (H^*)^{\widehat{\circ}\,1}\,\widehat{\circ}\,G^*\ .$$

Suppose by induction that

$$(G\,\widehat{\circ}\,H^{\widehat{\circ}\,n})^* = (H^*)^{\widehat{\circ}\,n}\,\widehat{\circ}\,G^*\ . \tag{7.19}$$

Then, recalling the second and the first identity in (7.11) and using (7.19), one has

$$\begin{aligned}(G\,\widehat{\circ}\,H^{\widehat{\circ}\,(n+1)})^* =& \left((G\,\widehat{\circ}\,H^{\widehat{\circ}\,n}) \circ H\right)^* \\ =& H^* \circ (G\,\widehat{\circ}\,H^{\widehat{\circ}\,n})^* \\ =& H^* \circ ((H^*)^{\widehat{\circ}\,n}\,\widehat{\circ}\,G^*) \\ =& (H^*)^{\widehat{\circ}\,(n+1)}\,\widehat{\circ}\,G^*\ .\end{aligned}$$

Therefore (7.19) holds for all $n \in \mathbb{N}$. (7.18) then implies

$$\begin{aligned}\left(G\,\widehat{\circ}\,e^{\widehat{\circ}\,(-H)}\right)^* =& \sum_{n\geq 0}\frac{1}{n!}(-1)^n\left(G\,\widehat{\circ}\,H^{\widehat{\circ}\,n}\right)^* \\ =& \sum_{n\geq 0}\frac{1}{n!}(-1)^n\left((H^*)^{\widehat{\circ}\,n}\,\widehat{\circ}\,G^*\right) \\ =& e^{\widehat{\circ}\,(-H^*)}\,\widehat{\circ}\,G^*\ ,\end{aligned}$$

which is (7.15). Applying the identity (7.2) with $Y = a^\dagger B^* a$, $X = a^\dagger A a^\dagger$

$$\begin{aligned}e^{[a^\dagger Ba,\ \cdot]}a^\dagger A a^\dagger =& \left(e^{[-a^\dagger B^* a,\ \cdot]}aA^*a\right)^* \\ =^{(7.13)}& \left(a\left((A^*)\,\widehat{\circ}\,e^{\circ(-B^*)}\right)a\right)^* \\ =& a^\dagger\left((A^*)\,\widehat{\circ}\,e^{\circ(-B^*)}\right)^* a^\dagger \\ =^{(7.15)}& a\left((A^*)\,\widehat{\circ}\,e^{\circ(-B^*)}\right)^* a \\ =& a^\dagger\left(e^{\circ(-B)}\,\widehat{\circ}\,A\right)a^\dagger\ ,\end{aligned}$$

which is (7.16). Similarly,

$$\begin{aligned}e^{[a^\dagger Ba,\ \cdot]}a^\dagger B' a =& \sum_{n\geq 0}\frac{1}{n!}[a^\dagger Ba,\ \cdot]^n a^\dagger B' a \\ =& a^\dagger B' a + \sum_{n\geq 1}\frac{1}{n!}[a^\dagger Ba,\ \cdot]^n a^\dagger B' a \\ =& a^\dagger B' a + \sum_{n\geq 1}\frac{1}{n!}[a^\dagger Ba,\ \cdot]^{n-1}[a^\dagger Ba, a^\dagger B' a]\ .\end{aligned}$$

One has

$$[a^\dagger Ba, a^\dagger B'a][a^\dagger Ba,\ \cdot]a^\dagger B'a =^{(2.15)} a^\dagger[B, B']a = a^\dagger([B,\ \cdot]B')a\ .$$

Suppose by induction that

$$[a^\dagger Ba,\ \cdot]^n a^\dagger B'a = a^\dagger([B,\ \cdot]^n B')a\ . \tag{7.20}$$

Then

$$\begin{aligned}[a^\dagger Ba,\ \cdot]^{n+1} a^\dagger B'a =& [a^\dagger Ba, [a^\dagger Ba,\ \cdot]^n a^\dagger B'a] = [a^\dagger Ba, a^\dagger([B,\ \cdot]^n B')a] \\ =& a^\dagger([B, [B,\ \cdot]^n B']) = a^\dagger([B,\ \cdot]^{n+1} B')a\ .\end{aligned}$$

Therefore (7.20) holds for all $n \in \mathbb{N}$. It follows that

$$\begin{aligned} e^{[a^\dagger Ba,\ \cdot]} a^\dagger B'a =& \sum_{n\geq 0} \frac{1}{n!}[a^\dagger Ba,\ \cdot]^n a^\dagger B'a \\ =& \sum_{n\geq 0} \frac{1}{n!} a^\dagger([B,\ \cdot]^n B')a \\ =& a^\dagger \left(\sum_{n\geq 0} \frac{1}{n!}[B,\ \cdot]^n B' \right) a \\ =& a^\dagger \left(e^{[B,\ \cdot]} B' \right) a\ , \end{aligned}$$

which is equivalent to (7.17). □

Lemma 7.7. *If C and B commute and are both symmetric*

$$C \,\widehat{\circ}\, e^{\widehat{\circ}(-B)} = Ce^{-2B} = Ce^{-(B\circ 1)}\ . \tag{7.21}$$

