# Selected Problems on Matroid Minors

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# SELECTED PROBLEMS ON MATROID MINORS

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# Abstract

This dissertation begins with an introduction to matroids and graphs. In the first chapter, we develop matroid and graph theory definitions and preliminary results sufficient to discuss the problems presented in the later chapters. These topics include duality, connectivity, matroid minors, and Cunningham and Edmonds's tree decomposition for connected matroids.

One of the most well-known excluded-minor results in matroid theory is Tutte's characterization of binary matroids. The class of binary matroids is one of the most widely studied classes of matroids, and its members have many attractive qualities. This motivates the study of matroid classes that are close to being binary. One very natural such minor-closed class $\mathcal{Z}$ consists of those matroids $M$ such that the deletion or the contraction of $e$ is binary for all elements $e$ of $M$. Chapter 2 is devoted to determining the set of excluded minors for $\mathcal{Z}$.

Duality plays a central role in the study of matroids. It is therefore natural to ask the following question: which matroids guarantee that, when present as minors, their duals are present as minors? We answer this question in Chapter 3. We also consider this problem with additional constraints regarding the connectivity and representability of the matroids in question. The main results of Chapter 3 deal with 3-connected matroids.

# Chapter 1
# Introduction

This dissertation is primarily concerned with matroids. Except where otherwise noted, the matroid terminology will follow Oxley [13]. In Chapter 3, graph theory is used in the proofs of several results. The graph theory terminology will follow Diestel [4] except that we use the term *simple graph* to describe what Diestel calls a *graph*, and we use the term *graph* to describe what Diestel calls a *multigraph*. We also define graphs such that the vertex set of a graph cannot be empty, unlike Diestel. The remainder of this chapter is devoted to briefly discussing some important matroid properties that will be used throughout this dissertation. For a more complete introduction, or to see proofs of the facts stated in this introduction, the reader is referred to [13].

A matroid is a combinatorial object that generalizes the notion of independence as understood through linear algebra, graph theory, and affine geometry. The following definition of a matroid is taken from Oxley [13, p. 17]. A *matroid* $M$ is an ordered pair $(E, \mathcal{B})$ consisting of a finite set $E$ and a collection $\mathcal{B}$ of subsets of $E$ having the following two properties:

(B1) $\mathcal{B}$ is non-empty.

(B2) If $B_1$ and $B_2$ are in $\mathcal{B}$ and $x \in B_1 - B_2$, then there is an element $y$ of $B_2 - B_1$ such that $(B_1 - x) \cup y \in \mathcal{B}$.

The set $E$ is called the *ground set* of $M$ and the members of $\mathcal{B}$ are called *bases* of $M$. It is not uncommon to denote the set $\mathcal{B}$ more specifically by $\mathcal{B}(M)$. Let $\mathcal{B}^*(M)$ be $\{E(M) - B : B \in \mathcal{B}(M)\}$. Then the ordered pair $(E, \mathcal{B}^*(M))$ is a matroid called the *dual matroid* of $M$. We denote this matroid by $M^*$. Note that every matroid has a unique dual matroid whose set of bases can be obtained in this way. All the bases of $M$ have the same cardinality, which is called the *rank* $r(M)$ of $M$.

A subset $I$ of $E$ is *independent* in $M$ if $I$ is a subset of some member of $\mathcal{B}$. Clearly a matroid is uniquely determined by its set of independent sets. For any independent set $I$, we define the rank $r(I)$, or $r_M(I)$, of $I$ in $M$ to be $|I|$. If a set is not independent, we say it is *dependent*. We call a minimally dependent set a *circuit*. For any dependent set $D$, the rank of $D$ is equal to the size of a largest independent set contained in $D$. We define the *corank* $r^*(S)$ of any subset $S$ of $E$ to be the rank of $S$ in the dual matroid $M^*$.

For a subset $S$ of $E$, the *closure* $\mathrm{cl}(S)$ of $S$ consists of all elements $e$ of $E$ such that $r(S) = r(S \cup e)$. If $\mathrm{cl}(S){=}S$ for some subset $S$, we say that $S$ is a *closed set* or that $S$ is a *flat* of $M$. If $r(M) = r$, then we call a flat of rank $r - 1$ a *hyperplane* of $M$.

The class of uniform matroids will come up frequently in this document. Let $E$ be an $n$-element set and $r$ be an integer with $0 \leq r \leq n$. The *uniform matroid* $U_{r,n}$ of rank $r$ on $E$ has $E$ as its ground set and the collection of $r$-element subsets of $E$ as its set of bases.

### 1.1 Representable and Graphic Matroids

Let $A$ be an $n \times m$ matrix over a field $\mathbb{F}$. Let $E$ be the $m$-element set of column labels of $A$ and let $\mathcal{I}$ be the set of subsets $X$ of $E$ for which the multiset of columns labeled by $X$ is linearly independent in the vector space $V(n, \mathbb{F})$. Then $\mathcal{I}$ is the set of independent sets of a matroid $M$ on $E$. We say that the matrix $A$ is a *representation* or $\mathbb{F}$-*representation* of $M$, and the matroid $M$ is $\mathbb{F}$-*representable*. The matroid obtained in this way will often be denoted by $M[A]$. It is called the *vector matroid* of $A$.

Of particular importance in this dissertation is the class of matroids that are $GF(2)$-representable, that is, those matroids that have a matrix representation over the finite field of two elements. We call such matroids *binary*.

Now consider a graph $G$ and let $E$ be the set of edge labels of $G$. Let $\mathcal{B}$ be the set of edge sets of spanning forests of $G$. Then $\mathcal{B}$ is the set of bases of a matroid $M(G)$ on $E$. We call $M(G)$ the cycle matroid of $G$. A matroid that is isomorphic to the cycle matroid of a graph $G$ is called *graphic*. Graphic matroids are $\mathbb{F}$-representable for all fields $\mathbb{F}$.

## 1.2 Minors

In this section, we discuss the dual operations of element deletion and element contraction. Let $M = (E, \mathcal{B})$ be a matroid and take $T \subseteq E$. The *deletion* of $T$ from $M$, denoted by $M\backslash T$, is the matroid whose ground set is $E - T$ and $\mathcal{B}(M\backslash T)$ is the set of maximal members of $\{B - T : B \in \mathcal{B}(M)\}$. In the case where T=$\{e\}$, the matroid $M\backslash T$ is written $M\backslash e$. We sometimes refer to $M\backslash T$ as the *restriction* of $M$ to $E - T$ and denote this by $M|(E - T)$. We use duality to define contraction. The *contraction* of $T$ from $M$, denoted by $M/T$, is the matroid $(M^*\backslash T)^*$.

A matroid $N$ is a *minor* of a larger matroid $M$ if and only if $N$ can be obtained from $M$ by a possibly empty sequence of deletions and contractions. If the sequence is non-empty, then we say $N$ is a *proper minor* of $M$. If $M$ has $N$ as a minor, we say that $M$ has an $N$*-minor*. Some classes of matroids have the property that all their minors are also in the class. We say that such classes are *closed under minors* or are *minor-closed*. Note the class of $\mathbb{F}$-representable matroids is minor-closed for all fields $\mathbb{F}$.

Every minor-closed class can be described by a list of *excluded minors*, that is, matroids $M$ that are not in the class, but whose proper minors are all in the class. When the explicit list of all excluded minors for a class $\mathcal{M}$ is known, we call the result an *excluded-minor characterization* of $\mathcal{M}$. Some of the most celebrated results in matroid theory are excluded-minor characterizations. One such result is Tutte's characterization of binary matroids [19] below.

**Theorem 1.2.1.** *A matroid is binary if and only if it has no $U_{2,4}$-minor.*

The main result of Chapter 2 is an excluded-minor characterization of a class of matroids whose members are close to being binary.

## 1.3 Connectivity

The property of connectedness is of basic structural importance for matroids. We begin this section by noting that, for matroids, the terms connected and 2-connected are interchangeable. A matroid $M$ is *connected*, or 2-*connected*, if and only if, for every pair of distinct elements of $E(M)$, there is a circuit containing both.

We will also need the notion of 3-connectedness for matroids. Describing 3-connectedness will require us to introduce some new terminology. The discussion below follows [13, p. 293].

Let $M$ be a matroid with ground set $E$. If $X \subseteq E$, let $\lambda_M(X) = r(X) + r(E - X) - r(M)$. We call $\lambda_M$ the *connectivity function* of $M$. Let $k$ be a positive integer. For $X \subseteq E$, if $\lambda_M(X) < k$, then both $X$ and the pair $(X, E - X)$ are called $k$-*separating*. A 1-separating set is also called a *separator*. A $k$-separating pair $(X, E - X)$ for which $\min\{|X|, |E - X|\} \geq k$ is called a $k$-*separation* of $M$ with *sides* $X$ and $E - X$. A matroid $M$ is 3-*connected* if, for all $k \in \{1, 2\}$, it has no $k$-separations.

Let

Define a relation $\psi$ on $E$ by $e \ \psi \ f$ if either $e = f$, or $M$ has a circuit containing $\{e, f\}$. Then $\psi$ is an equivalence relation on $E$ and the $\psi$-equivalence classes are called *components* of $M$.

### 1.4 Tree Decomposition of Connected Matroids

We now introduce Cunningham and Edmonds's tree decomposition for connected matroids [3]. Our treatment of this material follows [13, pp. 307–310]. A *matroid-labeled tree* is a tree $T$ with vertex set $\{M_1, M_2, \ldots, M_k\}$ for some positive integer $k$ such that

(i) each $M_i$ is a matroid;

(ii) if $M_{j_1}$ and $M_{j_2}$ are joined by an edge $e_i$ of $T$, then $E(M_{j_1}) \cap E(M_{j_2}) = \{e_i\}$, and $\{e_i\}$ is not a separator of $M_{j_1}$ or $M_{j_2}$; and

(iii) if $M_{j_1}$ and $M_{j_2}$ are non-adjacent, then $E(M_{j_1}) \cap E(M_{j_2})$ is empty.

Let $e$ be an edge of a matroid-labeled tree $T$ and suppose $e$ joins vertices labeled by $M_1$ and $M_2$. Suppose that we contract $e$ and relabel by $M_1 \oplus_2 M_2$ the composite vertex that results from identifying the endpoints of $e$. Then we retain a matroid-labeled tree and we denote this tree by $T/e$. This process can be repeated and since the operation of 2-sum is associative, for every subset $\{e_{i_1}, e_{i_2}, \ldots, e_{i_m}\}$ of $E(T)$, the matroid-labeled tree $T/e_{i_1}, e_{i_2}, \ldots, e_{i_m}$ is well-defined.

A *tree decomposition* of a 2-connected matroid $M$ is a matroid-labeled tree $T$ such that if $V(T)$=$\{M_1,\ M_2,\ \ldots,\ M_k\}$ and $E(T)$=$\{e_1,\ e_2,\ \ldots,\ e_{k-1}\}$, then

(i) $E(M)$=$(E(M_1) \cup E(M_2) \cup \cdots \cup E(M_k)) - \{e_1, e_2, \ldots, e_{k-1}\}$;

(ii) $E(M_i) \geq 3$ for all $i$ unless $|E(M)| < 3$, in which case $k$=1 and $M_1$=$M$; and

(iii) $M$ labels the single vertex of $T/e_1, e_2, \ldots, e_{k-1}$.

In general, a tree decomposition of a matroid is not unique. However, Cunningham and Edmonds were able to guarantee uniqueness of the *canonical tree decomposition* described in the following theorem from [3].

**Theorem 1.4.1.** *Each 2-connected matroid $M$ has a tree decomposition $T$ in which every vertex is labeled by a 3-connected matroid, $U_{m-1,m}$ for some $m \geq 3$, or $U_{1,n}$ for some $n \geq 3$. Moreover, there are no two adjacent vertices that are both labeled by uniform matroids of rank one or are both labeled by uniform matroids of corank one, and $T$ is unique to within a relabeling of its edges.*

The canonical tree decomposition provides a unique way to decompose a 2-connected matroid $M$ into 3-connected pieces, uniform matroids of rank one, and uniform matroids of corank one. Moreover, we can reconstruct $M$ from these pieces using the 2-sum operation with the common elements between matroids as basepoints. A basic property of the 2-sum operation is that $M_1$ and $M_2$ are minors of $M_1 \oplus_2 M_2$. The following two results are well known; the first is trivial while the proof of the second is omitted.

**Lemma 1.4.2.** *Let $M_1$ and $M_2$ be matroids. Then $r(M_1 \oplus_2 M_2) = r(M_1) + r(M_2) - 1$ and $r^*(M_1 \oplus_2 M_2) = r^*(M_1) + r^*(M_2) - 1$.*

**Lemma 1.4.3.** *Let $M_1$ and $M_2$ label vertices in a tree decomposition $T$ of a connected matroid $M$. Let $P$ be the path in $T$ joining $M_1$ and $M_2$, and let $p_1$ and $p_2$ be the edges of $P$ meeting $M_1$ and $M_2$ respectively. In other words, $p_1$ and $p_2$ are basepoints for 2-sums in the reconstruction of $M$. Then $M$ has a minor isomorphic to the 2-sum of $M_1$ and $M_2$, where $p_1{=}p_2$ is the basepoint of the 2-sum.*

The dual of a 2-sum of two matroids is the 2-sum of the dual matroids. The following lemma [13, Lemma 8.3.8] is an immediate consequence of this fact.

**Lemma 1.4.4.** *If $T$ is a tree decomposition for a 2-connected matroid $M$ and every vertex label is replaced by its dual, then the resulting matroid-labeled tree $T^*$ is a tree decomposition for $M^*$.*

In Chapter 3, we will make frequent use of the following well-known result, illustrated in Figure 1.1. Its proof is omitted.

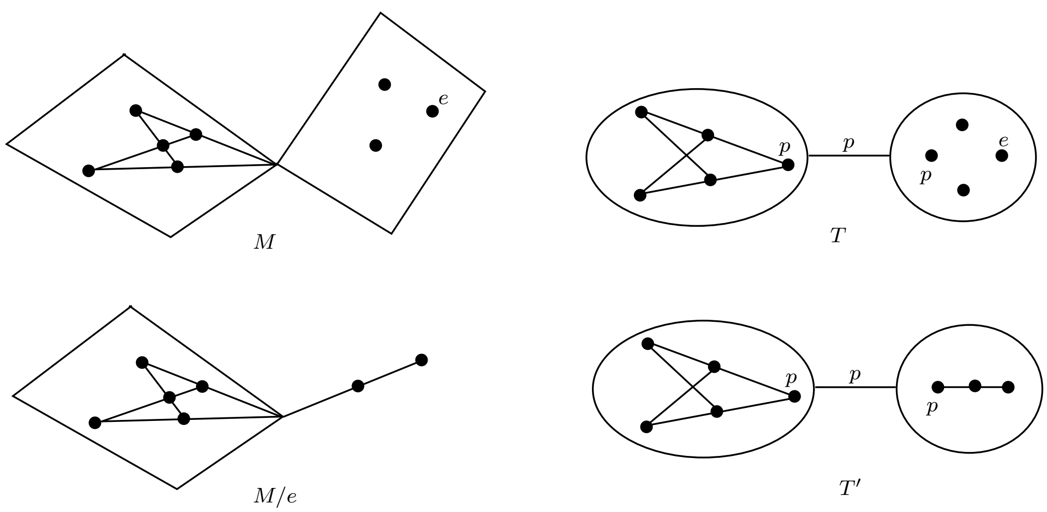



Figure 1.1. Contracting an element in a tree decomposition.

