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SCS 93: Refinement Monoids

Hans Dobbertin
Leibniz University Hannover, 30167, Hannover, Germany

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NAME: Hans Dobbertin | Date: M D Y — 5 15 84

TOPIC: Refinement monoids

We want to give a brief survey about some results, applications, and open questions appearing in the study of refinement monoids. The reader who is interested in details and further information is referred to the papers [2 - 6]. Independently Grillet [6] and Myers [8] have introduced refinement monoids. A *refinement monoid* is a commutative monoid in which a sum of non-zero elements is always non-zero and which has the *refinement property*, i.e., whenever $\sum x_i = \sum y_j$ then there are elements z_{ij} with $x_i = \sum_j z_{ij}$ and $y_j = \sum_i z_{ij}$. Examples of refinement monoids are

- *distributive sup-semilattices with zero,*
- *"extended" commutative groups, i.e., commutative groups with a new zero element added,*
- *positivity domains of lattice-ordered commutative groups,*
- *for each infinite cardinal number* κ, *the monoid* BA_κ *of all isomorphism types of Boolean algebras with at most* κ *elements, where the addition is induced by the direct product.*

The adequate morphisms between refinement monoids are not the usual (monoid) homomorphisms, but the so-called V-homomorphisms: a homomorphism $h : M \longrightarrow N$ between commutative monoids is called a *V-homomorphism* provided that

- $h(x) = 0$ iff $x = 0$ $(x \in M)$,
- whenever $h(x) = y_1 + y_2$ $(x \in M, \; y_i \in N)$ then there are $x_i \in M$ with $x = x_1 + x_2$ and $h(x_i) = y_i$.

The V-homomorphic image of a refinement monoid is again a refinement monoid. A submonoid N of a commutative monoid M is *hereditary*, in symbols $N \trianglelefteq M$, if the inclusion map from N into M is a V-homomorphism. By $\sigma(x)$ $(x \in M)$ we denote the cardinality of the set $\{(y,z) \in M^2 : x = y + z\}$. The *sum rank* $\sigma(M)$ of M is the supremum of all cardinalities $\sigma(x)$, $x \in M$.

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A main reason why we are interested in refinement monoids is that they have close connections with Boolean algebras, as we shall see subsequently. Vaught's Isomorphism Theorem for countable Boolean algebras can be formulated as follows:

> *Let* R *be a binary symmetric relation on* BA_ω *such that*
> *(i)* $x\,R\,0$ *iff* $x = 0$,
> *(ii) whenever* $x_1 + x_2\,R\,y$ *then there are* y_i *with* $y = y_1 + y_2$ and $x_i\,R\,y_i$.
> *Then* $x\,R\,y$ *implies* $x = y$.

A refinement monoid satisfying the implication of this theorem is called a *V-monoid*. The V-monoids form a natural axiomatic framework wherein isomorphism types of countable Boolean algebras can be studied For each infinite cardinality κ, there exists a V-monoid V_κ of sum rank κ, unique up to isomorphism, such that if M is any refinement monoid of sum rank less that or equal to κ then there is a (unique) V-homomorphism π_M from M into V_κ. Moreover, we can choose the V_κ such that $V_{\kappa'} \trianglelefteq V_\kappa$ for $\kappa' < \kappa$ (whence π_M does in fact not depend on κ). Since the kernel relation of a V-homomorphism satisfies the assumption of Vaught's Theorem, π_M is an embedding if M is a V-monoid. Therefore we can say that

> V_κ *is the greatest V-monoid of sum rank* κ.

By $\bar{M}$ we denote the image of M under π_M. Note that $\bar{M}$ is a V-monoid and $\bar{M} \trianglelefteq V_\kappa$ for $\kappa \geqq \sigma(\bar{M})$. As $\sigma(\overline{BA}_\kappa) = \kappa$, we have $\overline{BA}_\kappa \trianglelefteq V_\kappa$. It is a basic question whether $\overline{BA}_\kappa = V_\kappa$ for all κ. Later we shall see that the answer is yes for $\kappa = \omega$ and $\kappa = \omega_1$. However, the case $\kappa > \omega_1$ is still open. Several facts are indicating an affirmative answer. At least we know that $\overline{BA}_\kappa$ and V_κ both have 2^κ elements.

It is not very hard to prove that every commutative semigroup can be embedded into a refinement monoid. Ketonen has shown that every countable commutative semigroup can even be extended to a countable V-monoid. This is a deep result, and thus we cannot expect that our next question is easy to answer:

> *Can every infinite commutative semigroup be embedded into a V-monoid of the same cardinality?*

Suppose that B is a Boolean algebra, $\kappa \geq |B|$ is infinite, and let $\mu : B \longrightarrow BA_\kappa$ be the mapping which associates to $b \in B$ the isomorphism type of $B\restriction b$, then

- μ *is finitely additive,*
- *whenever* $\mu(b) = x_1 + x_2$ *then there is a disjoint decomposition* $b = b_1 + b_2$ *with* $\mu(b_i) = x_i$,
- $\mu(b) = 0$ *iff* $b = 0$.

A map μ on a Boolean algebra with values in a refinement monoid is said to be a *V-measure* provided that the preceding conditions are fulfilled (see Myers [9]). We call a refinement monoid M *measurable* if for each $x \in M$ there exists some Boolean algebra B and a V-measure $\mu : B \longrightarrow M$ with $\mu(1) = x$. Thus, of course, BA_κ and therefore $\overline{BA}_\kappa$ are measurable. In general, we do not know whether every refinement monoid is measurable. So far we only have positive answers for

(1) *refinement monoids of sum rank* $\leq \omega_1$,
(2) *distributive lattices with zero (under supremum),*
(3) *extended commutative groups,*
(4) *positivity domains of linearly ordered commutative groups.*

One can show that $\overline{BA}_\kappa = V_\kappa$ if V_κ is measurable. Hence (1) implies

$$\overline{BA}_\omega = V_\omega \quad \text{and} \quad \overline{BA}_{\omega_1} = V_{\omega_1}.$$

Note that Vaught's Theorem just states that $BA_\omega = \overline{BA}_\omega$. Therefore we conclude $BA_\omega = V_\omega$, or in other words:

BA_ω *is the greatest V-monoid of countable sum rank.*

A very special case of (1), namely the measurability of countable bounded distributive sup-semilattices, and (2) above can be used to show respectively:

- *the compact second countable Stone spaces are precisely the* T_0*-images of the Cantor set under continuous open mappings,*
- *every Heyting algebra can be embedded into the ideal lattice of some atomless Boolean algebra.*

Finally let us consider quite a different question, namely: what are the finite refinement monoids? An answer is given by the following theorem:

Suppose that I *is a finite partially ordered set,* G_i $(i \in I)$ *are finite commutative groups, and* $h_i^j : G_i \longrightarrow G_j$ $(i, j \in I,\ i \leq j)$ *are homomorphisms such that*

h_i^i *is the identity mapping on* G_i,

$h_j^k h_i^j = h_i^k \quad (i \leq j \leq k)$.

Then the commutative monoid generated by the disjoint union of the G_i *with respect to the relations*

$$u + v = h_i^j(u) + v \quad (u \in G_i,\ v \in G_j,\ i \leq j)$$

is a finite refinement monoid. Conversely, every finite refinement monoid can be obtained in this way.

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