Proof. Since C and B commute and are both symmetric, one has

$$C \cdot B = CB + (CB)^T = cB + B^c = 2CB\ .$$

Therefore

$$(C \circ B) \circ B = 2(2cB) = c(2B)^2\ .$$

Suppose by induction that

$$C \circ B^{\circ n} = C(2B)^n\ . \tag{7.22}$$

Then, since $(C \circ B^{\circ n})$ is symmetric and commutes with C,

$$C \circ B^{\circ(n+1)} = (C \circ B^{\circ n}) \circ B = (C(2B)^n) \circ B = 2(C(2B)^n)B = C(2B)^{n+1}\ .$$

Therefore by induction (7.22) holds for each $n \in \mathbb{N}$. In this case (7.12) becomes

$$\begin{aligned} C \,\widehat{\circ}\, e^{\widehat{\circ}(-B)} =& \sum_{n\geq 0} \frac{1}{n!}(-1)^n C \,\widehat{\circ}\, B^{\widehat{\circ}\, n} \\ =& \sum_{n\geq 0} \frac{1}{n!}(-1)^n C(2B)^n = C \sum_{n\geq 0} \frac{1}{n!}(-1)^n (2B)^n = Ce^{-2B}\ , \end{aligned}$$

which is (7.21). □

7.1. Commutation relations among exponentials. Recall that, according to the splitting lemma

$$e^{\left(a^\dagger A a^\dagger + a^\dagger B a + a C a\right)} = e^{-\frac{1}{2}\operatorname{Tr}(B) + \frac{1}{2}\operatorname{Tr}(g(A,B,C))} \cdot e^{\frac{1}{2} a^\dagger f(A,B,C) a^\dagger} e^{a^\dagger g(A,B,C) a} e^{\frac{1}{2} a \hat{h}(A,B,C) a} .$$

Lemma 7.8.

$$e^{aMa} e^{a^\dagger N a^\dagger} = e^{-\frac{1}{2}\ Tr(4MN) + \frac{1}{2}\ Tr(g(N,4MN,2(M\circ(NM))))} \cdot e^{a^\dagger N a^\dagger + 4\, a^\dagger NMa + 4a(M\circ(NM))a} e^{aMa} e^{\frac{1}{2} a^\dagger \hat{f}(N,4MN,4(M\circ(NM))) a^\dagger} \cdot e^{a^\dagger g(N,4MN,2(M\circ(NM)))a} e^{\frac{1}{2} a(M+\hat{h}(N,4MN,2(M\circ(NM))))a} , \tag{7.23}$$

$$e^{aMa} e^{a^\dagger N a} = e^{a^\dagger N a + a(M\circ N)a} e^{aMa} = e^{-\frac{1}{2}\ Tr(N) + \frac{1}{2}\, Tr(g(0,N,(M\circ N)))} e^{\frac{1}{2} a^\dagger f(0,N,(M\circ N)) a^\dagger} e^{a^\dagger g(0,N,(M\circ N))a} \cdot e^{a(M+\frac{1}{2}\hat{h}(0,N,(M\circ N)))a} , \tag{7.24}$$

$$e^{a^\dagger M a} e^{a^\dagger N a^\dagger} = e^{a^\dagger N a^\dagger} e^{a^\dagger (M\circ N) a^\dagger + a^\dagger M a} = e^{a^\dagger N a^\dagger} e^{-\frac{1}{2}\ Tr(M) + \frac{1}{2}\, Tr(g((M\circ N),M,0))} e^{\frac{1}{2} a^\dagger f((M\circ N),M,0) a^\dagger} \cdot e^{a^\dagger g((M\circ N),M,0)a} e^{\frac{1}{2} a\hat{h}((M\circ N),M,0)a} , \tag{7.25}$$

$$e^{a^\dagger B a} e^{a^\dagger A a^\dagger} = e^{a^\dagger \left(e^{\circ(-B)}\, \widehat{\circ}\, A\right) a^\dagger} e^{a^\dagger B a} . \tag{7.26}$$

Proof. From (7.6) and (7.3)

$$e^{aMa} e^{a^\dagger N a^\dagger} e^{-aMa} = e^{[aMa,\ \cdot]} e^{a^\dagger N a^\dagger} = e^{e^{[aMa,\ \cdot]} a^\dagger N a^\dagger} = e^{a^\dagger N a^\dagger + 4\, a^\dagger NMa + 2a(M\circ(NM))a} ,$$