**Lemma 1.4.5.** *Let $M$ be a matroid and let $T$ be its canonical tree decomposition. Assume there is a corank-one matroid $C$ that labels a vertex of $T$. Let $e$ be an element of $M$, where $e$ is an element contributed to $M$ by $C$ in $T$. Then the tree decomposition $T'$ of $M/e$ is obtained in one of the following ways.*

*(i) If* $|C| = 3$*, then* $T'$ *is obtained from* $T$ *by deleting the vertex labelled by* $C$ *and identifying any elements of* $C$ *that are basepoints of* $2$*-sums in* $T$*.*

*(ii) If* $|C| \geq 4$*, then* $T'$ *is obtained from* $T$ *by replacing the label* $C$ *with* $C/e$*.*

# Chapter 2
# Nearly Binary Matroids[1]

## 2.1 Introduction

The class of binary matroids is one of the most widely studied classes of matroids and its members have numerous attractive properties. This motivates the study of classes of matroids whose members are close to being binary. In this chapter, we consider one very natural such minor-closed class $\mathcal{Z}$, which consists of those matroids $M$ such that $M\backslash e$ or $M/e$ is binary for all elements $e$ of $M$. The main result of the chapter is an excluded-minor characterization of $\mathcal{Z}$. This theorem can be restated in terms of matroid fragility, which has enjoyed a recent surge of research interest. Let $N$ be a matroid. A matroid $M$ is *$N$-fragile* if, for each element $e$ of $E(M)$, at least one of $M\backslash e$ and $M/e$ has no $N$-minor (see, for example, [8]). The class of $N$-fragile matroids is clearly minor-closed. The main result of this chapter determines the set of excluded minors for the class of $U_{2,4}$-fragile matroids.

It is well known that if $H$ is a circuit and a hyperplane of a matroid $M$, then there is another matroid $M'$ on $E(M)$ whose bases are the bases of $M$ together with $H$. We say that $M'$ is obtained from $M$ by *relaxing* the *circuit-hyperplane* $H$ and call $M'$ a *relaxation* of $M$. A class of matroids that arises naturally in determining the excluded minors for $\mathcal{Z}$ is $\mathcal{R}$, those matroids $M$ such that $M$ is binary or $M$ is a relaxation of a binary matroid.

The rank-three whirl is denoted by $\mathcal{W}^3$, while $P_6$ is the six-element rank-three matroid that has a single triangle as its only non-spanning circuit. Let $Q_6$ and $R_6$ be the six-element matroids of rank three for which geometric representations are given in Figure 2.1. Evidently $R_6 \cong U_{2,4} \oplus_2 U_{2,4}$. Let $K$ be the seven-element rank-two matroid that is obtained by adding elements in parallel to three of the elements of $U_{2,4}$. The matroid $K$ is depicted with its dual

[1]This chapter is substantially the same as the following paper: J. Oxley and J. Taylor, On two classes of nearly binary matroids, European J. Combin. 36 (2014) 251-260. See Appendix B for publisher's permission to reprint.

in Figure 2.2. In Section 2.2, we note that both $\mathcal{Z}$ and $\mathcal{R}$ are minor-closed and dual-closed classes of matroids and establish some excluded minors of each. We also introduce some preliminaries.

Let $\mathcal{D}$ denote the collection of all matroids that are obtained from connected binary matroids by relaxing two disjoint circuit-hyperplanes that partition the ground set. The collection $\mathcal{D}$ is in both our sets of excluded minors. Section 2.3 is devoted to proving the main result and another related result, both of which are stated next.

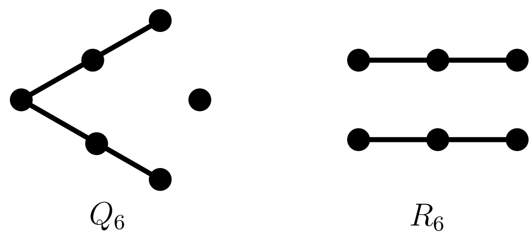



Figure 2.1. Geometric representations of the six-element rank-three matroids $Q_6$ and $R_6$.

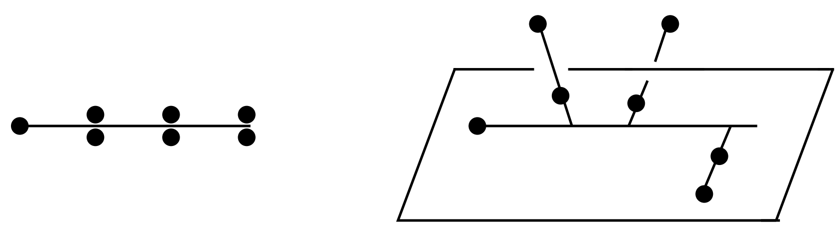


Figure 2.2. Representations of the matroid $K$ and its (rank-5) dual $K^*$.

**Theorem 2.1.1.** *The set of excluded minors for the class of matroids* $\mathcal{Z}=\{M : M\backslash e$ *or* $M/e$ *is binary for all* $e$ *in* $E(M)\}$ *is* $\{Q_6, P_6, U_{3,6}, R_6, U_{2,4} \oplus U_{1,1}, U_{2,4} \oplus U_{0,1}\} \cup \mathcal{D}$.

**Theorem 2.1.2.** *The set of excluded minors for the class* $\mathcal{R}$ *of matroids* $M$ *such that* $M$ *is binary or can be obtained from a binary matroid by relaxing a circuit-hyperplane, is* $\{U_{2,5}, U_{3,5}, K, K^*, R_6, U_{2,4} \oplus U_{1,1}, U_{2,4} \oplus U_{0,1}\} \cup \mathcal{D}$.

For an even integer $r$ exceeding two, let $M_r$ be the rank-$r$ tipless binary spike, that is, the vector matroid of the binary matrix $[I_r|J_r - I_r]$ where $J_r$ is the matrix of all ones. Labeling the columns of this matrix $e_1, e_2, \ldots, e_{2r}$ in order, we see that $\{e_2, e_3, \ldots, e_r, e_{r+1}\}$ and its complement are both circuit-hyperplanes of $M_r$. By relaxing these circuit-hyperplanes, we obtain a member of $\mathcal{D}$. Thus the sets of excluded minors in Theorems 2.1.1 and 2.1.2 are both infinite. However, these doubly relaxed spikes are not the only members of $\mathcal{D}$. In Section 2.4, we further discuss the complexity of $\mathcal{D}$.

As $\mathcal{D}$ shows, the class of matroids that can be obtained from binary matroids by relaxing at most two circuit-hyperplanes does contain an infinite antichain. Geelen, Gerards, and Whittle announced in 2009 that the class of binary matroids itself contains no infinite antichains. These observations raise the interesting question, which was asked by a referee of [14], as to whether or not the class $\mathcal{R}$ contains an infinite antichain. It is not difficult to check using, for example, [6, Lemma 2.6], that $\mathcal{Z}$ contains an infinite antichain if and only if $\mathcal{R}$ does.

## 2.2 Preliminaries

This section first notes that both $\mathcal{Z}$ and $\mathcal{R}$ are minor- and dual-closed, and then determines some excluded minors for each class.

**Lemma 2.2.1.** *The classes $\mathcal{Z}$ and $\mathcal{R}$ are both closed under duality and the taking of minors.*

*Proof.* Take $M$ in $\mathcal{Z}$. Let $x$ be in $E(M)$. At least one of $M/x$ and $M\backslash x$ is binary, so assume $M/x$ is. Then $M/x$ is certainly in the class. As for $M\backslash x$, let $y$ be in $E(M\backslash x)$. We know that at least one of $M/y$ and $M\backslash y$ is binary. Thus at least one of $M\backslash x/y$ and $M\backslash x\backslash y$ is binary. Hence $\mathcal{Z}$ is minor-closed.

Now consider $N'$ in $\mathcal{R}$. Either $N'$ is binary, or $N'$ can be obtained from a binary matroid $N$ by relaxing a circuit-hyperplane. If $N'$ is binary, then all minors of $N'$ will be in $\mathcal{R}$. Assume $N'$ can be obtained from a binary matroid $N$ by relaxing a circuit-hyperplane $X$. For any $f \in X$, Lemma 2.2.2 tells us $N'\backslash f{=}N\backslash f$ and, unless $N$ has $f$ as a loop, $N'/f$ is obtained from $N/f$ by relaxing the circuit-hyperplane $X - f$ of the latter. If $f$ is a loop of $N$, then $f = X$ and $r(N) = 1$, so $N'$ is binary. So both $N'\backslash f$ and $N'/f$ are in $\mathcal{R}$. Now take $e \in E(N) - X$. Lemma 2.2.2 tells us $N'/e{=}N/e$ and, unless $N$ has $e$ as a coloop, $N'\backslash e$ is obtained from $N\backslash e$

by relaxing the circuit-hyperplane $X$ of the latter. And if $e$ is a coloop of $N$, then $N'$ is a circuit, which is binary. Thus both $N'\backslash e$ and $N'/e$ are in $\mathcal{R}$, hence $\mathcal{R}$ is minor-closed. □

This lemma is immediate for $\mathcal{Z}$ and is a straightforward consequence of the following result of Kahn [7] for $\mathcal{R}$ (see also [13, p. 115]).

**Lemma 2.2.2.** *Let $X$ be a circuit-hyperplane of a matroid $M$ and let $M'$ be the matroid obtained from $M$ by relaxing $X$. Then $(M')^*$ is obtained from $M^*$ by relaxing the circuit-hyperplane $E(M) - X$ of the latter. Moreover, when $e \in E(M) - X$, $M/e$ and $M'/e$ are equal and, unless $M$ has $e$ as a coloop, $M'\backslash e$ is obtained from $M\backslash e$ by relaxing the circuit-hyperplane $X$ of the latter, and the dual situation holds when $e \in X$.*

It is not difficult to deduce from the above result that the class $\mathcal{R}$ is contained in the class $\mathcal{Z}$. We say a matroid $N$ is a *series extension* of a matroid $M$ if $M = N/T$ and every element of $T$ is in series with some element of $M$. We call $N$ a *parallel extension* of $M$ if $N^*$ is a series extension of $M^*$. Note that this differs from the terminology used in [13]. The following result from [12] will be used extensively throughout the chapter.

**Theorem 2.2.3.** *A matroid $M$ is non-binary and in $\mathcal{Z}$ if and only if*

*(i) both $r(M)$ and $r^*(M)$ exceed two and $M$ can be obtained from a connected binary matroid by relaxing a circuit-hyperplane; or*

*(ii) $M$ is isomorphic to a parallel extension of $U_{2,n}$ for some $n \geq 5$; or*

*(iii) $M$ is isomorphic to a series extension of $U_{n-2,n}$ for some $n \geq 5$; or*

*(iv) $M$ can be obtained from $U_{2,4}$ by series extension of a subset $S$ of $E(U_{2,4})$ and parallel extension of a disjoint subset $T$ of $E(U_{2,4})$ where $S$ or $T$ may be empty.*

Let $EX(\mathcal{M})$ denote the class of excluded minors for a class of matroids $\mathcal{M}$. Some excluded minors for $\mathcal{Z}$ and $\mathcal{R}$ are easy to identify. We omit the routine argument that establishes the following.

**Lemma 2.2.4.** *The matroids $U_{2,4} \oplus U_{1,1}$, $U_{2,4} \oplus U_{0,1}$, and $R_6$ are in both $EX(\mathcal{Z})$ and $EX(\mathcal{R})$.*

*Proof.* First consider the matroids $U_{2,4} \oplus U_{1,1}$ and $U_{2,4} \oplus U_{0,1}$. Clearly they are not in $\mathcal{Z}$, as deleting or contracting the coloop and loop, respectively, gives the matroid $U_{2,4}$. Therefore they are also not in $\mathcal{R}$. Now consider their minors. Deleting or contracting the coloop or loop element gives the matroid $U_{2,4}$, which can be obtained from $U_{2,3} \oplus_2 U_{1,3}$ by relaxing the parallel elements. Thus $U_{2,4}$ is in $\mathcal{R}$, hence also in $\mathcal{Z}$. Now consider deleting or contracting any other element. The result will be a binary matroid, which will be in both $\mathcal{Z}$ and $\mathcal{R}$. Thus $U_{2,4} \oplus U_{1,1}$ and $U_{2,4} \oplus U_{0,1}$ are in both $EX(\mathcal{Z})$ and $EX(\mathcal{R})$.

Now we consider $R_6$. Deletion of any element gives the five-element rank-3 matroid whose only non-spanning circuit is a triangle, and contraction of any element gives the matroid $U_{2,4} \oplus_2 U_{1,3}$. Neither of these matroids is binary, thus $R_6$ is not in $\mathcal{Z}$, nor $\mathcal{R}$. Let $e$ be in $E(R_6)$, and consider $R_6/e$. The matroid $R_6/e$ can be realized as the relaxation of a triangle in which parallel elements are added to two sides. Therefore $R_6/e$ is in $\mathcal{R}$, hence also in $\mathcal{Z}$. As for $R_6 \backslash e$, it can be realized as the relaxation of the five-element rank-three matroid consisting of two triangles that share one common element. Hence $R_6 \backslash e$ is in $\mathcal{R}$, hence also in $\mathcal{Z}$, and $R_6$ is an excluded minor for both $\mathcal{Z}$ and $\mathcal{R}$. □

The following three results will also be useful, the first is from [7]; the second is elementary; the third follows from the first two.

**Lemma 2.2.5.** *Let $M'$ be obtained from $M$ by relaxing a circuit-hyperplane.*

*(i) If $M$ is connected, then $M'$ is non-binary; and*

*(ii) if $M$ is $n$-connected, then so is $M'$.*

**Lemma 2.2.6.** *The only disconnected matroids having a circuit-hyperplane are $U_{n-1,n} \oplus U_{1,k}$, for integers $n$, $k \geq 1$.*

*Proof.* Assume $M$ is a disconnected matroid having a circuit-hyperplane $C$. If $C$ is a loop, then the result holds, so assume not. Then $M = M_1 \oplus M_2$ for some matroids $M_1$ and $M_2$. We know $C$ is contained in one of $M_1$ and $M_2$, say $M_1$. As $C$ is a hyperplane, it follows that $M$ has no loops and that $r(M_2) = 1$ and $C{=}M_1$. $\square$

**Corollary 2.2.7.** *Let $M$ be a binary matroid, $H$ be a circuit-hyperplane of $M$, and $M'$ be obtained from $M$ by relaxing $H$. Then $M'$ is binary if and only if $M$ is $U_{n-1,n} \oplus U_{1,k}$, for integers $n$, $k \geq 1$.*

Note that, in Lemma 2.2.6 and Corollary 2.2.7, the disconnected matroids are graphic and carry the name *enlarged* 1-*wheels* in [15].

Recall, $\mathcal{D}$ is the collection of all matroids that are obtained from connected binary matroids by relaxing two disjoint circuit-hyperplanes that partition the ground set.

**Lemma 2.2.8.** *All matroids in $\mathcal{D}$ are in both $EX(\mathcal{Z})$ and $EX(\mathcal{R})$.*

*Proof.* Take a matroid $M_2$ in $\mathcal{D}$. Let $X$ and $Y$ be the disjoint circuit-hyperplanes of the connected binary matroid $M$ that are relaxed to obtain $M_2$. Let $M_X$ and $M_Y$ denote the matroids obtained from $M$ by relaxing $X$ and $Y$, respectively, and take $e$ in $E(M_2)$. Note that the case with $e \in X$ is symmetric to the case with $e \in Y$; both $\mathcal{Z}$ and $\mathcal{R}$ are dual-closed classes, and since $X$ and $Y$ are complementary circuit-hyperplanes of $M$, they are so for $M^*$ as well.

Suppose $e \in X$. By Lemma 2.2.2, $M_2/e$ is obtained from $M_Y/e$ by relaxing the circuit-hyperplane $X-e$ of the latter and $M_Y/e{=}M/e$. If $M/e$ is connected, then $M_2/e$ is non-binary by Lemma 2.2.5. Now assume $M/e$ is disconnected. Then $M/e{=}U_{n-1,n} \oplus U_{1,k}$ for some $n$, $k \geq 1$, by Lemma 2.2.6. But $Y$ is a spanning circuit in $M/e$, which is a contradiction since $M/e$ has no spanning circuits. We conclude that $M_2/e$ is non-binary. By symmetry and duality the same argument holds for $M_2\backslash e$, and for both $M_2/f$ and $M_2\backslash f$ when $f \in Y$.

Any deletion $M_2\backslash z$ equals $M_Y\backslash z$ or $M_X\backslash z$. By symmetry we only need to consider the case with $z \in X$. The matroid $M_Y\backslash z$ can be obtained by relaxing a circuit-hyperplane in a binary matroid. By duality, the same holds for $M_2/z$. Therefore any minor of $M_2$ is in $\mathcal{R}$ and so is in $\mathcal{Z}$. Thus $M_2$ is in $EX(\mathcal{R})$ and in $EX(\mathcal{Z})$. □

The next two lemmas list matroids that are excluded minors for exactly one of $\mathcal{R}$ and $\mathcal{Z}$.

