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SCS 92: Products of Continuous Partially Ordered Sets

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TOPIC: Products of continuous partially ordered sets

Recently the concepts of continuous lattices in the sense of Scott and of upper continuous lattices in the sense of Crawley and Dilworth have been extended to so-called up-complete posets. While the class of continuous, resp., upper continuous complete lattices is closed under arbitrary products, an analogous result fails for the general setting of posets. It is shown that (i) a direct product of nonempty posets P_j is Scott continuous if and only if each factor P_j has this property and for almost all j, the components of P_j have least elements; (ii) a direct product of nonempty posets P_j is upper continuous if and only if each factor P_j has this property and for almost all j, the components of P_j are filtered (= down-directed). A common generalization of (i) and (ii) is presented for so-called ($\dot{\iota}$,m)-continuous posets.

Given a poset P and a subset Y of P, let

$$\downarrow Y = \{ x \in P : x \leq y \text{ for some } y \in Y \}$$

denote the *lower set* generated by Y; the system $\theta(P)$ of all lower sets is an Alexandroff-discrete topology, referred to as the *lower A-topology* of P. The system $\dot{\iota}(P)$ of all directed (nonempty!) lower sets plays a fundamental role in the representation theory of *algebraic posets*; for more details on this subject, see [3],[6] and [8]. If P^* denotes the dual poset of P then $\theta(P^*)$ is the *upper A-topology* of P, and its members are called *upper sets*.

In the last years, the following generalization of Scott's *continuous lattices* [9] has attracted the attention of mathematicians working in the field of order theory and foundations of computer sciences: a poset P is said to be *up-complete* if each nonempty chain of P (or, equivalently, each member of $\dot{\iota}(P)$) has a join; further, an up-complete poset is called *continuous* (cf. [5],[6],[7]) if for each $x \in P$ the *way-below set* $\Downarrow x = \bigcap\{ Y \in \dot{\iota}(P) : x \leq \bigvee Y \}$ is directed and has join x; in other words, if for each $x \in P$ there exists a least $D \in \dot{\iota}(P)$ with $x \leq \bigvee D$. Alterating this definition slightly, we call an up-complete poset P *upper continuous* if for each $x \in P$ and each $Y \in \dot{\imath}(P)$ with $x \le \bigvee Y$ there exists a $D \in \dot{\imath}(P)$ such that $D \subseteq Y$ and $x = \bigvee D$. It is easy to see that this definition extends in fact the notion of upper continuous lattices as introduced in the book of Crawley and Dilworth [1]("meet-continuous lattices" in the sense of [5]). A common generalization of continuous posets and upper continuous posets has been proposed in [4]: Given an infinite cardinal m, write $Y \underset{m}{\subseteq} X$ if Y is a subset of X with less than m elements. Thus, denoting the least infinite cardinal by ω, $Y \underset{\omega}{\subseteq} X$ means that Y is a finite (possibly empty) subset of X. Now an up-complete poset P is called *$(\dot{\imath}, m)$-continuous* if for all $x \in P$ and all subsystems $\mathfrak{Y} \underset{m}{\subseteq} \dot{\imath}(P)$ with $x \le \bigvee Y$ for all $Y \in \mathfrak{Y}$, there exists a $D \in \dot{\imath}(P)$ such that $D \subseteq \bigcap \mathfrak{Y}$ and $x = \bigvee D$. Thus "continuous" means "$(\dot{\imath},m)$-continuous for one (resp.,all) $m > |P|$", and "upper continuous" means "$(\dot{\imath},\omega)$-continuous".

It is the main purpose of the present note to give a necessary and sufficient condition for a direct product of posets to be $(\dot{\imath},m)$-continuous and to draw special conclusions for continous, resp., upper continuous posets. (The case of continuous posets with least elements has already been treated in [2]).