which is equivalent to

$$e^{aMa} e^{a^\dagger N a^\dagger} = e^{a^\dagger N a^\dagger + 4\, a^\dagger NMa + a2(M\circ(NM))a} e^{aMa}$$
$$= e^{-\frac{1}{2}\operatorname{Tr}(4MN) + \frac{1}{2}\operatorname{Tr}(g(N,4MN,2(M\circ(NM))))} e^{\frac{1}{2} a^\dagger f(N,4MN,4(M\circ(NM))) a^\dagger} e^{a^\dagger g(N,4MN,2(M\circ(NM)))a} e^{\frac{1}{2} a\hat{h}(N,4MN,2(M\circ(NM)))a} e^{aMa}$$
$$= e^{-\frac{1}{2}\operatorname{Tr}(4MN) + \frac{1}{2}\operatorname{Tr}(g(N,4MN,2(M\circ(NM))))} e^{\frac{1}{2} a^\dagger f(N,4MN,4(M\circ(NM))) a^\dagger} e^{a^\dagger g(N,4MN,2(M\circ(NM)))a} e^{\frac{1}{2} a(M+\hat{h}(N,4MN,2(M\circ(NM))))a} ,$$

which is (7.23). Recalling (2.10) one verifies that

$$2(M\circ(NM)) = 2(M(NM) + (M(NM))^T) = 2(MNM + (NM)^T M^T) = 2(MNM + M^T N^T M^T) = 2(MNM + (MNM)^T) .$$

Comparing (7.23) with equation (2.13) in the paper, i.e.

$$\begin{aligned} e^{aMa}e^{a^\dagger Na^\dagger} =& e^{\frac{1}{2}\mathrm{Tr}\left(g\left(N,4NM,2\left(MNM+(MNM)^T\right)\right)\right)} \\ &\cdot e^{\frac{1}{2}a^\dagger \hat{f}\left(N,4NM,2\left(MNM+(MNM)^T\right)\right)a^\dagger} \\ &\cdot e^{a^\dagger g\left(N,4NM,2\left(MNM+(MNM)^T\right)\right)a} \\ &\cdot e^{\frac{1}{2}a\hat{h}\left(N,4NM,2\left(MNM+(MNM)^T\right)\right)a} , \end{aligned}$$

we see that in the scalar term,

$$-\frac{1}{2}\,\mathrm{Tr}(4MN)$$

is missing, and in the a–a–term, $M+$ is missing. From (7.5) one has

$$\begin{aligned} e^{aMa}e^{a^\dagger Na}e^{-aMa} =& e^{[aMa,\ \cdot]}e^{a^\dagger Na} = e^{e^{[aMa,\ \cdot]}a^\dagger Na} = e^{a^\dagger Na + a(M\circ N)a} \\ =& e^{-\frac{1}{2}\mathrm{Tr}(N)+\frac{1}{2}\mathrm{Tr}(g(0,N,(M\circ N)))}e^{\frac{1}{2}a^\dagger f(0,N,(M\circ N))a^\dagger} \\ &\cdot e^{a^\dagger g(0,N,(M\circ N))a}e^{\frac{1}{2}a\hat{h}(0,N,(M\circ N))a} . \end{aligned}$$

This implies

$$\begin{aligned} e^{aMa}e^{a^\dagger Na} =& e^{a^\dagger Na + a(M\circ N)a} \\ =& e^{-\frac{1}{2}\mathrm{Tr}(N)+\frac{1}{2}\mathrm{Tr}(g(0,N,(M\circ N)))}e^{\frac{1}{2}a^\dagger f(0,N,(M\circ N))a^\dagger} \\ &\cdot e^{a^\dagger g(0,N,(M\circ N))a}e^{\frac{1}{2}a\hat{h}(0,N,(M\circ N))a}e^{aMa} , \end{aligned}$$

or equivalently

$$\begin{aligned} e^{aMa}e^{a^\dagger Na} =& e^{a^\dagger Na + a(M\circ N)a} \\ =& e^{-\frac{1}{2}\mathrm{Tr}(N)+\frac{1}{2}\mathrm{Tr}(g(0,N,(M\circ N)))}e^{\frac{1}{2}a^\dagger f(0,N,(M\circ N))a^\dagger} \\ &\cdot e^{a^\dagger g(0,N,(M\circ N))a}e^{a(M+\frac{1}{2}\hat{h}(0,N,(M\circ N)))a} , \end{aligned}$$

which is (7.24). Comparing (7.24) with equation (2.14) in the paper, i.e.

$$\begin{aligned} e^{aMa}e^{a^\dagger Na} =& e^{\mathrm{Tr}\left(-\frac{1}{2}N+\frac{1}{2}g\left(0,N,MN+(MN)^T\right)\right)} \\ &\cdot e^{a^\dagger g\left(0,N,MN+(MN)^T\right)a}\cdot e^{a\left(M+\frac{1}{2}\hat{h}\left(0,N,MN+(MN)^T\right)\right)a} , \end{aligned}$$

we see that they coincide. Taking the adjoint of (7.24) one finds

$$(e^{aMa}e^{a^\dagger Na})^* = (e^{a^\dagger Na + a(M\circ N)a}e^{aMa})^* ,$$

which is equivalent to

$$e^{a^\dagger N^* a}e^{a^\dagger M^* a^\dagger}e^{a^\dagger M^* a^\dagger}e^{a^\dagger N^* a + a^\dagger (M\circ N)^* a^\dagger} = e^{a^\dagger M^* a^\dagger}e^{a^\dagger N^* a + a^\dagger (N^*\circ M^*)a^\dagger} .$$