**Lemma 2.2.9.** *The matroids $U_{2,5}$, $U_{3,5}$, $K$, and $K^*$ are excluded minors for the class $\mathcal{R}$.*

*Proof.* First note that neither $U_{2,5}$ nor $U_{3,5}$ is binary. Choose any basis of the matroid $U_{2,5}$ and tighten it. The result is the matroid $U_{2,4} \oplus_2 U_{1,3}$, which is not binary. Thus $U_{2,5}$ may only be realized as a relaxation of a non-binary matroid, so it is not in $\mathcal{R}$. Now let $e$ be in $E(U_{2,5})$ and consider minors of $U_{2,5}$. If we delete $e$ we get the matroid $U_{2,4}$, which we discussed above. If we contract $e$ we get the matroid $U_{1,4}$, which is binary. Thus $U_{2,5}$ is an excluded minor for $\mathcal{R}$. By duality, we also know that $U_{3,5}$ is an excluded minor for $\mathcal{R}$.

Now consider $K$, which is non-binary. Every basis of $K$ contains an element in a non-trivial parallel class. Therefore, by tightening any basis we cannot achieve a hyperplane since the other element in the parallel class will be in the closure of the tightened basis. Thus there is no way to tighten a basis of $K$ to obtain a circuit-hyperplane. Hence, $K$ cannot be realized as a relaxation and it is not in $\mathcal{R}$. Let $e$ denote the element in the trivial parallel class, and let $f$ denote any other element of $K$. First note that $K\backslash e$, $K/e$, and $K/f$ are all binary matroids. So consider $K\backslash f$. This matroid can be realized as a relaxation of the binary matroid that is a triangle where all sides are in parallel classes of size two. Thus $K$ is an excluded minor for $\mathcal{R}$. By duality, we also know that $K^*$ is an excluded minor for $\mathcal{R}$. □

**Lemma 2.2.10.** *The matroids $Q_6$, $P_6$, and $U_{3,6}$ are excluded minors for the class $\mathcal{Z}$.*

*Proof.* Consider the matroids $Q_6$, $P_6$, and $U_{3,6}$ as depicted in Figure 2.3. Note that all three of these matroids are non-binary. Clearly, by deleting or contracting the element labeled 1 in each of these matroids we obtain a non-binary matroid. Therefore none of $Q_6$, $P_6$, and $U_{3,6}$ are in $\mathcal{Z}$.

First we show that $Q_6$ is an excluded minor. Note that $Q_6\backslash 1 \cong Q_6\backslash 2 \cong Q_6\backslash 4 \cong Q_6\backslash 5$, and $Q_6/1 \cong Q_6/2 \cong Q_6/4 \cong Q_6/5$. So we only consider minors obtained by deleting and contracting element 1 in the set $\{1, 2, 4, 5\}$. By deleting 1 in $Q_6$ we get the five-element rank-three matroid whose only non-spanning circuit is a triangle. This matroid can be realized as a relaxation of the five-element rank-three matroid consisting of two triangles that share one common element, which implies $Q_6/1$ is in $\mathcal{Z}$. By contracting 1 in $Q_6$, we get the matroid $U_{2,4} \oplus_2 U_{1,3}$. Contraction of any element in $U_{2,4} \oplus_2 U_{1,3}$ gives the matroid $U_{1,4}$, so $Q_6/1$ is also in $\mathcal{Z}$. Next consider deleting element 3. This gives the matroid $U_{3,5}$, and deleting any element from $U_{3,5}$ gives a binary matroid. Contracting element 3 gives a binary matroid, so finally consider element 6. By deleting element 6 we get a binary matroid, and by contracting element 6 we get $U_{2,5}$. Contracting any element in $U_{2,5}$ gives the binary matroid $U_{1,4}$, hence $Q_6$ is an excluded minor of $\mathcal{Z}$.

Next we look at $P_6$. Note that $P_6\backslash 1 \cong P_6\backslash 2 \cong P_6\backslash 3$, and $P_6\backslash 4 \cong P_6\backslash 5 \cong P_6\backslash 6$, and we have the analogous situation for contraction. Therefore we only need to consider deleting and contracting the elements 1 and 4. Note that $P_6\backslash 1 \cong Q_6\backslash 1$, so $P_6\backslash 1$ is in $\mathcal{Z}$. Contracting 1 gives the matroid $U_{2,5}$, which we discussed above. Deleting element 4 gives the matroid $U_{3,5}$, which we also discussed above. And $P_6/4 \cong Q_6/1$, so $P_6/4$ is in $\mathcal{Z}$. Thus $P_6$ is an excluded minor for $\mathcal{Z}$.

Finally we look at $U_{3,6}$. The deletion of any element yields the matroid $U_{3,5}$, and the contraction of any element yields the matroid $U_{2,5}$, both of which are in $\mathcal{Z}$. Therefore $U_{3,6}$ is also an excluded minor of $\mathcal{Z}$. $\square$

A class $\mathcal{N}$ of matroids is 1-*rounded* [17] if every member of $\mathcal{N}$ is connected and, whenever $e$ is an element of a connected matroid $M$ having an $\mathcal{N}$-minor, $M$ has an $\mathcal{N}$-minor using

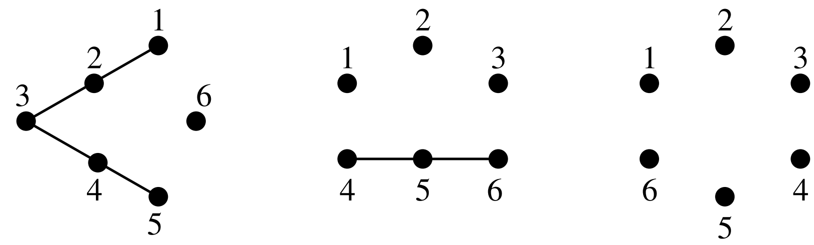



Figure 2.3. Geometric representations of the matroids $Q_6$, $P_6$, and $U_{3,6}$.

$e$. The following three results will be useful in our proofs, they come from [2], [17], and [9], respectively.

**Lemma 2.2.11.** *The set* $\{U_{2,4}\}$ *is* 1*-rounded.*

**Lemma 2.2.12.** *The set* $\{M(K_4), U_{2,4}\}$ *is* 1*-rounded.*

**Lemma 2.2.13.** *The set* $\{\mathcal{W}^3, P_6, Q_6, U_{3,6}\}$ *is* 1*-rounded.*

The following two results will also be needed. The first is basic and its proof is omitted. The second result comes from [11].

**Lemma 2.2.14.** *The class of binary matroids is closed under the operation of* 2*-sum.*

**Lemma 2.2.15.** *The following statements are equivalent for a* 3*-connected matroid* $M$ *having rank and corank at least three:*

*(i)* $M$ *has a* $U_{2,5}$*-minor;*

*(ii)* $M$ *has a* $U_{3,5}$*-minor;*

*(iii)* $M$ *has a minor isomorphic to one of* $P_6$, $Q_6$, *or* $U_{3,6}$.

### 2.3 Main Result

In this section we prove the main results of the chapter, Theorems 2.1.1 and 2.1.2. We begin by finding all the disconnected excluded minors of each class. Due to the similarity of the proofs for each class, we combine the arguments where possible.

**Lemma 2.3.1.** *Suppose $\mathcal{U} \in \{\mathcal{Z}, \mathcal{R}\}$. The only disconnected members of $EX(\mathcal{U})$ are $U_{2,4} \oplus U_{1,1}$ and $U_{2,4} \oplus U_{0,1}$.*

*Proof.* By Lemma 2.2.4, both matroids are in $EX(\mathcal{U})$. Now let $M$ be an arbitrary disconnected member of $EX(\mathcal{U})$. As $M$ is non-binary and disconnected, it has distinct components $M_1$ and $M_2$ where $M_1$ is non-binary. Since $M_1$ has a $U_{2,4}$-minor and $M_2$ has a $U_{0,1}$- or $U_{1,1}$-minor, the lemma follows. □

The following result from [11] will be useful in our proofs.

**Theorem 2.3.2.** *Let $M$ be a 3-connected matroid having rank and corank exceeding two.*

*(i) If $M$ is binary, then $M$ has an $M(K_4)$-minor.*

*(ii) If $M$ is non-binary, then $M$ has one of $\mathcal{W}^3$, $Q_6$, $P_6$, and $U_{3,6}$ as a minor.*

Before finding the complete list of 2-connected excluded minors, we need the following lemmas. The first lemma comes from [13, Section 1.5, Exercise 14].

**Lemma 2.3.3.** *The following statements are equivalent for a matroid $M$:*

*(a) $M$ is a relaxation of some matroid,*

*(b) $M$ has a basis $B$ such that $B \cup e$ is a circuit of $M$ for every $e$ in $E(M) - B$ and neither $B$ nor $E(M) - B$ is empty.*

*Proof.* Assume $M$ is a relaxation of some matroid $N$. Then $M$ has a basis $B$ that is a circuit-hyperplane in $N$. Assume $B \cup e$ is not a circuit for some $e \in E(M) - B$. Let $C$ denote the circuit contained in $B \cup e$. Consider the circuit-hyperplane $B$ in $N$. As all of $C$ is contained in $B$ except $e$, the element $e$ should be in the closure of $B$ in the matroid $N$. But $B$ is a closed set and $e \notin B$, a contradiction. Thus $B \cup e$ is a circuit of $M$ for all $e \in E(M) - B$. Clearly $B$ is non-empty, and since $B$ is a circuit-hyperplane in $N$ there must be some element $x \in E(N) - B$ that is not spanned by $B$ in $N$, thus $E(M) - B$ is also non-empty.

Conversely assume $M$ has a basis $B$ such that $B \cup e$ is a circuit of $M$ for every $e$ in $E(M) - B$ and neither $B$ nor $E(M) - B$ is empty. By tightening the basis $B$ to make a new matroid $N$, we make $B$ a circuit of rank $r(M) - 1$ automatically. Assume that $B$ is not a hyperplane. Then there is some element $e \in E(M) - B$ such that a proper subset of $B$ together with $e$ forms a circuit $C$. But then $C_M(e, B)=C$, a contradiction. Thus $B$ is a circuit-hyperplane of $N$, and $M$ is a relaxation of the matroid $N$. □

**Lemma 2.3.4.** *Let $M$ be a matroid that can be obtained from a binary matroid $N$ by relaxing a circuit-hyperplane $X$ of the latter. If $M$ contains a $\mathcal{W}^k$-minor for some $k \geq 3$, then, in every $\mathcal{W}^k$-minor of $M$, the rim elements are contained in $X$ and no element of $X$ is a spoke.*

*Proof.* Let $M_1$ be a $\mathcal{W}^k$-minor of $M$. If $e$ is in the rim of $M_1$, then $M_1/e$ is non-binary. But, for all $f$ in $E(M) - X$, by Lemma 2.2.2, $M/f$ is binary. Therefore $e \in X$. The assertion about spokes follows by duality. □

**Lemma 2.3.5.** *Let $M$ be a connected non-binary matroid. Either $M$ has an $R_6$-, $U_{2,4} \oplus U_{0,1}$-, or $U_{2,4} \oplus U_{1,1}$-minor, or $M$ is obtained from a 3-connected non-binary matroid $M_0$ by parallel and series extension of disjoint subsets $T$ and $S$ of $E(M_0)$, where both $S$ and $T$ are possibly empty.*

*Proof.* Consider the canonical tree decomposition $T$ of $M$. As $M$ is non-binary, by Lemma 2.2.14 there must be a non-binary matroid $M_0$ in $T$. Assume there is another vertex labeled by a non-binary matroid $M_1$. Then, by Lemma 1.4.3, we see that $M$ has an $M_0 \oplus_2 M_1$-minor. Let $p_1$ be the basepoint of this 2-sum. Each of $M_0$ and $M_1$ is connected and non-binary, so by Lemma 2.2.11 each of $M_0$ and $M_1$ has a $U_{2,4}$-minor that uses $p_1$. Thus $M$ has an $R_6$-minor, and the lemma holds when $M_1$ exists.

We may now assume that $M_0$ is the unique non-binary matroid labeling a vertex of $T$. Suppose there is a vertex labeled by a 3-connected binary matroid $M_2$ with at least four elements. Then $M$ has an $M_0 \oplus_2 M_2$-minor. Now $M_0$ has a $U_{2,4}$-minor and, as $M_2$ is 3-connected and binary, Theorem 2.3.2 tells us that $M_2$ has an $M(K_4)$-minor. Let $p_2$ be the basepoint of $M_0 \oplus_2 M_2$. As above, $M_0$ has a $U_{2,4}$-minor using $p_2$. By Lemma 2.2.12, $M_2$ has an $M(K_4)$-minor using $p_2$. Thus $M$ has a $U_{2,4} \oplus_2 M(K_4)$-minor and therefore has a $U_{2,4} \oplus U_{1,1}$-minor. Hence the lemma holds when $M_2$ exists.

We may now assume all matroids other than $M_0$ labeling vertices in $T$ are $U_{1,n}$ or $U_{m-1,m}$ for varying $n$, $m \geq 3$. If we have a path in $T$ beginning at $M_0$ that has the form $M_0$—$U_{m-1,m}$—$U_{1,n}$, then $M$ has a $U_{2,4} \oplus U_{0,1}$-minor. By duality, we may not have a path of the form $M_0$—$U_{1,n}$—$U_{m-1,m}$. Therefore we may assume the only non-trivial paths beginning at $M_0$ in $T$ are of the form $M_0$—$U_{m-1,m}$, or $M_0$—$U_{1,n}$. In other words, $M$ is obtained from $M_0$ by parallel and series extension of disjoint subsets of $E(M_0)$. $\square$

Recall that the matroid $K$ is the matroid obtained from $U_{2,4}$ by adding elements in parallel to three of its elements.

**Lemma 2.3.6.** *The matroid $R_6$ is the only connected, but not 3-connected, member of $EX(\mathcal{Z})$. The connected, but not 3-connected, members of $EX(\mathcal{R})$ are $R_6$, $K$, and $K^*$.*

*Proof.* Suppose $\mathcal{U} \in \{\mathcal{Z}, \mathcal{R}\}$. By Lemmas 2.2.4 and 2.2.9, $R_6$ is in $EX(\mathcal{U})$ and $K$ and $K^*$ are in $EX(\mathcal{R})$. Let $M$ be a 2-connected member of $EX(\mathcal{U})$ that is not 3-connected and is not $R_6$, $K$, or $K^*$. By Lemma 2.3.5, $M$ is obtained from a 3-connected non-binary matroid $M_0$ by parallel and series extension of disjoint subsets $T$ and $S$ of $E(M_0)$ where $S \cup T \neq \emptyset$.

Let $M_0 \cong U_{2,4}$. If $\mathcal{U}=\mathcal{Z}$, then $M$ is in $\mathcal{U}$, as it satisfies (iv) in Theorem 2.2.3, which is a contradiction, so let $\mathcal{U}=\mathcal{R}$. As $M$ has neither $K$ nor $K^*$ as a minor, both $S$ and $T$ have size less than three. By duality, we may assume that $0 \leq |T| \leq |S| \leq 2$. In each case, $M$ can be realized as a relaxation of a binary matroid. For example, when $|S| = |T| = 2$, assume the non-trivial series classes have sizes $s_1$ and $s_2$, and the non-trivial parallel classes have sizes $p_1$ and $p_2$. We can obtain $M$ by relaxing the circuit-hyperplane in $M(G)$ where $G$ is a graph on three vertices $\{a, b, c\}$ with $p_1$ parallel edges between $a$ and $c$, $p_2$ parallel edges between $b$ and $c$, and two internal vertex disjoint paths with sizes $s_1$ and $s_2$ between $a$ and $b$. The other cases can be checked similarly. We deduce contradictorily that $M \in \mathcal{U}$.