Call a poset P *m-filtered* if every subset $Y \underset{m}{\subseteq} P$ has a lower bound in P. Thus "ω-filtered" means "filtered" (="down-directed") in the usual sense. On the other extreme, a poset P is m-filtered for one (resp., all) $m > |P|$ if and only if P has a least element. Now we say a poset P is *componentwise m-filtered* if every component (that is, every maximal connected subset) of P is m-filtered. In order to illustrate the significance of this definition, we give some alternative characterizations of componentwise m-filtered posets in the extremal cases $m = \omega$ and $m > |P|$, respectively.

PROPOSITION 1. *The following statements on a poset* P *are equivalent:*

(a) P *is componentwise filtered.*

(b) *Every component* C *of* P *is an irreducible space in its lower* A*-topology* $\theta(C)$*, i.e., any two nonempty lower subsets of* C *intersect.*

(c) P *is a normal space in its upper* A*-topology* $\theta(P^*)$*; that is, any two disjoint lower sets can be seperated by disjoint upper sets.*

(d) *If* x *and* y *have a common upper bound in* P *then they also have a common lower bound.*

(e) *Every nonempty finite connected subset of* P *has a lower bound.*

(f) *Every finite subset of* P *possessing an upper bound must also have a lower bound.*

PROOF. (a) => (b): If A and B are nonempty lower subsets of a component C then, choosing $a \in A$ and $b \in B$, we get $\emptyset \neq {\downarrow}a \cap {\downarrow}b \subseteq A \cap B$ because C is filtered. (As usual, we write ${\downarrow}x$ for ${\downarrow}\{x\}$).
(b) => (c): P is the disjoint union of its components, and each component is both open and closed (being both an upper and a lower set). Now, given disjoint lower subsets A and B of P, we see that for each component C, either $A \cap C = \emptyset$ or $B \cap C = \emptyset$ (because $A \cap C$ and $B \cap C$ are disjoint $\theta(C)$-open subsets of the irreducible space C). Setting
$U := \bigcup\{C : C \text{ is a component with } A \cap C \neq \emptyset\}$,
$V := \bigcup\{C : C \text{ is a component with } B \cap C \neq \emptyset\}$,
we obtain disjoint upper sets U and V with $A \subseteq U$ and $B \subseteq V$, as desired.
(c) => (d): If x and y have no common lower bound, then ${\downarrow}x$ and ${\downarrow}y$ are disjoint $\theta(P^*)$-closed subsets of P, and by normality of $(P, \theta(P^*))$, we find disjoint upper sets U and V with ${\downarrow}x \subseteq U$ and ${\downarrow}y \subseteq V$. But then x and y cannot possess a common upper bound either, since $x \leq z$ and $y \leq z$ would imply $z \in U \cap V = \emptyset$, a contradiction.
(d) => (e): Use induction on the cardinality of the connected finite sets $Z \subseteq P$. If $|Z| = n+1$ $(n \geq 1)$ then we may choose a $z \in Z$ such that $Y = Z \setminus \{z\}$ is connected and $z \leq y$ or $y \leq z$ for some $y \in Y$. By the induction hypothesis, we may assume that Y has a lower bound x. If $y \leq z$ then x is also a lower bound of Z. Otherwise y is a common upper bound of x and z, whence x and z have a common lower bound, and this is a lower bound of Z.
The implications (e) => (f) => (d) and (e) => (a) are clear. □

A bit easier is the proof of

PROPOSITION 2. *For any poset* P *with* $|P| < m$, *the following conditions are equivalent:*

(a) P *is componentwise* m*-filtered.*

(b) *Every component* C *of* P *has a least element.*

(c) *Every component* C *of* P *is completely irreducible in the lower A-topology* $\theta(C)$*, that is, arbitrary intersections of nonempty lower* *subsets of* C *are nonempty.*

(d) P *is uniquely minimized; that is, every element of* P *dominates a unique minimal element.*

(e) *Every nonempty connected subset of* P *has a lower bound.*

The proof details are left to the reader.□

Now let $(P_j : j \in J)$ be a family of nonempty posets and $P = \prod_{j\in J} P_j$ their direct product. The projection from P onto P_j is denoted by π_j. Thus $\pi_j(x) = x_j$ for $x \in P$.