With the changes

$$N^* \to M\,,\ M^* \to N\ ,$$

one finds

$$\begin{aligned} e^{a^\dagger Ma}e^{a^\dagger Na^\dagger} =& e^{a^\dagger Na^\dagger}e^{a^\dagger(M\circ N)a^\dagger + a^\dagger Ma} \\ =& e^{a^\dagger Na^\dagger}e^{-\frac{1}{2}\operatorname{Tr}(M)+\frac{1}{2}\operatorname{Tr}(g((M\circ N),M,0))}e^{\frac{1}{2}a^\dagger f((M\circ N),M,0)a^\dagger} \\ &\cdot e^{a^\dagger g((M\circ N),M,0)a}e^{\frac{1}{2}a\hat{h}((M\circ N),M,0)a} \ , \end{aligned}$$

which is (7.25). Finally, from the identities

$$e^{a^\dagger Ba}e^{a^\dagger Aa^\dagger}e^{-a^\dagger Ba} = e^{[a^\dagger Ba,\ \cdot]}e^{a^\dagger Aa^\dagger} = e^{e^{[a^\dagger Ba,\ \cdot]}a^\dagger Aa^\dagger} =^{(7.16)} e^{a^\dagger\left(e^{\circ(-B)}\,\widehat{\circ}\,A\right)a^\dagger}$$

one deduces

$$e^{a^\dagger Ba}e^{a^\dagger Aa^\dagger} = e^{a^\dagger\left(e^{\circ(-B)}\,\widehat{\circ}\,A\right)a^\dagger}e^{a^\dagger Ba} \ ,$$

which is (7.26). □

7.2. Normal order of products of elements in type 2 coordinates.

Theorem 7.9. *For $A, C, A', C' \in M_{d,sym}(\mathbb{C})$, $B, B' \in M_d(\mathbb{C})$, the normally ordered form of the product*

$$e^{a^\dagger Aa^\dagger}e^{a^\dagger Ba}e^{aCa}e^{a^\dagger A'a^\dagger}e^{a^\dagger B'a}e^{aC'a} \ ,$$

is

$$c_1c_2e^{a^\dagger A_4a^\dagger}e^{a^\dagger Ba}e^{a^\dagger B_1a}e^{a^\dagger B_2a}e^{aC_3a} \ , \tag{7.27}$$

with c_1 given by (7.29)*, c_2 by* (7.32)*, A_4, B_1, B_2 and C_3 respectively by* (7.37)*,* (7.30)*,* (7.34)*,* (7.36)*.*

Proof. . One has

$$e^{a^\dagger Aa^\dagger}e^{a^\dagger Ba}\left(e^{aCa}e^{a^\dagger A'a^\dagger}\right)e^{a^\dagger B'a}e^{aC'a} \ . \tag{7.28}$$

Use (7.23) to write

$$e^{aCa}e^{a^\dagger A'a^\dagger} = c_1e^{a^\dagger A_1a^\dagger}e^{a^\dagger B_1a}e^{aC_1a} \ ,$$

with

$$c_1 = e^{-\frac{1}{2}\operatorname{Tr}(4CA')+\frac{1}{2}\operatorname{Tr}\left(g(A',4CA',2(C\circ(A'C)))\right)} \tag{7.29}$$

$$\begin{aligned} A_1 =& \frac{1}{2}\hat{f}(A', 4CA', 4(C\circ(A'C))) \\ B_1 =& g(A', 4CA', 2(C\circ(A'C))) \\ C_1 =& \frac{1}{2}(C+\hat{h}(A', 4CA', 2(C\circ(A'C)))) \ . \end{aligned} \tag{7.30}$$

Thus (7.28) becomes

$$c_1e^{a^\dagger Aa^\dagger}e^{a^\dagger Ba}e^{a^\dagger A_1a^\dagger}e^{a^\dagger B_1a}\left(e^{aC_1a}e^{a^\dagger Ba}\right)e^{aC'a} \ . \tag{7.31}$$

Use (7.24) to write

$$e^{aC_1a}e^{a^\dagger Ba} = c_2e^{a^\dagger A_2a^\dagger}e^{a^\dagger B_2a}e^{aC_2a} \ ,$$

with

$$c_2 = e^{-\frac{1}{2}\operatorname{Tr}(B)+\frac{1}{2}\operatorname{Tr}(g(0,B,(C_1\circ B)))} \tag{7.32}$$