We may now assume $|E(M_0)| \geq 5$ and consider $\mathcal{U} \in \{\mathcal{Z}, \mathcal{R}\}$. By switching to the dual if necessary, we may also assume that $M$ has at least one non-trivial parallel class and let $\{x, y\}$ be in that class.

**2.3.6.1.** *The matroid $M\backslash x$ can be obtained from a binary matroid by relaxing a circuit-hyperplane.*

*Proof.* We know $M\backslash x$ is in $\mathcal{U}$. Thus it satisfies one of (i)-(iv) in Theorem 2.2.3. If $M\backslash x$ satisfies (i), then the result follows. Assume $M\backslash x$ satisfies (ii). Then $M\backslash x$ is a parallel extension of $U_{2,n}$, for some $n \geq 5$. Hence $M$ is also a parallel extension of this matroid and $M \in \mathcal{Z}$, and $M$ has a $U_{2,5}$-minor, which contradicts Lemma 2.2.9 if $\mathcal{U}$=$\mathcal{R}$. Next assume that $M\backslash x$ satisfies (iii). Then $M\backslash x$ is a series extension of $U_{n-2,n}$ for some $n \geq 5$, and $M$ is a parallel extension of this series extension. Then $M\backslash x$, and hence $M$, contains the excluded minor $U_{2,4} \oplus U_{0,1}$. Lastly, assume $M\backslash x$ satisfies (iv). Let $U = E(U_{2,4}) - S - T$, where $S$ and $T$ are as defined in Theorem 2.2.3. Recall that $\{x, y\}$ is a circuit of $M$. If $y$ is in a non-trivial series class of $M\backslash x$, then $M$ contains the excluded minor $U_{2,4} \oplus U_{0,1}$, so $y \in T \cup U$. Therefore $M$ satisfies (iv), so $M \in \mathcal{Z}$ and we assume $\mathcal{U}$=$\mathcal{R}$. As $M$ has neither $K$ nor $K^*$ as a minor, $|S| < 3$ and $|T \cup y| < 3$. As noted above, in these cases $M$ can be realized as a relaxation of a binary matroid. $\square$

If $r(M_0) = 2$, then $M_0$ has a $U_{2,5}$-minor, so assume $\mathcal{U}$=$\mathcal{Z}$. It is not hard to check that we get a contradiction in this case by establishing that either $M \in \mathcal{Z}$, or $M$ contains a $U_{2,4} \oplus U_{1,1}$-minor. Thus we may assume $\mathcal{U} \in \{\mathcal{Z}, \mathcal{R}\}$, $r(M_0) \geq 3$ and, by duality, $r^*(M_0) \geq 3$. As $M_0$ is non-binary and 3-connected, and all of $P_6$, $Q_6$, and $U_{3,6}$ are either in $EX(\mathcal{U})$ or contain members of $EX(\mathcal{U})$, Theorem 2.3.2 implies that $M_0$ contains a $\mathcal{W}^3$-minor. By 2.3.6.1,

$M\backslash x$ is a relaxation of a binary matroid $N$, so let $B$ be the circuit-hyperplane relaxed in $N$ to produce $M\backslash x$. Assume $y \notin B$ and let $N_1$ be obtained from $N$ by adding $x$ back in parallel to $y$. Then $B$ is a circuit-hyperplane of $N_1$ whose relaxation is $M$, a contradiction.

We may now assume $y \in B$. Since $M_0$ has a $\mathcal{W}^3$-minor and no $P_6$-, $Q_6$-, or $U_{3,6}$-minor, by Lemma 2.2.13 $M_0$ has a $\mathcal{W}^3$-minor $M_y$ using $y$. By Lemma 2.3.4, we know that $y$ is a rim element of $M_y$. This implies that $M$ has a $\mathcal{W}^3$-minor in which one of the rim elements is replaced by the parallel class containing $\{x, y\}$. This is a contradiction since it implies $M$ has a $U_{2,4} \oplus U_{0,1}$-minor. $\square$

In finding the complete list of 3-connected excluded minors for each class, we use the following lemma.

**Lemma 2.3.7.** *If a matroid $N'$ is obtained from a non-binary matroid $N$ by relaxing a circuit-hyperplane $X$, then*

*(i) $N'$ has a $U_{2,5}$- or $U_{3,5}$-minor; or*

*(ii) $N'$ has a matroid in the class $\mathcal{D}$ as a minor.*

*Proof.* As $N$ is non-binary and has a circuit-hyperplane, $|E(N)| \geq 5$. If $|E(N)| = 5$, then either $N$ is $U_{2,4} \oplus_2 U_{1,3}$, in which case $N'$ is $U_{2,5}$, or $N$ is $U_{2,4} \oplus_2 U_{2,3}$, in which case $N'$ is $U_{3,5}$. Thus the result holds if $|E(N)| = 5$. Now assume that the result holds for $|E(N)| < k$, and consider the case where $|E(N)| = k \geq 6$. If $r(N) = 2$, then $N'$ has a $U_{2,5}$-minor and the result holds. Dually, the result holds if $r^*(N) = 2$, so assume $r(N), r^*(N) \geq 3$.

Take $e \in X$ and consider $N/e$. By Lemma 2.2.2, $N'/e$ is obtained from $N/e$ by relaxing $X - e$. If $N/e$ is non-binary, then we invoke the induction hypothesis to see that the result holds. Hence $N/e$ is binary for all $e \in X$. By duality, $N\backslash e$ is binary for all $e \notin X$. Thus, for every $e \in E(N)$, at least one of $N\backslash e$ and $N/e$ is binary. By Theorem 2.2.3, we deduce that one of (i)-(iv) holds for $N$.

As $r(N), r^*(N) \geq 3$, we know $N$ cannot satisfy (ii) or (iii). Assume $N$ satisfies (iv). If $|S| = 0$ or $|T| = 0$, we contradict our rank or corank assumptions, so $|S|, |T| \geq 1$. It is straightforward to check that $N$ cannot have a circuit-hyperplane, which is a contradiction.

Finally, assume $N$ satisfies (i). Then $N$ can be obtained from some connected binary matroid $M$ by relaxing a circuit-hyperplane $Y$ in $M$. Assume $X \cap Y \neq \emptyset$ and take $e \in X \cap Y$. Then $N/e$ is binary and is obtained from the binary matroid $M/e$ via relaxation. By Corollary 2.2.7, $M/e \cong U_{n-1,n} \oplus U_{1,k}$ for some $n, k \geq 1$, and $N/e \cong U_{n,n+1} \oplus_2 U_{1,k+1}$. However, this implies $N/e$ has no circuit-hyperplane unless $n = 2$ and $k = 2$, so we assume these values for $n$ and $k$. But this means $N'/e \cong U_{2,4}$, which is a contradiction since $r(N'), r^*(N') \geq 3$. Thus $X \cap Y = \emptyset$ and, by duality, $(E(M) - X) \cap (E(N) - Y) = \emptyset$. As both $(X, E(N) - X)$ and $(Y, E(N) - Y)$ partition the ground set, $X = E(N) - Y$ and $E(N) - X = Y$. Hence $N'$ is obtained from the connected binary matroid $M$ by relaxing the two disjoint circuit-hyperplanes $X$ and $Y$, so $N'$ is in $\mathcal{D}$ and the result holds. □

**Lemma 2.3.8.** *The complete list of* 3*-connected members of* $EX(\mathcal{Z})$ *is* $Q_6$*,* $P_6$*,* $U_{3,6}$*, and the matroids in* $\mathcal{D}$*. The complete list of* 3*-connected members of* $EX(\mathcal{R})$ *is* $U_{2,5}$*,* $U_{3,5}$*, and the matroids in* $\mathcal{D}$

*Proof.* Suppose $\mathcal{U} \in \{\mathcal{Z}, \mathcal{R}\}$. Let $M$ be a 3-connected excluded minor of $\mathcal{U}$ that is not $Q_6$, $P_6$, $U_{3,6}$, $U_{2,5}$, $U_{3,5}$, or any of the matroids in $\mathcal{D}$. Clearly $r(M) \geq 3$ and $r^*(M) \geq 3$. Either (a) $M$ is a relaxation of a non-binary matroid; or (b) $M$ is not a relaxation of any matroid at all. Case (a) follows immediately by Lemmas 2.3.7 and 2.2.15.

Now consider case (b). By Theorem 2.3.2, $M$ must contain one of $\mathcal{W}^3$, $Q_6$, $P_6$, and $U_{3,6}$. As all of these except $\mathcal{W}^3$ contain excluded minors of $\mathcal{U}$, we know that $M$ has a $\mathcal{W}^3$-minor. Let $\mathcal{W}^k$ be the largest whirl-minor of $M$. We use Seymour's Splitter Theorem [16] to grow $M$ from $\mathcal{W}^k$. Let $x$ be the element added with the last move. By duality, we may assume that $x$ is added via extension. Thus $M \backslash x$ is a non-binary 3-connected member of $\mathcal{U}$. If $\mathcal{U}=\mathcal{R}$, then $M\backslash x$ is a relaxation of a binary matroid. If $\mathcal{U}=\mathcal{Z}$, then $M\backslash x$ satisfies one of (i)-(iv) in Theorem 2.2.3. As $M\backslash x$ is 3-connected, it cannot satisfy (ii)-(iv). Hence, in both cases, $M\backslash x$ is a relaxation of a binary matroid $N_1$.

Let $B$ be the special basis in $M\backslash x$ that is a circuit-hyperplane in $N_1$. For all $e \in E(M\backslash x) - B$, the set $B \cup e$ is a circuit in $M\backslash x$. Now $B$ is also a basis of $M$ and $B \cup e$ is a circuit of $M$ for all $e \in E(M) - (B \cup x)$. If $B \cup x$ is a circuit of $M$, then $M$ can be realized as a relaxation of some matroid by Lemma 2.3.3, which is a contradiction. Thus there is some $y \in B$ such that $y$ is not in the circuit contained in $B \cup x$. Now, by Lemma 2.2.2, $M\backslash x/y$ can be obtained by relaxing the circuit-hyperplane $B - y$ in $N_1/y$.

Assume that $M\backslash x/y$ is binary. It follows from Corollary 2.2.7 that $M\backslash x/y$ can be obtained from a circuit $C$ by adding some, possibly empty, set of elements in parallel with some element $z$ of $C$ where $C - z = B - y$. As $M\backslash x$ is 3-connected, it has no non-trivial series classes. Hence $|C - z| = 1$, so $r(M\backslash x/y) = 1$, which contradicts the fact that $r(M) \geq 3$. Therefore $M\backslash x/y$ must be non-binary, and so $M/y$ is also non-binary.

**2.3.8.1.** *The matroid $M/y$ can be obtained from a binary matroid via relaxation.*

*Proof.* This is certainly true if $\mathcal{U}=\mathcal{R}$, so assume $\mathcal{U}=\mathcal{Z}$. Then $M/y$ satisfies one of (i)-(iv) in Theorem 2.2.3. First note that $M/y$ cannot satisfy (iii), because a connected single-element coextension of a series extension of $U_{n-2,n}$ has corank two, and so has no $\mathcal{W}^3$-minor. Assume $M/y$ satisfies (ii). Then $r(M) = 3$. As $M$ has a $\mathcal{W}^3$-minor, it is not hard to check that we must coextend $M/y$ by $y$ in a way that creates a matroid having a $U_{2,4} \oplus U_{1,1}$-, $Q_6$-, or $P_6$-minor. Now assume $M/y$ satisfies (iv). As $M$ is 3-connected, $M/y$ cannot have any non-trivial series classes. A routine check shows that either $M$ contains an excluded minor or $M$ can be realized as a relaxation of a binary matroid, both of which are contradictions. Thus (i) holds, and so does the result. □

We now revert to working in generality, where $\mathcal{U} \in \{\mathcal{Z},\mathcal{R}\}$. We know that $M\backslash x$ is obtained by relaxing a circuit-hyperplane $B$ in a binary matroid $N_1$, and $M/y$ is obtained by relaxing a circuit-hyperplane $B'$ in a binary matroid $N_2$. We show next that

**2.3.8.2.** *$B'=B-y$.*

*Proof.* By Lemma 2.2.2, $M\backslash x/y$ is obtained by relaxing the circuit-hyperplane $B-y$ in $N_1/y$. Consider $(M\backslash x/y)\backslash e$ for $e \notin B-y$, and assume $(M\backslash x/y)\backslash e$ is binary. Then, as $(M\backslash x/y)\backslash e$ is a relaxation of a binary matroid, we know $(M\backslash x/y)\backslash e \cong U_{t-1,t} \oplus_2 U_{1,v}$ for some $t, v \geq 1$. Now $E(M\backslash x/y\backslash e)$ has a partition $(S, P)$ where $S$ is the relaxed set $B-y$ and $P$ is its complement. Then $S \cup y$ is the relaxed set of $M\backslash x$ and $P \cup e$ is the relaxed set of $(M\backslash x)^*$. As $r_{M\backslash x}(P \cup y) = 2$, we know $|P| \leq 2$, else the matroid $N_1$ would be non-binary, and, by duality, $|S| \leq 2$. However, $r(M\backslash x) = |S \cup y|$ and $r^*(M\backslash x) = |P \cup e|$. Thus, as $M\backslash x$ has a $\mathcal{W}^k$-minor for some $k \geq 3$, we know $|S \cup y| = |P \cup e| = 3$. Therefore, $M\backslash x \cong \mathcal{W}^3$. The only 3-connected single-element extension of $\mathcal{W}^3$ that does not contain an excluded minor is $F_7^-$, depicted in Figure 2.4. But $F_7^-$ is a relaxation of the Fano plane, which is binary, giving us a contradiction. Therefore we may assume $(M\backslash x/y)\backslash e$ is non-binary for all $e \notin B-y$. Then, for all such $e$, the matroid $M/y\backslash e$ is also non-binary. By Lemma 2.2.2, for every $e \in B'$ the matroid $M/y\backslash e$ is binary. Thus if $e$ is not in $B-y$, then it is not in $B'$. Therefore, $B' \subseteq B-y$. As $B-y$ and $B'$ are both bases for $M/y$, the result holds. $\square$

We know $B \cup e$ is a circuit of $M$ for all $e \in E(M)-(B \cup x)$, and that $(B-y) \cup e$ is a circuit of $M/y$ for all $e \in E(M)-B$. Thus, since $B \cup x$ is not a circuit of $M$, we see $(B-y) \cup x$ is a circuit of $M$. As $M\backslash x$ has a $\mathcal{W}^3$-minor, but no $Q_6$-, $P_6$, or $U_{3,6}$-minor, Lemma 2.2.13 implies that $M\backslash x$ has a $\mathcal{W}^3$-minor using $y$. By Lemma 2.3.4, as $y$ is in $B$, it follows that this $\mathcal{W}^3$-minor has $y$ as a rim element. Hence by adding $x$ back, $M$ has a single-element extension of $\mathcal{W}^3$ as a minor. Let $\{b_1, b_2, y\}$ be the set of rim elements in $\mathcal{W}^3$. There are only two single-element extensions of $\mathcal{W}^3$ that do not contain excluded minors, and they are $F_7^-$

and a parallel extension of a spoke element of $\mathcal{W}^3$ (see Figure 2.4). But, in each of them, the set $\{b_1, b_2, x\}$ should be a circuit because $(B - y) \cup x$ is a circuit of $M$ and the only elements of $M \backslash x$ that can be contracted to produce the $\mathcal{W}^3$-minor must belong to $B$. This contradiction completes the proof. $\square$

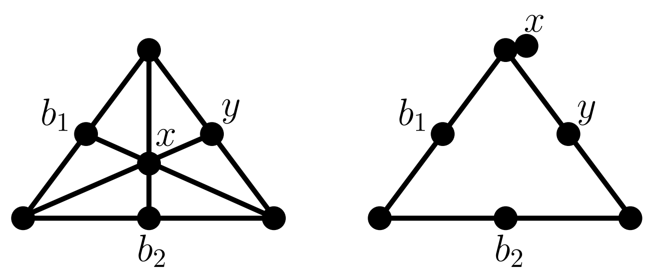



Figure 2.4. Representations of $F_7^-$, and a parallel extension of a spoke element of $\mathcal{W}^3$.