We wish to characterize (ι,m)-continuity of P by means of the factors P_j. For that purpose, we need the following definitions. For $x,y \in P$, write $x \lesssim y$ if the set $\{j \in J : x_j \nleq y_j\}$ is finite. Evidently $\lesssim$ is a quasiorder (i.e., a reflexive and transitive relation) but not a partial order. We call the product poset P m-*normal* if for all $x \in P$ and all $Y \subseteq_m P$ such that for each $y \in Y$, x and y have a common upper bound (depending on x and y!) there exists an element $z \in P$ such that $z \lesssim x$ and $z \lesssim y$ for all $y \in Y$. Obviously P is m-normal whenever J is finite. The term "*for almost all* $j \in J$" will have the meaning "*for all but a finite number of indices* $j \in J$". In the sequel, the axiom of choice will often apply without particular emphasis.

PROPOSITION 3. *Let* $(P_j : j \in J)$ *be a family of nonempty posets such that for almost all* $j \in J$, P_j *is componentwise* m-*filtered. Then the product poset* $P = \prod_{j\in J} P_j$ *is* m-*normal. The converse implication holds whenever* $m = \omega$ *or* $m > |P|$.

PROOF. Define $K := \{j \in J : P_j$ is not componentwise m-filtered$\}$. Choose $x \in P$ and $Y \subseteq_m P$ such that x and y have a common upper bound for each $y \in P$. Then $Y_j := \pi_j[\{x\} \cup Y]$ is contained in a component of P_j, and for $j \in J \setminus K$, $Y_j \subseteq_m P_j$ implies that Y_j has a lower bound. Hence we find a $z \in P$ such that $z_j \leq x_j$ and $z_j \leq y_j$ for all $y \in Y$ and all $j \in J \setminus K$. Accordingly, if K is finite then P is m-normal.

Now suppose P is ω-normal. By definition of K and Proposition 1, we find elements $x,y \in P$ such that x and y have a common upper bound, but for no $j \in K$, x_j and y_j have a common lower bound in P_j. Since P is ω-normal, we find a $z \in P$ with $z \lesssim x$ and $z \lesssim y$, so K must be finite.

Third, assume $m > |P|$ and P is m-normal. By definition of K and Proposition 2, we may choose an element $x \in P$ such that for no $j \in K$ there exist a unique minimal element dominated by x_j. Now, applying the definition of m-normality to the case $Y = \downarrow x$, we find an element $z \in P$ with $z \lesssim y$ for all $y \leq x$. But there exists a $y \in P$ such that $y_j = x_j$ for $j \in J \setminus K$, while $y_j \leq x_j$ and $z_j \not\leq y_j$ for $j \in K$. (Indeed, either z_j is minimal in $\downarrow x_j$, then let y_j be a different minimal element of $\downarrow x_j$; or z_j is not minimal in $\downarrow x_j$, then choose $y_j < z_j$). Accordingly, K must be finite.$_\square$

COROLLARY 1. *Let* $(P_j : j \in J)$ *be a family of nonempty posets and* P *their direct product.*

(1) *For* $m = \omega$, P *is* m*-normal if and only if for almost all* j, P_j *is componentwise filtered (i.e. a normal space in its upper* A*-topology).*

(2) *For* $m > |P|$, P *is* m*-normal if and only if for almost all* j, P_j *is uniquely minimized.*

A sligthly different situation holds for the first uncountable cardinal Ω:

PROPOSITION 4. *Let* $(P_j : j \in \omega)$ *be a countable family of nonempty posets. Then the direct product* $P = \prod_{j\in\omega} P_j$ *is* Ω*-normal if and only if for almost all* $j \in \omega$, P_j *is componentwise filtered.*

PROOF. Let $x \in P$ and $Y \subseteq_\Omega P$, say $Y = \{y^{(n)} : n \in \omega\}$, such that for each n, x and $y^{(n)}$ have a common upper bound. Suppose the set K of all indices $j \in \omega$ for which P_j fails to be componentwise filtered is finite. For $j \in \omega \setminus K$, we find a lower bound for the set $\{x_j\} \cup \{y_j^{(n)} : n \leq j\}$ which is contained in a component of P_j. Thus there exists an element $z \leq x$ such that $z_j \leq y_j^{(n)}$ for $j \in \omega \setminus K$ and $n \leq j$, whence $z \lesssim y^{(n)}$ for all $n \in \omega$. This proves Ω-normality of P.$_\square$

According to Propositions 1 and 4, ω-normality and Ω-normality are equivalent properties for countable products of posets.