$$A_2 = \frac{1}{2}f(0,B,(C_1\circ B)) \tag{7.33}$$

$$B_2 = g(0,B,(C_1\circ B)) \tag{7.34}$$

$$C_2 = C_1 + \frac{1}{2}\hat{h}(0,B,(C_1\circ B)) \ . \tag{7.35}$$

Thus (7.31) becomes

$$\begin{aligned}
& c_1 e^{a^\dagger A a^\dagger} e^{a^\dagger B a} e^{a^\dagger A_1 a^\dagger} e^{a^\dagger B_1 a} \left(c_2 e^{a^\dagger A_2 a^\dagger} e^{a^\dagger B_2 a} e^{a C_2 a} \right) e^{a C' a} \\
=& c_1 c_2 e^{a^\dagger A a^\dagger} e^{a^\dagger B a} e^{a^\dagger A_1 a^\dagger} e^{a^\dagger B_1 a} e^{a^\dagger A_2 a^\dagger} e^{a^\dagger B_2 a} e^{a C_2 a} e^{a C' a} \\
=& c_1 c_2 e^{a^\dagger A a^\dagger} e^{a^\dagger B a} e^{a^\dagger A_1 a^\dagger} e^{a^\dagger B_1 a} e^{a^\dagger A_2 a^\dagger} e^{a^\dagger B_2 a} e^{a (C_2 + C') a} \\
=& c_1 c_2 e^{a^\dagger A a^\dagger} e^{a^\dagger B a} e^{a^\dagger A_1 a^\dagger} \left(e^{a^\dagger B_1 a} e^{a^\dagger A_2 a^\dagger} \right) e^{a^\dagger B_2 a} e^{a C_3 a} \\
=&^{(7.26)} c_1 c_2 e^{a^\dagger A a^\dagger} e^{a^\dagger B a} e^{a^\dagger A_1 a^\dagger} e^{a^\dagger \left(e^{\circ(-B_1)} \,\widehat{\circ}\, A_2 \right) a^\dagger} e^{a^\dagger B_1 a} e^{a^\dagger B_2 a} e^{a C_3 a} \\
=& c_1 c_2 e^{a^\dagger A a^\dagger} e^{a^\dagger B a} e^{a^\dagger \left(A_1 + e^{\circ(-B_1)} \,\widehat{\circ}\, A_2 \right) a^\dagger} e^{a^\dagger B_1 a} e^{a^\dagger B_2 a} e^{a C_3 a} \\
=& c_1 c_2 e^{a^\dagger A a^\dagger} e^{a^\dagger B a} e^{a^\dagger A_3 a^\dagger} e^{a^\dagger B_1 a} e^{a^\dagger B_2 a} e^{a C_3 a} \ ,
\end{aligned}$$

with

$$C_3 := C_2 + C' \ ; \ A_3 := A_1 + e^{\circ(-B_1)} \,\widehat{\circ}\, A_2 \ . \tag{7.36}$$

Finally

$$\begin{aligned}
& c_1 c_2 e^{a^\dagger A a^\dagger} \left(e^{a^\dagger B a} e^{a^\dagger A_3 a^\dagger} \right) e^{a^\dagger B_1 a} e^{a^\dagger B_2 a} e^{a C_3 a} \\
=&^{(7.26)} c_1 c_2 \left(e^{a^\dagger A a^\dagger} e^{a^\dagger \left(e^{\circ(-B)} \,\widehat{\circ}\, A_3 \right) a^\dagger} \right) e^{a^\dagger B a} e^{a^\dagger B_1 a} e^{a^\dagger B_2 a} e^{a C_3 a} \\
=& c_1 c_2 \left(e^{a^\dagger \left(A + e^{\circ(-B)} \,\widehat{\circ}\, A_3 \right) a^\dagger} \right) e^{a^\dagger B a} e^{a^\dagger B_1 a} e^{a^\dagger B_2 a} e^{a C_3 a} \\
=& c_1 c_2 e^{a^\dagger A_4 a^\dagger} e^{a^\dagger B a} e^{a^\dagger B_1 a} e^{a^\dagger B_2 a} e^{a C_3 a} \ ,
\end{aligned}$$

with

$$A_4 := A + e^{\circ(-B)} \,\widehat{\circ}\, A_3 \ , \tag{7.37}$$

which is (7.27). □

8. Exponentiability of the Quadratic Algebra in the Fock Representation

In this section we prove an estimate (see Theorem 8.5 below) which implies that, in the Fock representation, any vector in the dense sub–space linearly spanned by the number vectors is analytic for any multiple of any skew–adjoint element of the quadratic algebra. Thus, by Nelson's analytic vector theorem, the hermitian elements of this algebra are essentially self–adjoint and their exponential series converges strongly on the linear span of the number vectors. From this the existence and unitarity of the quadratic Weyl operators follows.

In the Fock representation the Boson creation–annihilation operators $a_j^{\pm}$ are realized on $\Gamma_{sym}(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}^{\widehat{\otimes} n}$, where $\mathcal{H}$ is a d–dimensional complex Hilbert space and

$$\mathcal{H}^{\widehat{\otimes} 0} := \mathbb{C} \cdot \Phi \, , \; \|\Phi\| = 1 \; . \tag{8.1}$$

The vectors

$$\{a_d^{+n_d} \cdots a_1^{+n_1} \Phi \; : \; (n_1, \dots, n_d) \in \mathbb{N}^d\} \; ,$$

are total in $\Gamma_{sym}(\mathcal{H})$ and their scalar product is uniquely determined by (8.1) and the conditions

$$[a_j^-, a_k^\dagger] = \delta_{jk} \, , \; a_j^- \Phi = 0 \; .$$

The following is a known result whose proof is included for completeness.