*Proofs of Theorems 2.1.1 and 2.1.2.* These follow immediately by combining Lemmas 2.3.1, 2.3.6, and 2.3.8. $\square$

### 2.4 The Complexity of $\mathcal{D}$

Jim Geelen asked (private communication) whether members of $\mathcal{D}$ could contain arbitrarily large projective geometries. In this section, we observe that they can. Note that all sums in this section are modulo two. Let $A$ be a $k \times (2^k - 1)$ matrix representing the rank-$k$ binary projective geometry $PG(k-1, 2)$, where $k$ is odd. Let $n = 2^k + k + 1$, let $t = 2^k + k - 1$, and consider the rank-$n$ binary matrix $Z$ in Figure 2.5. The entries $\alpha_i$ and $\beta_j$ are defined the next paragraph.

Let $z_{sc}$ denote the entry in row $s$ and column $c$ of $Z$. Let $\alpha_i{=}\Sigma_{s=1}^{n-2} z_{s(n+1+i)}$, for $1 \leq i \leq t$. Let $\beta_j{=}\Sigma_{c=n+2}^{2n-1} z_{jc}$, for $1 \leq j \leq t$, and let $\gamma = 1 + \Sigma_{j=1}^{t} \beta_j$. Let $Z'$ be the submatrix of $Z$ whose columns are labeled by $n+1$, $n+2$, $\ldots$, $2n$. Then each column in $Z'$ is contained in the hyperplane of $PG(n-1, 2)$ consisting of those vectors whose coordinates sum to zero. Moreover, no other column of $Z$ is in this hyperplane. The definitions ensure that all the rows of $Z'$, except possibly row $n-1$, sum to zero. To see that row $n-1$ also sums to zero,

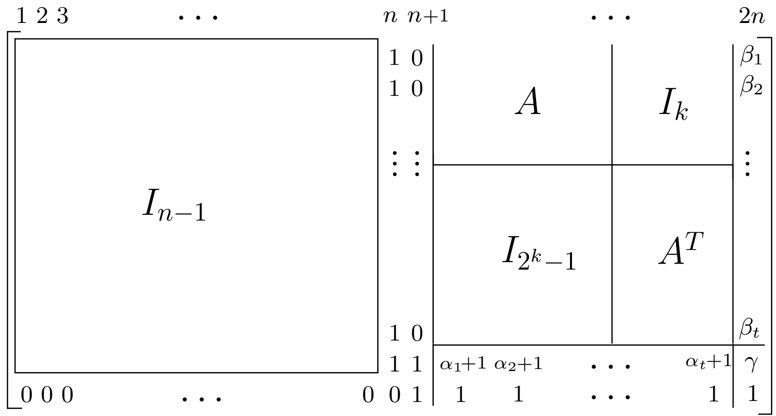



Figure 2.5. The matrix $Z$.

note that $\Sigma_{j=1}^{t}\beta_j=\Sigma_{i=1}^{t}\alpha_i$ since both of these sums count the number of non-zero entries in the same submatrix. We know that $\Sigma_{c=n+2}^{2n-1}z_{(n-1)c}=1+\Sigma_{i=1}^{t}(\alpha_i+1)=1+t+\Sigma_{i=1}^{t}\alpha_i$. As $t$ is even, $1+t+\Sigma_{i=1}^{t}\alpha_i=1+\Sigma_{i=1}^{t}\alpha_i=1+\Sigma_{j=1}^{t}\beta_j=\gamma$. Thus $\{n+1,n+2,\ldots,2n\}$ is a circuit-hyperplane of $M[Z]$ and it is easy to see that its complement is as well. By relaxing both these circuit-hyperplanes, we get a member of $\mathcal{D}$ that contains a $PG(k-1,2)$-minor.

# Chapter 3
# Dual-closed Matroids

## 3.1 Introduction

The goal of this chapter is to determine which matroids, if any, guarantee that their duals are present as minors whenever they themselves are present as minors. Clearly, for matroids $M$ and $N$, in order for $M$ to have both $N$ and $N^*$ as minors, we must have $\min\{r(M), r^*(M)\} \geq \max\{r(N), r^*(N)\}$. Subject to these *obligatory rank constraints*, we want to find all matroids $N$ such that $M$ has an $N$-minor if and only if $M$ has an $N^*$-minor. The main results of the chapter, stated below, deal with the case where $N$ and $M$ are 3-connected.

**Theorem 3.1.1.** *Let $N$ be a 3-connected matroid that is not self-dual, and let $M$ be a 3-connected matroid for which*

$$\min\{r(M), r^*(M)\} \geq \max\{r(N), r^*(N)\}.$$

*The following are equivalent:*

*(i) $M$ has an $N$-minor if and only if $M$ has an $N^*$-minor.*

*(ii) The matroid $N$ has fewer than four elements, or is $U_{2,5}$ or $U_{3,5}$.*

**Theorem 3.1.2.** *Let $N$ be a 3-connected binary matroid that is not self-dual, and let $M$ be a 3-connected binary matroid for which*

$$\min\{r(M), r^*(M)\} \geq \max\{r(N), r^*(N)\}.$$

*The following are equivalent:*

*(i) $M$ has an $N$-minor if and only if $M$ has an $N^*$-minor.*

*(ii) The matroid $N$ has fewer than four elements, or is $F_7$ or $F_7^*$.*

We let $P$ and $K_5\backslash e$, respectively, denote the prism graph and the complete graph on five vertices with a single edge deleted. These two dual graphs are depicted in Figure 3.1.

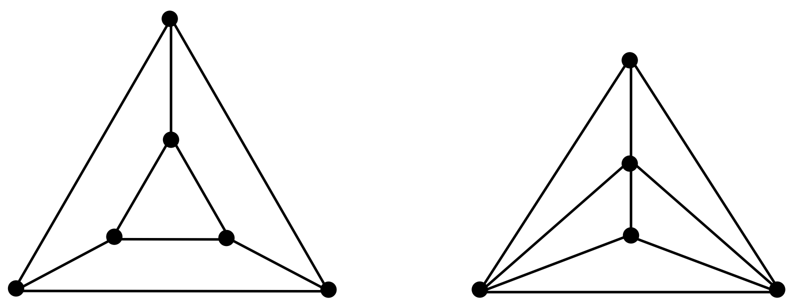
**Theorem 3.1.3.** *Let $N$ be a 3-connected graphic matroid that is not self-dual, and let $M$ be a 3-connected graphic matroid for which*

$$\min\{r(M), r^*(M)\} \geq \max\{r(N), r^*(N)\}.$$

*The following are equivalent:*

*(i) $M$ has an $N$-minor if and only if $M$ has an $N^*$-minor.*

*(ii) The matroid $N$ has fewer than four elements, or is $M(P)$ or $M(K_5\backslash e)$.*



$P$ $\qquad$ $K_5\backslash e$

Figure 3.1. The dual graphs $P$ and $K_5\backslash e$.

We also consider this question with other connectivity and representability constraints. In Section 3.2, we solve the problem with no restrictions on connectivity or representability and introduce some preliminaries. Section 3.3 is devoted to the case where $N$ and $M$ are 2-connected while, in Section 3.4, we prove Theorem 3.1.1 and the analogous result for matroids representable over infinite fields. In Section 3.5, we solve the problem for 3-connected, $GF(q)$-representable matroids and, in Section 3.6, we prove Theorem 3.1.3.

## 3.2 Preliminaries

We begin with a few observations. First, consider a self-dual matroid $N$. Clearly, every matroid $M$ that contains $N$ as a minor also contains $N^*$ as a minor. Therefore, we restrict our interest to matroids that are not self-dual. Now let $N$ be an arbitrary matroid. We define $\delta(N) = r(N) - r^*(N)$. If $\delta(N) = 0$, then it is easy to see that either $N$ is self-dual, or there is a matroid $M$ that satisfies the obligatory rank constraints and has $N$ but not $N^*$ as a minor. Thus, without loss of generality, we may assume that $r(N) > r^*(N)$, and so $\delta(N) > 0$.

We now describe a strategy that will be used in many proofs throughout this chapter. Beginning with a matroid $N$, we construct a matroid $M$ by adjoining $\delta(N)$ elements to $N$ in such a way that we do not increase the rank of $N$. Then we increase the corank of $N$ by $\delta(N)$. This matroid $M$ satisfies the obligatory rank constraints. Note that, when we employ this strategy, we must contract $\delta(N)$ elements from $M$ if we hope to obtain an $N^*$-minor. Furthermore, each contraction must lower the rank of the matroid. This implies that we must contract an independent set of size $\delta(N)$. By carefully adjoining these $\delta(N)$ elements, we show that, unless $N$ has a specific structure, we can construct such a matroid $M$ that has $N$ but not $N^*$ as a minor. We now solve the general problem.

**Proposition 3.2.1.** *Let $\mathcal{M}$ be a class of matroids that is minor-closed and closed under duality. Let $N$ and $M$ be matroids in $\mathcal{M}$ such that $N$ is not self-dual and*

$$\min\{r(M), r^*(M)\} \geq \max\{r(N), r^*(N)\}.$$

*The following are equivalent:*

*(i) $M$ has an $N$-minor if and only if $M$ has an $N^*$-minor.*

*(ii) The matroid $N$ is $U_{0,n}$ or $U_{n,n}$, for some $n \geq 1$.*

*Proof.* Clearly $(ii)$ implies $(i)$. To see that $(i)$ implies $(ii)$, assume $\delta(N) > 0$ and let $c_N$ denote the number of coloops in $N$ and let $l_N$ denote the number of loops in $N$. We begin by showing that

**3.2.1.1.** $\delta(N) = c_N - l_N$.

Construct a matroid $M$ from $N$ by adding $\delta(N) = r(N) - r^*(N)$ loops to $N$. Then $c_M = c_N$ and $l_M = l_N + \delta(N)$. This matroid $M$ satisfies the obligatory rank constraints and thus must have an $N^*$-minor. We must contract $\delta(N)$ elements from $M$ to obtain $N^*$, and each contraction must drop the rank. Thus $l_N = c_{N^*} \geq c_M - \delta(N) = c_N - \delta(N)$ and $c_N = l_{N^*} \geq l_M = l_N + \delta(N)$. These inequalities can be rewritten as $\delta(N) \geq c_N - l_N$ and $\delta(N) \leq c_N - l_N$. Thus 3.2.1.1 holds.

Suppose that $N$ has a 2-connected component $K$ with at least two elements. Construct a new matroid $M$ by adding $\delta(N)$ elements to a parallel class in $K$. Then $c_N = c_M$ and $l_N = l_M$. By 3.2.1.1, we know that $\delta(N) = c_N - l_N = c_N - c_{N^*}$. Therefore $N^*$ has $\delta(N)$ fewer coloops than $N$, so, since $c_N = c_M$, we must contract $\delta(N)$ coloops from $M$ to obtain $N^*$. However, contracting coloops does not create any loops, which contradicts the fact that $M$ has an $N^*$-minor. Thus, each component of $N$ consists of a single element and $N \cong U_{0,s} \oplus U_{n,n}$ for some $s, n \geq 0$.

Consider the graph $G$ that consists of a path of length $n$ with $s$ loops adjoined to one end. Clearly $M(G) = N$. Create a graph $G'$ as follows: add an edge $e$ to $G$ such that the path of length $n$ becomes a cycle of length $n + 1$. Add the other $\delta(N) - 1$ edges in parallel to $e$. Let $M(G') = M$. This new matroid $M$ satisfies the obligatory rank constraints and has no coloops. Since contraction cannot create new cocircuits, this implies that $s = 0$ and $N \cong U_{n,n}$ for some $n \geq 0$. The result holds by duality. □

### 3.3 2-connected Case

In this section, we impose the additional constraints that $N$ and $M$ are 2-connected. Specifically, we determine the set of 2-connected matroids $N$ such that if a 2-connected matroid $M$ satisfies the obligatory rank constraints, then $M$ has an $N$-minor if and only if $M$ has an $N^*$-minor. We also solve the problem in the 2-connected case with the additional assumption that $N$ and $M$ are graphic. The next theorem deals with small matroids.

**Theorem 3.3.1.** *Let $\mathcal{M}$ be a class of matroids that is minor-closed and closed under duality. Take $n \in \{2, 3\}$ and let $N$ and $M$ be $n$-connected matroids in $\mathcal{M}$ such that $N$ is not self-dual, $|E(N)| < 4$, and*

$$\min\{r(M), r^*(M)\} \geq \max\{r(N), r^*(N)\}.$$

*The following are equivalent:*

*(i) $M$ has an $N$-minor if and only if $M$ has an $N^*$-minor.*

*(ii) The matroid $N$ is $U_{0,1}$, $U_{1,1}$, $U_{1,3}$, or $U_{2,3}$.*

*Proof.* First assume (i) holds. If $|E(N)| = 1$, then $N$ is $U_{0,1}$ or $U_{1,1}$, so assume $|E(N)| \geq 2$. If $|E(N)| = 2$, then $N$ is either disconnected or self-dual, a contradiction. If $|E(N)| = 3$, then, as $N$ is 2-connected, $N$ is $U_{1,3}$ or $U_{2,3}$. Thus (ii) holds.

Now assume (ii) holds and $\delta(N) > 0$. If $N$ is $U_{1,1}$, then any matroid $M$ that has an $N$-minor and satisfies the obligatory rank constraints has corank at least one and so has a $U_{0,1}$-minor. Thus we may assume $N$ is $U_{2,3}$. We now consider $n = 2$ and $n = 3$ separately.

Suppose $n = 2$ and consider extending $U_{2,3}$ by a single element to construct a 2-connected matroid $M$. Either $M \cong U_{2,4}$ or $M \cong U_{2,3} \oplus_2 U_{1,3}$. In both cases, $M$ contains a $U_{1,3}$-minor, so the result holds when $n = 2$.

Now assume $n = 3$ and consider extending $U_{2,3}$ to obtain a 3-connected matroid $M$. Either $M$ contains a $U_{2,4}$-minor or an $M(K_4)$-minor [9, Theorem 2.5], both of which contain $U_{1,3}$ as a minor. Thus the result holds by duality. □

In light of this result, for the remainder of the chapter, we restrict our attention to matroids with at least four elements. To each tree decomposition $T$ of a matroid $M$, we associate an ordered triple $\nu(T) = (n_1, n_2, n_3)$, where $n_1$ is the number of corank-one matroids in $V(T)$ and $n_2$ is the number of rank-one matroids in $V(T)$. We let $n_3 = |V(T)| - n_1 - n_2$. Evidently, $n_3$ represents the number of 3-connected matroids in $V(T)$ that have rank and corank larger than one. We will call such 3-connected matroids *large*. Note that if $\nu(T) = (c, d, k)$, then $\nu(T^*) = (d, c, k)$. If the matroid $K$ labels a vertex in the tree decomposition $T$, we say $K$ is in $V(T)$. The next lemma restricts the structure of the canonical tree decomposition $T$ for any 2-connected matroid $N$ with the property that its dual is present as a minor whenever $N$ is present as a minor.

**Lemma 3.3.2.** *Let $\mathcal{M}$ be a class of matroids that is minor-closed and closed under duality. Let $N$ and $M$ be 2-connected matroids in $\mathcal{M}$ such that $N$ is not self-dual, $|E(N)| \geq 4$, and*

$$\min\{r(M), r^*(M)\} \geq \max\{r(N), r^*(N)\}.$$

*Let $T$ be the canonical tree decomposition of $N$ and assume $M$ contains an $N$-minor if and only if $M$ contains an $N^*$-minor.*

*(i) If $G$ is a large 3-connected matroid in $V(T)$, then $G^*$ is also in $V(T)$; and*

*(ii) $T$ meets $\{U_{1,n} : n \geq 3\}$ if and only if $T$ avoids $\{U_{m-1,m} : m \geq 3\}$; and*

*(iii) if $\delta(N) \geq 1$, then $V(T)$ contains exactly one corank-one matroid $C$. Moreover, $C$ is a leaf of $V(T)$.*

*Proof.* First we show that (i) holds. Assume $\delta(N) > 0$ and add $\delta(N)$ elements in parallel to any element of $N$ to get a matroid $M$. Then $M$ is 2-connected, contains $N$ as a minor, and satisfies the obligatory rank constraints. Clearly, the large 3-connected vertex labels of the canonical tree decompositions, $T$ and $T_M$, of $N$ and $M$ are the same. As $N$ guarantees its dual as a minor, we know $M$ contains an $N^*$-minor. Using Lemma 1.4.4, we obtain the tree decomposition $T^*$ of $N^*$ from $T$. By construction, the large 3-connected matroids that label vertices of the tree decompositions for $M$, $N$, and $N^*$ have the same size and number. Furthermore, the 3-connected matroids in $V(T^*)$ must be precisely the duals of the 3-connected matroids in $V(T)$, and must all be present in $V(T_M)$. Hence, the large 3-connected matroids in $V(T)$ and $V(T^*)$ are the same, so (i) holds.