Although the first implication in Proposition 3 cannot be inverted in general, it is possible to describe m-normality by a weak kind of componentwise m-filtration:

PROPOSITION 5. *A product* P *of nonempty posets* P_j $(j \in J)$ *is* m-*normal if and only if each component* C *of* P *is* m-*filtered by the quasiorder* $\lesssim$ *(that is, for each* $Y \subseteq_m C$ *there exists a* $z \in C$ *with* $z \lesssim y$ *for all* $y \in Y$*).*

PROOF. Suppose each component of P is m-filtered by $\lesssim$. Given $x \in P$ and $Y \subseteq_m P$ such that each $y \in Y$ has a common upper bound with x, we conclude that $\{x\} \cup Y$ is contained in a component of P, and $|\{x\} \cup Y| < m$. Hence there exists a $z \in P$ with $z \lesssim x$ and $z \lesssim y$ for all $y \in Y$.
Conversely, suppose P is m-normal, C is a component of P, and $Y \subseteq_m C$. We may assume $Y \neq \emptyset$ and choose a fixed $x \in Y$. By Proposition 1, there exists a finite set $K \subseteq J$ such that for all $j \in J \setminus K$, P_j is componentwise filtered. Obviously $\pi_j[Y]$ is contained in a component of P_j, so we find for each $y \in Y$ an element $\tilde{y} \in C$ with $\tilde{y} \leq x$ and $\tilde{y}_j \leq y_j$ for all $j \in J \setminus K$ (for $j \in K$, we may take $\tilde{y}_j = x_j$). Put $\tilde{Y} := \{\tilde{y} : y \in Y\}$. Then $\tilde{Y} \subseteq_m P$, and by m-normality of P, we find a $z \in P$ with $z \lesssim x$ and $z \lesssim \tilde{y} \lesssim y$ for all $y \in Y$. □

Now we come to the crucial result of this note.

THEOREM. *A direct product* P *of nonempty posets* P_j $(j \in J)$ *is* (ι,m)-*continuous if and only if each factor* P_j *is* (ι,m)-*continuous and* P *is* m-*normal.*

PROOF. First suppose P is (ι,m)-continuous. Then it is easy to see that each P_j must also be (ι,m)-continuous. In order to prove m-normality of P, consider an element $x \in P$ and a set $Y \subseteq_m P$ such that for all $y \in Y$, there exists an upper bound $\bar{y}$ of the set $\{x,y\}$. The sets

$$D_y := \{ z \in P : z \leq \bar{y} \text{ and } z \lesssim y \} \qquad (y \in Y)$$

are directed lower sets, and $\bigvee D_y = \bar{y}$. Thus we obtain a system $\mathfrak{Y} := \{D_y : y \in Y\} \subseteq_m \iota(P)$ with $x \leq \bar{y} = \bigvee D_y$ for all $y \in Y$. Hence, by (ι,m) - continuity of P, there exists a directed set $D \subseteq \bigcap \mathfrak{Y}$ such that $x = \bigvee D$. In particular, choosing an arbitrary $z \in D$ $(\neq \emptyset!)$, we get $z \lesssim x$ and $z \lesssim y$ for all $y \in Y$, as desired.
Conversely, assume P is m-normal and each P_j is (ι,m)-continuous.