Lemma 8.1. *Uniformly in $d \in \mathbb{N}$ one has, for all $k \in \{1, \dots, d\}$and $n \in \mathbb{N}$,*

$$\left\| a_k^- \Big|_{\mathcal{H}^{\widehat{\otimes} n}} \right\| \le \sqrt{n} \, , \; \left\| a_k^\dagger \Big|_{\mathcal{H}^{\widehat{\otimes} n}} \right\| \le \sqrt{n+1} \; , \tag{8.2}$$

where here and in the following $\Big|$ *denotes restriction.*

Proof.

$$\begin{aligned} a_k^- a_d^{+n_d} \cdots a_1^{+n_1} \Phi =& [a_k^-, a_d^{+n_d} \cdots a_1^{+n_1}] \Phi = \sum_{h=0}^{d-1} \cdots [a_k^-, a_{d-h}^{+n_{d-h}}] \cdots a_1^{+n_1} \Phi \\ =& \sum_{h=0}^{d-1} \delta_{k,d-h} n_{d-h} \cdots a_{d-h}^{+(n_{d-h}-1)} \cdots a_1^{+n_1} \Phi \; . \end{aligned}$$

In particular, if $k = d$,

$$a_d^- a_d^{+n_d} \cdots a_1^{+n_1} \Phi = n_d a_d^{+(n_d-1)} a_{d-1}^{+(n_{d-1})} \cdots a_1^{+n_1} \Phi \; ,$$

so that, by induction

$$a_d^{-n_d} a_d^{+n_d} \cdots a_1^{+n_1} \Phi = n_d! a_{d-1}^{+(n_{d-1})} \cdots a_1^{+n_1} \Phi \; ,$$

hence, again by induction,

$$\|a_d^{+n_d} \cdots a_1^{+n_1} \Phi\|^2 = n_d! n_{d-1}! \cdots n_1! = \prod_{j=1}^{d} n_j! \; .$$

From this it follows that, for any finite set $F_n \subset \mathbb{N}^d$ and scalars $x_{n_1,\ldots,n_d} \in \mathbb{C}$, $(n_1, \ldots, n_d) \in F_n$,

$$\left\| a_k^- \sum_{(n_1,\ldots,n_d)\in F_n} x_{n_1,\ldots,n_d} a_d^{+n_d} \cdots a_1^{+n_1} \Phi \right\|^2$$

$$= \left\| \sum_{(n_1,\ldots,n_d)\in F_n} x_{n_1,\ldots,n_d} \sum_{h=0}^{d-1} \delta_{k,d-h} n_{d-h} \cdots a_{d-h}^{+(n_{d-h}-1)} \cdots a_1^{+n_1} \Phi \right\|^2$$

$$= \left\| \sum_{h=0}^{d-1} \delta_{k,d-h} n_{d-h} \sum_{(n_1,\ldots,n_d)\in F_n} x_{n_1,\ldots,n_d} \cdots a_{d-h}^{+(n_{d-h}-1)} \cdots a_1^{+n_1} \Phi \right\|^2$$

$$\leq \left(\sum_{h=0}^{d-1} n_{d-h} \left\| \sum_{(n_1,\ldots,n_d)\in F_n} x_{n_1,\ldots,n_d} \cdots a_{d-h}^{+(n_{d-h}-1)} \cdots a_1^{+n_1} \Phi \right\| \right)^2$$

$$= \left(\sum_{h=0}^{d-1} n_{d-h} \left(\left\| \sum_{(n_1,\ldots,n_d)\in F_n} x_{n_1,\ldots,n_d} \cdots a_{d-h}^{+(n_{d-h}-1)} \cdots a_1^{+n_1} \Phi \right\|^2 \right)^{1/2} \right)^2$$

$$= \left(\sum_{h=0}^{d-1} n_{d-h} \left(\sum_{(n_1,\ldots,n_d)\in F_n} |x_{n_1,\ldots,n_d}|^2 \prod_{j=1}^{d-h-1} n_j! \, (n_{d-h}-1)! \prod_{j=d-h+1}^{d} n_j! \right)^{1/2} \right)^2$$

$$\leq \left(\sum_{h=0}^{d-1} n_{d-h} \left(\sum_{(n_1,\ldots,n_d)\in F_n} |x_{n_1,\ldots,n_d}|^2 \prod_{j=1}^{d} n_j! \right)^{1/2} \right)^2$$

$$= \left(\sum_{h=1}^{d} n_h \left(\sum_{(n_1,\ldots,n_d)\in F_n} |x_{n_1,\ldots,n_d}|^2 \prod_{j=1}^{d} n_j! \right)^{1/2} \right)^2$$

$$= \left(n \left(\left\| \sum_{(n_1,\ldots,n_d)\in F_n} x_{n_1,\ldots,n_d} a_d^{+n_d} \cdots a_1^{+n_1} \Phi \right\|^2 \right)^{1/2} \right)^2$$

$$= n^2 \left\| \sum_{(n_1,\ldots,n_d)\in F_n} x_{n_1,\ldots,n_d} a_d^{+n_d} \cdots a_1^{+n_1} \Phi \right\|^2 ,$$

which is equivalent to the first inequality in (8.2). The second inequality in (8.2) is proved similarly □