To see that (ii) holds, let $\nu(T) = (c, d, k)$ and assume $c, d \geq 1$. Further assume that $c \neq d$, and add $\delta(N)$ parallel elements to any rank-one matroid in $V(T)$ to obtain a tree decomposition $T_M$ for a matroid $M$. Then $M$ satisfies the obligatory rank constraints, so $M$ has an $N^*$-minor. By construction, $\nu(T) = \nu(T_M)$, and $\nu(T^*) = (d, c, k)$. Recall that we must contract an independent set to obtain $N^*$ from $M$. Note that, since $N^*$ is 3-connected, we cannot contract an element in a rank-one matroid in $V(T^*)$, and, by (i), we cannot contract an element in a large 3-connected matroid. This is a contradiction since $c \neq d$ and, by Lemma 1.4.5, we do not change the number of rank-one matroids in the tree decomposition when contracting elements from corank-one matroids to obtain $N^*$. Thus $c = d$.

Now add $\delta(N)$ elements to a trivial parallel class of $N$ to get a new matroid $M$. We know such a parallel class exists since $r(N) > r^*(N)$. Then the resulting tree decomposition $T_M$ of $M$ has more rank-one matroids than corank-one matroids. But $M$ has an $N^*$-minor and, since $N^*$ is 2-connected, we cannot decrease the number of rank-one matroids via contracting an independent set to obtain $N^*$. This contradiction completes the proof of (ii).

Construct a matroid $M$ from $N$ by adding $\delta(N)$ elements in a parallel class. Then $V(T^*)$ must contain a rank-one matroid and, by duality, $V(T)$ contains a corank-one matroid. Then, by (ii), $V(T)$ contains no rank-one matroids. Assume $V(T)$ contains at least two corank-one matroids. Add $\delta(N)$ elements in parallel to some element of $N$ to get $M$. Then, by (ii), in the tree decomposition $T_M$ of $M$, there is a single rank-one matroid. However, $M$ contains an $N^*$-minor which has at least two rank-one matroids in its tree decomposition, which is a contradiction by Lemma 1.4.5. Therefore, $V(T)$ has exactly one matroid $C$ with corank one. Furthermore, $C$ must be a leaf in $T$, otherwise we can build a matroid $M$ by adding a parallel element to $N$ such that we create a rank-one matroid as a leaf in $T_M$. This gives a matroid $M$ with no $N^*$-minor, a contradiction. Thus (iii) holds. $\square$

The next two lemmas will be useful in the proof of the theorem that follows; the proofs of the lemmas are omitted. The first lemma is elementary and is surely well known. The second lemma comes from [13, Prop. 7.3.9].

**Lemma 3.3.3.** *Let $M$ be a connected matroid of rank at least three. Then $M$ has a $U_{3,4}$-minor.*

**Lemma 3.3.4.** *Let $e$ be an element of a matroid $N$ and suppose that $e$ is not a loop. Then $e$ is free in $N$ if and only if $e$ is in every dependent flat of $N^*$.*

**Theorem 3.3.5.** *Let $N$ be a 2-connected matroid that is not self-dual such that $|E(N)| \geq 4$, and let $M$ be a 2-connected matroid for which*

$$\min\{r(M), r^*(M)\} \geq \max\{r(N), r^*(N)\}.$$

*The following are equivalent:*

*(i) $M$ has an $N$-minor if and only if $M$ has an $N^*$-minor.*

*(ii) The matroid $N$ is $U_{1,4}$ or $U_{3,4}$.*

*Proof.* Assume $\delta(N) \geq 1$. Using Lemma 3.3.3, it is not hard to check that (ii) implies (i). To see that (i) implies (ii), we consider the canonical tree decomposition $T$ of $N$. We know, by Lemma 3.3.2 (ii), that there cannot be both corank-one and rank-one matroids in $V(T)$. By Lemma 3.3.2 (iii), $V(T)$ has exactly one corank-one matroid $C$, and $C$ labels a degree-one vertex of $T$. Next we show that

**3.3.5.1.** *$N$ is the circuit $C$.*

Assume that this is false. If $\delta(N) > 1$, then we can add $\delta(N)$ elements to $N$ to give a matroid $M$ having two non-trivial parallel classes. This creates two rank-one matroids in $V(T_M)$, a contradiction by Lemma 1.4.5. Hence $\delta(N) = 1$. Thus, as $C$ labels a degree-one vertex of $T$, we see that $N \cong C \oplus_2 K$ for some matroid $K$. Note that $K$ is made by 2-summing large 3-connected matroids. Thus, by Lemmas 1.4.2 and 3.3.2 (i), we have $r(K) = r^*(K)$. Moreover, since $\delta(N) = 1$, we know $C = U_{2,3}$.

Construct a new matroid $M$ by adding a parallel element to one of the series elements of $N$ contributed by $C$. We must contract the other series element that is present in $N$ to destroy the corank-one matroid in $V(T_M)$, as $V(T^*)$ has no such matroids. Thus, $N^* \cong U_{1,3} \oplus_2 K$. Hence

**3.3.5.2.** *$K$ is self-dual.*

Construct a new matroid $M$ from $N$ by adding an element $f$ freely to $N$. As $N^*$ has parallel elements and $\delta(N) = 1$, we know we must contract an element in a triangle of $M$ to get $N^*$. Therefore, either $f$ is free in $N^*$, or we must contract $f$ to obtain $N^*$. However, in the latter case, $r(N) = 2$ which contradicts the composition of $T$. Next we show that

**3.3.5.3.** *$N$ has two free elements in series.*

As $f$ is free in $N^*$, by Lemma 3.3.4, $f$ is in every dependent flat of $N$. If $f$ is one of the series elements contributed to $N$ by $C$, then $N$ has two free elements in series, so we assume not. Construct a new matroid $M$ by adding an element in parallel to $f$. We must contract one of the series elements contributed by $C$ to obtain $N^*$, otherwise $V(T^*)$ has a corank-one matroid, a contradiction. Thus, $N^*$ has a parallel pair $\{e, f\}$ that is in every dependent flat. Hence, by Lemma 3.3.4, the elements $e$ and $f$ are free in $N$ and are in series. Thus 3.3.5.3 holds.

As there is only one non-trivial series class in $N$, this implies that $\{e, f\} \subseteq C$. Construct a new matroid $M$ by adding an element $x$ in parallel to $e$. As $V(T^*)$ contains no corank-one matroids, we must contract $f$ to obtain $N^*$. Upon contraction, by Lemma 3.3.4, the unique parallel pair $\{e, x\}$ must be in every dependent flat. Consider our construction of $N^*$. Essentially, we contracted a free element $f$, giving us a matroid $N'$ having $e$ as a free element, and then added $x$ in parallel to $e$. Since $e$ and $x$ must be in every dependent flat, this implies that, in $N'$, the element $e$ is free and is in every dependent flat. Thus $K$ is a uniform matroid in addition to being self-dual, which implies $K \cong U_{n,2n}$ for some $n \geq 2$.

To complete the proof of 3.3.5.1, construct a new matroid $M$ by adding an element freely to the span of $K$ in $N$. As $N^*$ has no elements in series, we must contract one element in the series pair $\{e, f\}$ to obtain $N^*$ from $M$. However, this contraction yields no parallel elements, which contradicts the fact that $N^*$ has a parallel pair. This contradiction completes the proof of 3.3.5.1.

Assume $|C| \geq 5$. Then construct a matroid $M$ by adding one element in parallel to each of $|C| - 2 = \delta(N)$ distinct elements of $C$. This gives a matroid $M$ with no $C^*$-minor. Therefore $|C| \leq 4$ and, since $|E(N)| \geq 4$, we have $N \cong U_{3,4}$. The result holds by duality. □

We now consider the 2-connected problem with the additional assumption that $N$ and $M$ are graphic. Note that if $N^*$ is a minor of the graphic matroid $M$, then $N^*$ is also graphic. This implies that $N^*$ is both graphic and cographic, which means that $N^*$ is planar graphic. Therefore, we need only consider matroids $N$ that are planar graphic. Note that a graph is *maximally plane* if it is a plane graph in which every face is a triangle. The following lemma is well known; its routine proof is omitted.

**Lemma 3.3.6.** *Let $G$ be a maximally plane graph. Then $|E(G)| = 3|V(G)| - 6$ and $|F(G)| = 2|V(G)| - 4$.*

Let $G$ and $G^*$ be the dual graphs depicted in Figure 3.2. We now prove the following.

**Theorem 3.3.7.** *Let $N$ be a 2-connected graphic matroid that is not self-dual such that $|E(N)| \geq 4$, and let $M$ be a 2-connected graphic matroid for which*

$$\min\{r(M), r^*(M)\} \geq \max\{r(N), r^*(N)\}.$$

*The following are equivalent:*

*(i) $M$ has an $N$-minor if and only if $M$ has an $N^*$-minor.*

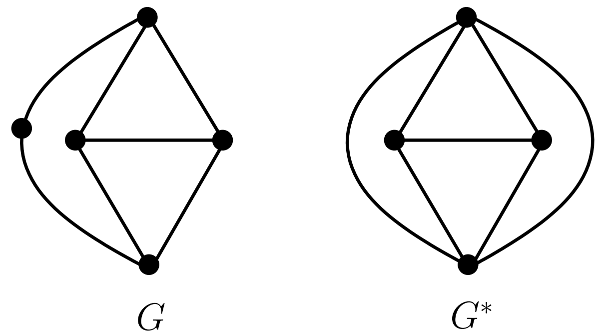



Figure 3.2. The graphs $G$ and $G^*$.

*(ii) The matroid $N$ is $U_{1,4}$, $U_{3,4}$, $M(G)$, or $M(G^*)$.*

*Proof.* Assume $\delta(N) \geq 1$. First we show that (ii) implies (i). It is not hard to check, using Lemma 3.3.3, that if $N$ is $U_{3,4}$ then (i) holds. Thus we may assume $N$ is $M(G)$. First note that adding an element in parallel to any element of $M(G)$ always produces a matroid with an $M(G^*)$-minor. Thus we consider extending $M(G)$ by an element $e$ such that $e$ is not a parallel element. There are only two ways, up to isomorphism, to add $e$ to $M(G)$ and get a 2-connected planar graphic matroid $M$. The graphs $G_1$ and $G_2$ underlying these extensions are depicted in Figure 3.3. By contracting the edge labeled $f$ in each of $G_1$ and $G_2$, we obtain $G^*$ as a minor. Thus we see that

**3.3.7.1.** *every connected single-element extension of $M(G)$ contains an $M(G^*)$-minor.*

Now consider an arbitrary matroid $M$ that satisfies the obligatory rank constraints and contains an $N$-minor. We know that $M/I\backslash I^* \cong N$ for some independent set $I$ and some coindependent set $I^*$. Consider $M/I$. If $I^*$ is not a set of loops in $M/I$, then $M$ has a connected single-element extension of $N$ as a minor and (i) holds by 3.3.7.1. Thus we may assume that $I^*$ is a set of loops in $M/I$. Then for some set $A \subset I$, the matroid $M/A$ has a non-trivial parallel class and an $M(K_4)$-minor. Let $e$ be an element in a non-trivial parallel class of $M/A$. By Lemma 2.2.12, $M/A$ has an $M(K_4)$-minor that uses $e$. Thus, $M$ has a parallel extension of $M(K_4)$ as a minor, that is, $M$ has an $N^*$-minor. Thus (i) holds.

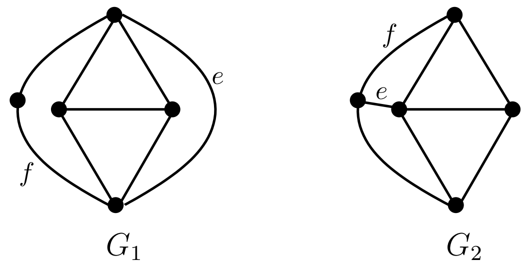



Figure 3.3. The graphs $G_1$ and $G_2$.

To see that (i) implies (ii), first note that Lemma 3.3.2 (ii) and (iii) respectively imply that $N$ has a non-trivial series class and no parallel elements, and $N$ has exactly one non-trivial series class.

Suppose the tree decomposition $T$ of $N$ has exactly one vertex. Then $N \cong U_{n-1,n}$ for some $n \geq 4$. Suppose $n \geq 5$. By adding an element to each of $n-2$ distinct parallel classes of $U_{n-1,n}$, we get a matroid $M$ that satisfies the obligatory rank constraints, but contains no $U_{1,n}$-minor. This contradiction implies that $n = 4$, so $N \cong U_{3,4}$, and the result holds.

Now assume $T$ has more than one vertex. By Lemma 3.3.2 (iii), we know that $N \cong U_{n-1,n} \oplus_2 K$ for some matroid $K$ that is a 2-sum of 3-connected matroids. By Lemmas 1.4.2 and 3.3.2 (i), $r(K) = r^*(K)$. Note that this implies that $\delta(N) = n-2$. Suppose $\delta(N) \geq 2$. Construct a new matroid $M$ by adding a single element to each of $\delta(N)$ distinct parallel classes of $K$. This $M$ satisfies the obligatory rank constraints and has more than one non-trivial parallel class. To obtain $N^*$, we cannot contract any parallel elements. However, $N^*$ does not have multiple non-trivial parallel classes, a contradiction. We deduce that $\delta(N) = 1$. This implies that the non-trivial series class in $N$ has exactly two elements. Thus we know that $N \cong U_{2,3} \oplus_2 K$ for some matroid $K$ that is a 2-sum of 3-connected planar graphic matroids.

Consider a graph $G_K$ such that $M(G_K) = K$. If $G_K$ is not a maximally plane graph, then create a new planar graph $H$ by adding an edge $e$ to $G_K$ such that $e$ is not parallel to any edge of $G_K$. Let the matroid $M = U_{2,3} \oplus_2 M(H)$. Then $M$ satisfies the obligatory rank constraints, so it must contain an $N^*$-minor. However, as $N^*$ contains no series elements, we must contract an element from the non-trivial series class to obtain $N^*$. But this contraction will not create any parallel elements, a contradiction. Thus $G_K$ is a maximally plane graph. By Lemma 3.3.6, we know that $|F(G_K)| = 2|V(G_K)| - 4$. Since $r(N) = r^*(N) + 1$, we have $|V(G_K)| = |F(G_K)|$. Thus we see that $|V(G_K)| = 2|V(G_K)| - 4$, so $|V(G_K)| = 4$. Thus, since $G_K$ is maximal planar with four vertices, $G_K = K_4$, the complete graph on four vertices. Therefore, $N \cong U_{2,3} \oplus_2 M(K_4) = G$. The result holds by duality. $\square$

### 3.4 General 3-connected Case

In this section, we prove Theorem 3.1.1 and solve the problem in the 3-connected case with the additional constraint that $N$ and $M$ are $\mathbb{F}$-representable for an infinite field $\mathbb{F}$. We denote by $\mathcal{N}_3$ the set of 3-connected matroids $N$ such that, for 3-connected matroids $M$ that satisfy the obligatory rank constraints, $M$ contains an $N$-minor if and only if $M$ contains an $N^*$-minor. Similarly, we denote by $\mathcal{N}_{3,\mathbb{F}}$ the set of 3-connected $\mathbb{F}$-representable matroids $N$ such that, for 3-connected $\mathbb{F}$-representable matroids $M$ that satisfy the obligatory rank constraints, $M$ contains an $N$-minor if and only if $M$ contains an $N^*$-minor.