Consider an element $x \in P$ and a system $\mathfrak{Y} \subseteq_m \iota(P)$ with $x \leq \bigvee Y$ for all $Y \in \mathfrak{Y}$. As joins are formed coordinatewise, we get $x_j \leq \bigvee \pi_j[Y]$ for all $Y \in \mathfrak{Y}$, and $\mathfrak{Y}_j := \{\pi_j[Y] : Y \in \mathfrak{Y}\} \subseteq_m \iota(P_j)$. By (ι,m)-continuity of P_j, we find directed lower sets $D_j \subseteq \bigcap \mathfrak{Y}_j$ with $x_j = \bigvee D_j$ $(j \in J)$. Choose arbitrary elements $d \in \prod_{j\in J} D_j$ and $\xi \in \prod_{Y\in\mathfrak{Y}} Y$ (i.e. $\xi(Y) \in Y$ for all $Y \in \mathfrak{Y}$). Then $\{\xi(Y) : Y \in \mathfrak{Y}\} \subseteq_m P$, $d \leq x \leq \bigvee Y$ and $\xi(Y) \leq \bigvee Y$ for all $Y \in \mathfrak{Y}$. Hence, by m-normality of P, there exists a $z \in P$ with $z \lesssim d$ and $z \lesssim \xi(Y)$ for all $Y \in \mathfrak{Y}$. The set

$$D := \{y \in \prod_{j\in J} D_j : y \lesssim z\}$$

is directed and has join x since $\pi_j[D] = D_j$ for all $j \in J$. It remains to prove the inclusion $D \subseteq Y$ for each $Y \in \mathfrak{Y}$. For $y \in D$, the set $K := \{j \in J : y_j \nleq \xi(Y)_j\}$ is finite since $y \lesssim z \lesssim \xi(Y)$. For each $j \in K$, there exists a $y^{(j)} \in Y$ with $\pi_j(y^{(j)}) = y_j$ (because $y_j \in D_j \subseteq \bigcap \mathfrak{Y}_j \subseteq \pi_j[Y]$). But the set Y is directed, so we find an element $\bar{y} \in Y$ with $\bar{y} \geq y^{(j)}$ for all $j \in K$ and $\bar{y} \geq \xi(Y)$. Thus $y_j = \pi_j(y^{(j)}) \leq \bar{y}_j$ for $j \in K$ and $y_j \leq \xi(Y)_j \leq \bar{y}_j$ for $j \in J \setminus K$, whence $y \leq \bar{y}$; finally, using the fact that Y is a lower set, we get $y \in Y$, as desired.$_\square$

Summarizing the previous results, we obtain a number of immediate corollaries.

COROLLARY 2. *Let* P *be a direct product of nonempty* (ι,m)*-continuous posets* P_j $(j \in J)$. *If the number of indices* j *for which* P_j *is not componentwise* m*-filtered is finite then* P *is* (ι,m)*-continuous.*

COROLLARY 3. *The class of* (ι,m)*-continuous posets is closed under finite products, and the class of* (ι,m)*-continuous posets with least elements is closed under arbitrary products.*

COROLLARY 4. *For a product of nonempty posets* P_j *to be upper continuous, it is necessary and sufficient that each* P_j *is upper continuous and almost all* P_j *are componentwise filtered.*

COROLLARY 5. *For a product of countably many nonempty posets* P_j *to be* (ι,Ω)*-continuous, it is necessary and sufficient that each* P_j *is* (ι,Ω)*-continuous and almost all* P_j *are componentwise filtered.*

COROLLARY 6. *For a product of nonempty posets* P_j *to be continuous, it is necessary and sufficient that each* P_j *is continuous and almost all* P_j *are uniquely minimized.*

A similar result for algebraic posets has been derived in [3].

EXAMPLE. Let C denote the chain $\{-n : n \in \omega\}$. C satisfies the ascending chain condition, and is therefore algebraic, in particular continuous. By Corollary 6 no infinite power of C is continuous, while by Corollary 4, every power of C is upper continuous, and by Corollary 5, every countable power of C is $(\dot{\imath},\Omega)$-continuous. But C^{ω} is not $(\dot{\imath},m)$-continuous for $m > 2^{\omega} = |C^{\omega}|$. Hence, accepting the continuum hypothesis, we may conclude that Ω is the greatest cardinal m such that C^{ω} is $(\dot{\imath},m)$-continuous. Notice that C^{ω} is Ω-normal (by Proposition 4) although C is not (componentwise) Ω-filtered.

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