Denoting for any $n \in \mathbb{N}$ and $A = (A_{j,k}) \in M_{d\times d}(\mathbb{C})$,

$$|A| := d^2 \max\{|A_{j,k}|\} ,$$

one has

$$\left\| B_0^2(A)\big|_{\mathcal{H}^{\widehat{\otimes} n}} \right\| = \left\| \sum_{j,k} A_{j,k}\ a_j^{\dagger} a_k^{\dagger} \Big|_{\mathcal{H}^{\widehat{\otimes} n}} \right\| \leq |A| \sqrt{(n+1)(n+2)}\ , \tag{8.3}$$

$$\left\| B_2^0(A)\big|_{\mathcal{H}^{\widehat{\otimes} n}} \right\| = \left\| \sum_{j,k} A_{j,k}\ a_j^{-} a_k^{-} \big|_{\mathcal{H}^{\widehat{\otimes} n}} \right\| \leq |A| \sqrt{n(n-1)\ ,} \tag{8.4}$$

$$\left\| B_A^0\big|_{\mathcal{H}^{\widehat{\otimes} n}} \right\| = \left\| \sum_{j,k} A_{j,k}\ a_j^{\dagger} a_k^{-} \Big|_{\mathcal{H}^{\widehat{\otimes} n}} \right\| \leq |A|\, n\ . \tag{8.5}$$

In particular

$$\left\| a_j^{\epsilon} a_k^{\epsilon'} \Big|_{\mathcal{H}^{\widehat{\otimes} n}} \right\| \leq n+2,\ \forall \{j,k\} \subset \{1,\dots,d\}\ ,\ n \in \mathbb{N} \text{ and } \epsilon, \epsilon' = \pm\ ,$$

and, for any $n \in \mathbb{N}$, $A = (A_{j,k}) \in M_{d\times d}(\mathbb{C})$ and $\epsilon \in \{0, \pm\}$,

$$\left\| B_A^{\epsilon}\big|_{\mathcal{H}^{\widehat{\otimes} n}} \right\| \leq |A|\,(n+2)\ .$$

Proposition 8.2. *For any $A \in M_{d\times d}(\mathbb{C})$, for any $n, m \in \mathbb{N}$ and $\xi \in \mathcal{H}^{\widehat{\otimes} n}$*

$$\left\| \left(B_0^2(A)\right)^m \xi \right\| \leq |A|^m \sqrt{(n+1)(n+2)\dots(n+2m-1)(n+2m)}\, \|\xi\|\ , \tag{8.6}$$

$$\tag{8.7}$$

$$\left\| \left(B_2^0(A)\right)^m \xi \right\| \leq \chi_{\frac{n-1}{2}]}(m)\, |A|^m \cdot \sqrt{n(n-1)\dots(n-2(m-1))(n-2(m-1)-1)}\, \|\xi\|\ , \tag{8.8}$$

$$\tag{8.9}$$

$$\left\| \left(B_A^0\right)^m \xi \right\| \leq |A|^m\, n^m\, \|\xi\|\ . \tag{8.10}$$

Proof. It is clear that for any $\xi \in \mathcal{H}^{\widehat{\otimes} n}$, $\left(B_2^0(A)\right)^m \xi$ differs from zero only if $2m-1 < n$, i.e. $m < \frac{n+1}{2}$, or equivalently, $m \leq \frac{n+1}{2} - 1 = \frac{n-1}{2}$. We know from (8.3), (8.4) and (8.5) that the thesis is true for $m = 1$. Noting that, for any $n, m \in \mathbb{N}$, $\xi \in \mathcal{H}^{\widehat{\otimes} n}$, one has $\left(B_A^0\right)^m \xi \in \mathcal{H}^{\widehat{\otimes} n}$, $\left(B_0^2(A)\right)^m \xi \in \mathcal{H}^{\widehat{\otimes}(n+2m)}$ and $\left(B_2^0(A)\right)^m \xi \in \mathcal{H}^{\widehat{\otimes}(n-2m)}$, where $\mathcal{H}^{\widehat{\otimes}(n-2m)} := \{0\}$ if $2m > n$, the thesis follows by induction. □

Remark 8.3. (8.6), (8.8) and (8.10) guarantee that for any $A \in M_{d\times d}(\mathbb{C})$, $\epsilon \in \{0, \pm\}$, $n, m \in \mathbb{N}$ and $\xi \in \mathcal{H}^{\widehat{\otimes} n}$,

$$\|(B_A^{\epsilon})^m \xi\| \leq |A|^m \sqrt{(n+1)(n+2)\dots(n+2m-1)(n+2m)}\, \|\xi\|\ . \tag{8.11}$$