We also prove a result in this section that applies to the 3-connected case with the more specific constraint that $N$ and $M$ are $GF(q)$-representable. For this reason, we now introduce the notation $\mathcal{N}_{3,q}$ to denote the set of 3-connected $GF(q)$-representable matroids $N$ such that, for 3-connected $GF(q)$-representable matroids $M$ that satisfy the obligatory rank constraints, $M$ contains an $N$-minor if and only if $M$ contains an $N^*$-minor, even though we do not discuss this problem in detail until the next section.

Due to the similarity of the proofs, we combine many of the results in this section. For $N \in \mathcal{N}_3 \cup \mathcal{N}_{3,\mathbb{F}}$, by Theorem 3.3.1, we assume $|E(N)| \geq 5$ and continue our assumption that $\delta(N) > 0$. The following well-known lemma will be useful in our proofs in this section. For completeness, we give a proof.

**Lemma 3.4.1.** *Let $\mathbb{F}$ be an infinite field, let $N$ be an $\mathbb{F}$-representable matroid, and let $F$ be a flat of $N$. Then the free extension of $N$ by a single element is also $\mathbb{F}$-representable. More generally, the matroid obtained from $N$ by adding a single element freely to $F$ is $\mathbb{F}$-representable.*

*Proof.* First we argue by induction on $r(N)$ that every $\mathbb{F}$-representation of $N$ can be extended to an $\mathbb{F}$-representation of the free extension of $N$. If $r(N) = 0$, then the result holds. Assume the result holds for $r(N) < r$ and let $r(N) = r$. Take an $\mathbb{F}$-representation $[I_r|A]$ for $N$, and let $b_r$ be the element that labels the $r$th column of $I_r$. Now $N/b_r$ is $\mathbb{F}$-representable and the free extension $N'$ of $N/b_r$ by $e$ is $\mathbb{F}$-representable by induction. Now consider the matrix in Figure 3.4, where removing the $r$th row and the $r$th column gives a representation of $N'$, while removing the last column gives an $\mathbb{F}$-representation of $N$. The entry $\zeta$ is a member of $\mathbb{F}$ whose value is yet to be determined.

$$\begin{array}{cccc} & b_r & & e \\ \end{array}$$
$$\left[\begin{array}{c|c|c|c} I_{r-1} & \begin{matrix}0\\0\\0\\\vdots\\0\end{matrix} & A' & \hat{z} \\ \hline 0\;0\;0\;\;\cdots\;\;0 & 1 & \alpha_1\;\alpha_2\;\alpha_3\;\alpha_4\;\cdots\;\alpha_{n-1}\;\alpha_n & \zeta \end{array}\right]$$

Figure 3.4. A possible free extension of $N$.

We want to choose $\zeta$ so that the matrix in Figure 3.4 is a representation of the free extension of $N$. The extension is not free if there is a subset $I$ of $E(N)$ such that $|I| \leq r-1$ and $I \cup e$ is a circuit. This means that $\Sigma_{i\in I}\, a_i(\hat{v}_i, \beta_i) = (\hat{z}, \zeta)$ for some non-zero $a_i$ in $\mathbb{F}$ where each $(\hat{v}_i, \beta_i)$ is a column of $[I_r|A]$. Now $\Sigma_{i\in I}\, a_i\hat{v}_i = \hat{z}$ means that $|I| = r-1$.

To prevent $I \cup e$ from being a circuit, we must choose $\zeta$ so that $\Sigma_{i\in I}\, a_i\beta_i \neq \zeta$. By considering all $(r-1)$-element subsets of the sets of columns of $[I_r|A]$, we get a finite set of inequations that $\zeta$ must satisfy. Since $\mathbb{F}$ is infinite, we can choose $\zeta$ so that none of these inequations hold. Thus we obtain an $\mathbb{F}$-representation of the free extension of $N$.

Now we consider the more general case, where we add $e$ freely to a flat $F$ of $N$. Let $r(N) = r$ and $r(F) = k$ for $k \leq r$. Take an $\mathbb{F}$-representation of $N$ such that the first $k$ columns are a basis for $F$. Then all other columns representing elements of $F$ have zero entries in rows $k+1$ through $r$, as depicted in Figure 3.5. Let $b_1, \ldots, b_r$ label the columns of $I_r$, and consider the matroid $F' = N/\{b_{k+1}, \ldots, b_r\}$.

$$
\begin{array}{c}
\begin{array}{cccc} b_1 \cdots b_k & b_{k+1} \cdots b_r & f_1 \cdots f_j & \phantom{A_1} \end{array} \\
\left[\begin{array}{c|c|c|c} I_k & 0 & \mathcal{F} & A_1 \\ \hline 0 & I_{r-k} & 0 & A_2 \end{array}\right]
\end{array}
$$

Figure 3.5. A representation of $N$.

Evidently, the matroid $F'$ contains all the elements of $F$; the elements of $F$ span $F'$; and the structure of the elements of $F$ is unchanged in $F'$, that is, $N|F = F'|F$. Now, using the proof above, add $e$ to $F'$ to get the free extension $P$ of $F'$, and consider the matrix in Figure 3.6 where removing the $(k+1)$st through $r$th rows and columns gives an $\mathbb{F}$-representation of $P$ while removing the last column gives an $\mathbb{F}$-representation of $N$. This matrix is an $\mathbb{F}$-representation of the matroid obtained from $N$ by adding an element freely to the flat $F$. $\square$

The next lemma provides information regarding adding elements to 3-connected $GF(q)$-representable matroids. The two lemmas that follow it determine structure for matroids in $\mathcal{N}_3 \cup \mathcal{N}_{3,\mathbb{F}}$. The following will be used extensively in Section 3.5.

**Lemma 3.4.2.** *Let $N$ be a rank-$r$ matroid in $\mathcal{N}_{3,q}$ and assume $\delta(N) \geq 1$ and $r \geq 2$. Then*

$$
\begin{array}{ccccc}
b_1 \cdots b_k & b_{k+1} \cdots b_r & f_1 \cdots f_j & & e \\
\end{array}
$$

$$
\left[\begin{array}{c|c|c|c|c}
I_k & 0 & \mathcal{F} & A_1 & \hat{g} \\
\hline
0 & I_{r-k} & 0 & A_2 & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}
\end{array}\right]
$$

Figure 3.6. Adding an element freely to a flat of $N$.

*(i) $N$ can be extended by $\delta(N)$ elements and remain 3-connected and $GF(q)$-representable; and*

*(ii) $\delta(N)-1$ elements can be added to any hyperplane of $N$ to obtain a 3-connected $GF(q)$-representable matroid.*

*Proof.* Consider $N$ as a restriction of $PG(r-1,q)$. We know $PG(r-1,q)$ has $1+q+q^2+\cdots+q^{r-1}$ elements and $N$ has $2r-\delta(N)$ elements. Now $1+q+q^2+\cdots+q^{r-1} \geq 1+2+2^2+\cdots+2^{r-1} = 2^r-1$. Thus, the result holds when $2^r-1 \geq 2r-\delta(N)+\delta(N) = 2r$. Note that $2^r-1-2r$ is an increasing function for $r \geq 2$ and, since every matroid in $\mathcal{N}_{3,q}$ with $\delta(N) \geq 1$ has rank at least three, (i) holds.

As $N$ is 3-connected, any hyperplane $H$ of $N$ has at most $2r-\delta(N)-3$ elements and has rank $r-1$. Using the same formulas as above, (ii) holds when $2^{r-1}-1 \geq 2r-\delta(N)-3+(\delta(N)-1) = 2(r-1)$. Since $2^{r-1}-1-2(r-1)$ is an increasing function for $r \geq 3$, (ii) holds. □

**Lemma 3.4.3.** *Let $N$ be a matroid in $\mathcal{N}_3 \cup \mathcal{N}_{3,\mathbb{F}}$. Then $N^*$ has a $(\delta(N)+2)$-element line and no triads. Hence $N$ has no triangles and has a $(\delta(N)+2)$-element set of which every three-element subset forms a triad.*

*Proof.* Add $\delta(N)$ elements freely on a line of $N$ to construct a matroid $M$. Observe that if $N \in \mathcal{N}_{3,\mathbb{F}}$, then so is $M$. As $N^*$ is 3-connected, we cannot contract any elements on the line to obtain $N^*$. Hence $N^*$ has a $(\delta(N)+2)$-element line. Construct a new matroid $M$ by adding $\delta(N)$ elements freely to $N$. By Lemma 3.4.1, we know $M$ is $\mathbb{F}$-representable if $N$ is. This matroid $M$ has no triads and, since contraction creates no new cocircuits, $N^*$ also has no triads. The results for $N$ follow by duality. $\square$

**Lemma 3.4.4.** *Let $N$ be a matroid in $\mathcal{N}_3 \cup \mathcal{N}_{3,\mathbb{F}} \cup \mathcal{N}_{3,q}$ such that $N$ has a triad. Then either $\delta(N) = 1$ or $N^*$ has a triad.*

*Proof.* Assume $\delta(N) \geq 2$ and let $r(N) = n$. Then $r^*(N) = n - \delta(N)$. Let $T^*$ be a triad $\{a, b, c\}$ in $N$ and let $H_1$ be the complementary hyperplane. Construct a matroid $M'$ by adding elements $b'$ and $c'$ to $N$ such that $\{a, b, b'\}$ and $\{a, c, c'\}$ are triangles, but $\{a, b, c\}$ remains a triad, as in Figure 3.7. Note that it is possible that one or both of $b'$ and $c'$ is already present in $N$, however, this will not pose a problem for our argument. To see that this construction is possible, consider performing a $Y$-$\Delta$ exchange on $\{a, b, c\}$, and let $\{a', b', c'\}$ denote the new triangle in the resulting matroid $N'$. Now consider performing a generalized parallel connection of $M(K_4)$ and $N'$ across the triangle $\{a', b', c'\}$, where $M(K_4)$ is labeled as shown in Figure 3.8. Then deleting $\{a', b', c'\}$ recovers $N$, but deleting only $a'$ gives the desired extension of $N$.

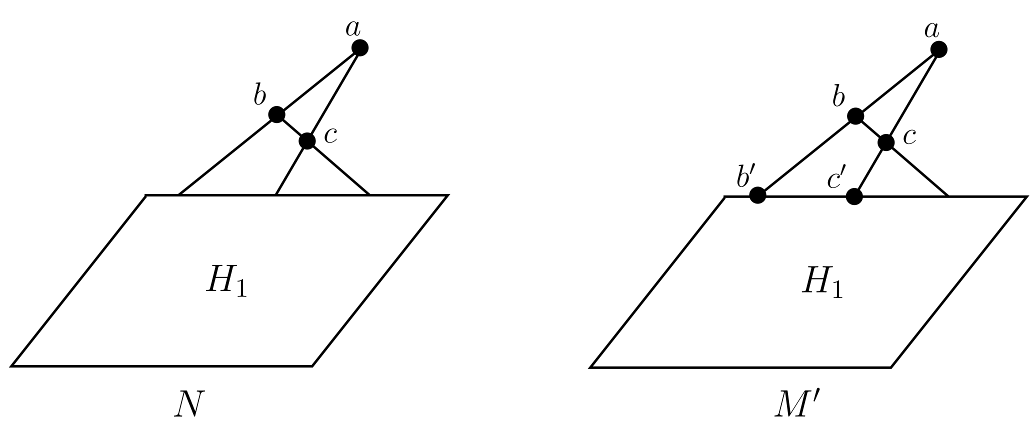



Figure 3.7. Geometric representations of the matroids $N$ and $M'$ in Lemma 3.4.4.

Finish constructing the matroid $M$ from $M'$ by adding another $\delta(N) - 2$, $\delta(N) - 1$, or $\delta(N)$ elements, depending on how many of $\{b', c'\}$ are originally present in $N$. We add these

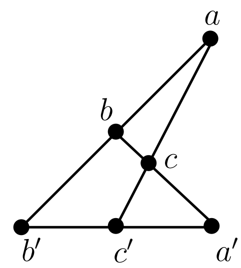



Figure 3.8. A geometric representation of the matroid $M(K_4)$.

remaining elements in $\mathrm{cl}(H_1)$ such that $M$ is 3-connected and $M$ is $\mathbb{F}$-representable if $N$ is. Clearly this is possible when $N \in \mathcal{N}_3 \cup \mathcal{N}_{3,\mathbb{F}}$. If $N \in \mathcal{N}_{3,q}$, then Lemma 3.4.2 guarantees the construction is also possible. Then $T^*$ is a triad of $M$ and we cannot contract any of its elements to obtain $N^*$, which implies that $N^*$ has a triad. $\square$

**Corollary 3.4.5.** *Let $N$ be a matroid in $\mathcal{N}_3 \cup \mathcal{N}_{3,\mathbb{F}}$. Then $\delta(N) = 1$.*

*Proof.* Combining Lemmas 3.4.3 and 3.4.4 gives the result immediately. $\square$

We now describe an argument that will be used several times throughout the rest of this chapter. Beginning with a matroid $N$, we construct a matroid $M_1$ by adjoining $\delta(N)$ elements to $N$ in such a way that we do not increase the rank of $N$. Under the right circumstances, we can construct $M_1$ such that we know which elements must be contracted from $M_1$ to obtain $N^*$. Assume that obtaining $N^*$ from this construction implies that $N^*$ has $j$ triangles. We then construct a matroid $M_2$ from $N$ such that $M_2$ satisfies the obligatory rank constraints and, as above, we know which elements must be contracted from $M_2$ to obtain $N^*$. If obtaining $N^*$ from this construction implies that $N^*$ has more or fewer than $j$ triangles, then we get a contradiction. This argument will be used in several proofs below to yield information about the structure of $N$. The next proposition determines the members of $\mathcal{N}_3$ and $\mathcal{N}_{3,\mathbb{F}}$.

**Proposition 3.4.6.** *A matroid $N$ with at least four elements is in $\mathcal{N}_3$ or $\mathcal{N}_{3,\mathbb{F}}$ if and only if $N$ is $U_{2,5}$ or $U_{3,5}$.*

*Proof.* First note that, by [11, Theorem 1.6], $U_{2,5}$ and $U_{3,5}$ are in both $\mathcal{N}_3$ and $\mathcal{N}_{3,\mathbb{F}}$. In the proof of the converse, we assume $r(N) > r^*(N)$ and, by Corollary 3.4.5, we know $\delta(N) = 1$. Several times throughout this proof we add an element freely to $N$ or add an element freely to a flat of $N$. By Lemma 3.4.1, we know this will not contradict $\mathbb{F}$-representability in the case where $N \in \mathcal{N}_{3,\mathbb{F}}$. Thus, we omit mention of the last lemma for the remainder of the proof.

Let $\{a, b, c\}$ be a triad of $N$ and construct a matroid $M$ by adding $e$ freely to the hyperplane $H_1$ of $N$, where $H_1 = E(N) - \{a, b, c\}$. As $N^*$ has no triads, we must contract one of $\{a, b, c\}$ to obtain $N^*$. As $N^*$ contains a triangle, but $N$ does not, either there is a four-circuit of $M$ that uses two of $\{a, b, c\}$ or $r(H_1) \leq 2$. Suppose the latter holds. If $r(H_1) \leq 1$, we immediately have a contradiction. Thus $r(H_1) = 2$; so $r^*(N) = 2$ and $r(N) = 3$. Hence $N \cong U_{3,5}$ and the result holds. We may now assume that $r(H_1) \geq 3$ and $M$ has a four-circuit $C$ that uses two of $\{a, b, c\}$.

Based on our construction of $M$, this implies that $C$ is also a circuit of $N$. Without loss of generality, let $C = \{a, b, f, g\}$ for some $\{f, g\} \subseteq E(N)$. By thinking in terms of $Y - \Delta$ exchanges we can construct a new matroid $M_1$ from $N$ by adding elements $a'$, $b'$, and $c'$ so that $\{a, b, b'\}$, $\{a, c, c'\}$, and $\{a', b, c\}$ are triangles. Then in $M_1$ we have one of the structures depicted in Figure 3.9 and circuit exchange verifies that $\{b', f, g\}$ is a circuit. Let $M = M_1 \backslash \{a', c'\}$. Then $M$ satisfies the obligatory rank constraints and so must have an $N^*$-minor. Now $\{a, b, b'\}$ and $\{b', f, g\}$ are triangles of $M$, so $N^*$ must have, as a restriction, either a five-point line or two triangles in rank three that share a common element.