Proposition 8.4. *For any $n, m \in \mathbb{N}$, $\{A_k\}_{k=1}^m \subset M_{d\times d}(\mathbb{C})$ and $\xi \in \mathcal{H}^{\widehat{\otimes} n}$, for any $\epsilon = (\epsilon(1), \dots, \epsilon(m)) \in \{0, \pm\}$,*

$$\left\| B_{A_1}^{\epsilon(1)} \dots B_{A_m}^{\epsilon(m)} \xi \right\| \leq \left(\max_{1\leq k\leq m} |A_k| \right)^m \cdot \sqrt{(n+1)(n+2)\dots(n+2m-1)(n+2m)}\, \|\xi\|\ .$$

Proof. For $m = 1$, (8.11) gives the thesis. Supposing by induction that the thesis is true for m consider the case $m+1$. By definition, $B_{A_{m+1}}^{\epsilon(m+1)}\xi$ belongs to $\mathcal{H}^{\widehat{\otimes} n'}$ with

$$n' = \begin{cases} n+2, & \text{if } \epsilon(m+1) = + \\ n, & \text{if } \epsilon(m+1) = 0 \\ n-2, & \text{if } \epsilon(m+1) = - \end{cases} .$$

So the assumption of induction gives

$$\begin{aligned} &\left\| B_{A_1}^{\epsilon(1)} \dots B_{A_m}^{\epsilon(m)} B_{A_{m+1}}^{\epsilon(m+1)} \xi \right\| \qquad (8.12) \\ &\le \left(\max_{1\le k\le m} |A_k| \right)^m \sqrt{(n'+1)(n'+2)\dots(n'+2m-1)(n'+2m)} \left\| B_{A_{m+1}}^{\epsilon(m+1)}\xi \right\| \\ &\le \left(\max_{1\le k\le m} |A_k| \right)^m \sqrt{(n+3)(n+4)\dots(n+2m+1)(n+2m+2)} \left\| B_{A_{m+1}}^{\epsilon(m+1)}\xi \right\| . \end{aligned}$$

Moreover, (8.11) tells us that

$$\left\| B_{A_{m+1}}^{\epsilon(m+1)}\xi \right\| \le |A_{m+1}| \sqrt{(n+1)(n+2)} \, \|\xi\| .$$

Therefore (8.12) becomes

$$\begin{aligned} &\left\| B_{A_1}^{\epsilon(1)} \dots B_{A_m}^{\epsilon(m)} B_{A_{m+1}}^{\epsilon(m+1)} \xi \right\| \\ &\le |A_{m+1}| \left(\max_{1\le k\le m} |A_k| \right)^m \\ &\cdot \sqrt{(n+1)(n+2)(n+3)(n+4)\dots(n+2m+1)(n+2m+2)} \, \|\xi\| \\ &\le \left(\max_{1\le k\le m+1} |A_k| \right)^{m+1} \\ &\cdot \sqrt{(n+1)(n+2)(n+3)(n+4)\dots(n+2(m+1)-1)(n+2(m+1))} \, \|\xi\| . \end{aligned}$$

The thesis then follows by induction. □

Theorem 8.5. *For any $n \in \mathbb{N}$, $\{A, C, D\} \subset M_{d\times d}(\mathbb{C})$ and $\xi \in \mathcal{H}^{\widehat{\otimes} n}$, the series*

$$\sum_{m=1}^{\infty} \frac{\left\| \left(uB_C^{\dagger} + vB_2^0(A) + wB_D^0 \right)^m \xi \right\|}{m!} z^m , \qquad (8.13)$$

has positive convergence radius.

Proof. For any $n \in \mathbb{N}$, $\{A, C, D\} \subset M_{d\times d}(\mathbb{C})$ and $\xi \in \mathcal{H}^{\widehat{\otimes} n}$, Proposition 8.4 gives

$$\begin{aligned} &\left\| \left(B_C^{\dagger} + B_2^0(A) + B_D^0 \right)^m \xi \right\| \\ &\le \sum_{\epsilon\in\{0,\pm\}^m} \left\| B_{A_1}^{\epsilon(1)} \dots B_{A_m}^{\epsilon(m)} B_{A_{m+1}}^{\epsilon(m+1)} \xi \right\| \\ &\le 3^m \left(\max_{1\le k\le m} |A_k| \right)^m \sqrt{(n+1)(n+2)\dots(n+2m-1)(n+2m)} \, \|\xi\| \\ &\le 3^m \left(\max\{|A|, |C|, |D|\} \right)^m \sqrt{(n+1)(n+2)\dots(n+2m-1)(n+2m)} \, \|\xi\| , \end{aligned}$$

where $A_k \in \{A, C, D\}$ for any k. So

$$\begin{aligned}\frac{\left\|\left(B_C^\dagger + B_2^0(A) + B_D^0\right)^m \xi\right\|}{m!} \leq & 3^m \left(\max\{|A|, |C|, |D|\}\right)^m \\ & \cdot \frac{(n+2)(n+4)\dots(n+2m)}{m!} \|\xi\| \\ & \leq (9n \max\{|A|, |C|, |D|\})^m \|\xi\| \ .\end{aligned}$$

Thus the series (8.13) converges uniformly for $z \in \mathbb{C}$ such that

$$|z| < (9n \max\{|A|, |C|, |D|\}) \ .$$

□

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