We now show that

**3.4.6.1.** *the result holds when $N^*$ has, as a restriction, a four-point line.*

Suppose $N^*$ has a four-point line. Then $N$ has a four-point set $\{w, x, y, z\}$ such that any three-element subset forms a triad. Let $H_1 = E(N) - \{w, x, y\}$ and $H_2 = E(N) - \{x, y, z\}$. Construct a new matroid $M$ by adding an element $e$ freely to the flat $H_1 \cap H_2$. If $r(H_1 \cap H_2) \geq$

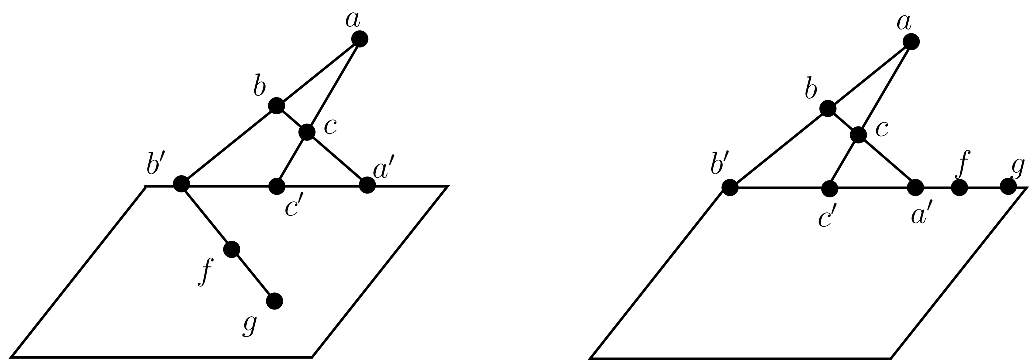



Figure 3.9. Adding $b'$ to show that $N^*$ contains one of two configurations.

2, then we can add $e$ in this way to get a simple matroid. But, since a single contraction cannot destroy all triads in $\{w, x, y, z\}$, we get a contradiction. Thus $r(H_1 \cap H_2) \leq 1$. But, if the rank is zero, we immediately get a contradiction. Therefore $r(H_1 \cap H_2) = 1$, and hence $r(N) = 3$ and $r^*(N) = 2$. So $N \cong U_{3,5}$ and 3.4.6.1 holds.

We may now assume $N^*$ contains a plane that is spanned by two triangles that meet in a point and that $N^*$ has no four-point lines. This implies $N$ has two triads $T_1^*$ and $T_2^*$ that share a common element $x$. Let $H_1$ and $H_2$ denote the hyperplanes that are complementary to $T_1^*$ and $T_2^*$. Suppose that $r(H_1 \cap H_2) > 2$. Then construct a new matroid $M$ by adding $e$ freely to $H_1 \cap H_2$. Note that $e$ creates neither a four-circuit that uses $x$ nor a triangle of any kind in $M$. To destroy the triads in $M$, we must contract $x$ to get $N^*$. Thus, every triangle in $N^*$ arises from a four-circuit of $N$ that uses $x$.

Now construct a new matroid $M$ by adding $e$ to $H_1 \cap H_2$ such that it creates a triangle. Again we must contract $x$ to obtain $N^*$. But now $N^*$ has an extra triangle, a contradiction. We conclude that $r(H_1 \cap H_2) \leq 2$.

If $r(H_1 \cap H_2) = 0$, then $|E(N)| = 5$, so $r(N^*) = 2$. This contradicts the fact that $N^*$ has two triangles in rank three. If $r(H_1 \cap H_2) = 1$, then $|E(N)| = 6$, a contradiction since $\delta(N) = 1$. Therefore, we assume $r(H_1 \cap H_2) = 2$, and so $|E(N)| = 7$, which implies $r(N) = 4$ and $r^*(N) = 3$. Recall our assumption that $N^*$ contains a plane that is spanned by two triangles that meet in a point. We now show

**3.4.6.2.** *$N^*$ has a free element.*

Assume $N^*$ has no free elements. Construct a new matroid $M$ from $N$ by adding $e$ freely to $N$. If we do not contract $e$ to obtain $N^*$, then $N^*$ has a free element, a contradiction. Thus we may assume that $N^*$ is the truncation of $N$, which contradicts the fact that $N$ has no triangles. Hence 3.4.6.2 holds.

Now $N^*$ has a free element and is obtained by taking: (1) a single-element extension of two triangles that meet in a point, together with (2) a freely placed element. As $N^*$ has no four-point lines, it is not hard to check that there are three possible such matroids, depicted in Figure 3.10. However, the matroids $D_1$, $D_2$, and $D_3$ in Figure 3.11 contain $S_1$, $S_2$, and $S_3$ as minors, but not $S_1^*$, $S_2^*$, and $S_3^*$, respectively. Thus these matroids are in neither $\mathcal{N}_3$ nor $\mathcal{N}_{3,\mathbb{F}}$. The result holds by duality.

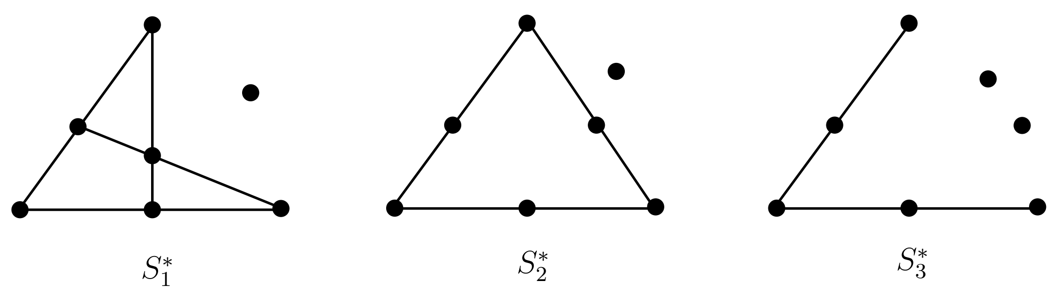



Figure 3.10. Geometric representations of possible matroids for $N^*$.

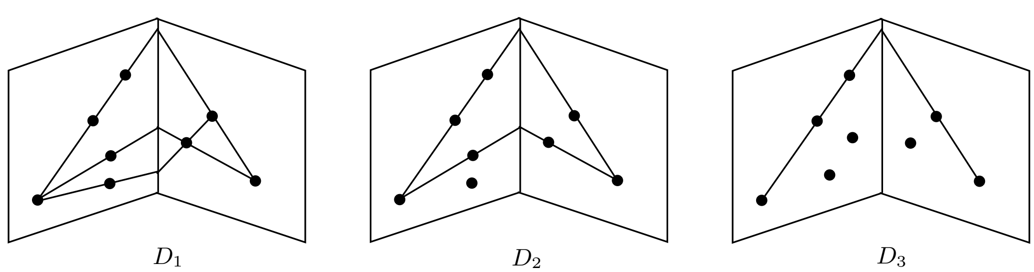



Figure 3.11. Geometric representations of matroids.

□

**Theorem 3.4.7.** *Let $\mathbb{F}$ be an infinite field. Let $N$ be a 3-connected $\mathbb{F}$-representable matroid that is not self-dual such that $|E(N)| \geq 4$. Let $M$ be a 3-connected $\mathbb{F}$-representable matroid for which*

$$\min\{r(M), r^*(M)\} \geq \max\{r(N), r^*(N)\}.$$

*The following are equivalent:*

*(i) $M$ has an $N$-minor if and only if $M$ has an $N^*$-minor.*

*(ii) The matroid $N$ is $U_{2,5}$ or $U_{3,5}$.*

*Proofs of Theorems 3.1.1 and 3.4.7.* Using Proposition 3.4.6 we obtain the results immediately. □

### 3.5 $GF(q)$-representable Case

In this section, we revisit the 3-connected problem with the additional requirement that the matroids are $GF(q)$-representable. For $N \in \mathcal{N}_{3,q}$, as before, we assume $|E(N)| \geq 5$ and $r(N) > r^*(N)$. The fact that a circuit and a cocircuit in a matroid cannot meet in exactly one element will be referred to as *orthogonality*. Let $N$ be a $GF(q)$-representable matroid with $r(N) = r$. Throughout this section, for a subset $H$ of $E(N)$, we let $\mathrm{cl}_{PG}(H)$ denote the closure of $H$ in the projective geometry $PG(r-1, q)$.

We begin by noting some members of $\mathcal{N}_{3,2}$.

**Lemma 3.5.1.** *The matroids $F_7$ and $F_7^*$ are in $\mathcal{N}_{3,2}$.*

*Proof.* To see that $F_7^*$ is in $\mathcal{N}_{3,2}$, note that Seymour [17] showed that $F_7^*$ has only two distinct 3-connected binary single-element extensions, namely, $S_8$ and $AG(3,2)$, both of which are self-dual and so contain $F_7$ as a minor. By duality, the result holds. □

The next lemma provides structure for the members of $\mathcal{N}_{3,q}$. Since $r(N) = r$, we see that $r^*(N) = r - \delta(N)$.

**Lemma 3.5.2.** *Let $N$ be a matroid in $\mathcal{N}_{3,q}$. Then $N$ has a triad and no triangles, so $N^*$ has a triangle and no triads.*

*Proof.* Either $N$ has a triangle, or we can add $e$ to $N$ to create a triangle and then add a further $\delta(N) - 1$ elements such that the resulting matroid $M$ is 3-connected and $GF(q)$-representable if $N$ is. This construction is guaranteed by Lemma 3.4.2. We cannot contract any of the elements in the triangle to obtain $N^*$. Thus $N^*$ has a triangle and, by duality, $N$ has a triad.

Let $T_1^*, T_2^*, \dots, T_k^*$ be all the triads of $N$, and let $N = M[A]$, where $A$ is a matrix over $GF(q)$. Choose $e_1, e_2, \dots, e_n$ from $T_1^* \cup T_2^* \cup \dots \cup T_k^*$ such that, for every $j \in \{1, \dots, k\}$, there is an $i \in \{1, \dots, n\}$ such that $e_i \in T_j^*$, and $n$ is as small as possible. We now prove that

**3.5.2.1.** *the result holds for $n \geq 2$.*

Suppose $n \geq 2$, and let $v_i$ be the column vector of $A$ that represents the element $e_i$ for all $i$ in $\{1, \dots, n\}$. Either there is a minimal such set $\{e_1, \dots, e_n\}$ that is independent, or every minimal such set contains a circuit.

Suppose the former holds. Then, letting $v = \Sigma_{i=1}^{n} v_i$, add $v$ as a new column to $A$, and let $M_0$ be the vector matroid obtained from the resulting matrix. We now show that

**3.5.2.2.** *$v$ is not parallel to any element of $M[A]$.*

We proceed by contradiction. Assume that there is a column $a_i$ of $A$ such that $v = a_i$ and let $a$ be the corresponding element in $M[A]$. Certainly $a \notin \{e_1, \dots, e_n\}$, otherwise $\{e_1, \dots, e_n\}$ contains a circuit, a contradiction. Consider the circuit $C_0 = \{e_1, \dots, e_n, a\}$. By orthogonality, each triad whose intersection with $C_0$ is non-empty meets $C_0$ in at least two elements.

We now show that this contradicts the minimality of $n$. Since $C_0$ intersects each triad in exactly two elements, the set $C_0 - \{a\}$ intersects every triad that does not contain $a$ in exactly two elements. Since, $n \geq 2$, we know $N$ has at least one triad that does not contain $a$. Let $T_1^*$

be a triad such that $T_1^* \cap C_0{=}\{e_i, e_j\}$. Then either $\{e_1, \ldots, e_n\} - \{e_i\}$ or $\{e_1, \ldots, e_n\} - \{e_j\}$ a set set of size smaller than $n$ that meets every triad or there are triads $T_i^*$ and $T_j^*$ such that $T_i^* \cap C_0{=}\{a, e_i\}$ and $T_j^* \cap C_0{=}\{a, e_j\}$. The former is a contradiction, so we may assume the latter holds.

Assume there is another triad $T_2^*$ such that $T_2^* \cap C_0 = \{e_k, e_m\}$ where $\{e_k, e_m\} \cap \{e_i, e_j\} = \emptyset$. Then, as above, we must have triads $T_k^*$ and $T_m^*$ such that $T_k^* \cap C_0{=}\{a, e_k\}$ and $T_m^* \cap C_0{=}\{a, e_m\}$. However, then $(\{e_1, \ldots, e_n\} - \{e_j, e_m\}) \cup \{a\}$ is a set of size smaller than $n$ that meets every triad, a contradiction. A similar argument shows that if there is a triad $T_3^*$ such that $T_3^* \cap C_0 = \{e_i, e_l\}$, then the set $(\{e_1, \ldots, e_n\} - \{e_j, e_l\}) \cup \{a\}$ provides a contradiction. Thus $T_1^*$ is the only triad of $N$ that meets $C_0$ in exactly two elements and does not contain $a$. Therefore $a$ is in every triad of $N$ except $T_1^*$, which implies that $n = 2$. Hence, $T_1^*$, $T_i^*$, and $T_j^*$ are the only triads of $N$ and $\{e_i, e_j, a\}$ is a triangle in $N$. Thus, in $N^*$, the only triangles are $T_1$, $T_i$, and $T_j$ and $\{e_i, e_i, a\}$ is a triad. Consider this situation as depicted in Figure 3.12. If $T_1$, $T_i$, and $T_j$ are the only triangles, then clearly $\{e_i, e_j, a\}$ cannot be a triad. Therefore 3.5.2.2 holds.

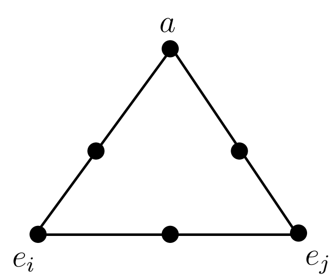



Figure 3.12. Representing the triangles of $N^*$.

Our constructed matroid $M_0$ has no triads. Let $r(N) = r(M_0) = r$. Then, viewing $M_0$ as a restriction of $PG(r-1, q)$, we can construct a new simple matroid $M$ by adding $\delta(N) - 1$ other elements in any available spots in $PG(r-1, q)$. This construction is guaranteed by Lemma 3.4.2. Thus $M$ has no triads and, therefore, $N^*$ has no triads, so 3.5.2.1 holds in this case.

We conclude that every minimum-sized set $\{e_1, \dots, e_n\}$ that meets every triad contains a circuit $C$. Again, by orthogonality, every triad whose intersection with $C$ is non-empty contains at least two elements of the set $\{e_1, \dots, e_n\}$. Taking $\{e_1, \dots, e_n\} - C$ together with one element from each triad that intersects $C$, we have a set that contradicts the minimality of $n$. Therefore 3.5.2.1 holds and we may now assume $n = 1$.

Let $e$ denote the element that is in every triad and assume $N$ has $k$ triads, where $k \geq 2$. Consider a set $\{f_1, \dots, f_k\}$, where $f_i \in T_i^* - \{e\}$. Let $u_i$ be the column vector of $A$ that represents the element $f_i$ for all $i$ in $\{1, \dots, k\}$. Letting $u = \Sigma_{i=1}^{k} u_i$, add $u$ as a new column to $A$, and let $M'$ be the vector matroid obtained from the resulting matrix. Clearly, since each $f_i$ is in exactly one triad, $u$ is not parallel to any element of $N$. The constructed $M'$ has no triads. By Lemma 3.4.2, we can then add another $\delta(N) - 1$ elements to construct a matroid that satisfies the obligatory rank constraints and has no triads, so the result holds in this case.

Thus we may assume that $N$ has a single triad $T^*$. Let $f$ be an element of $E - T^*$ such that $f$ is not in $\mathrm{cl}_N(T^*)$. Construct a new matroid $M_1$ by adding $g$ to $N$ such that $\{e, f, g\}$ forms a triangle. This matroid $M_1$ has no triads and, viewing $M_1$ as a restriction of $PG(r-1, q)$ and using Lemma 3.4.2, we can construct a matroid $M$ that satisfies the obligatory rank constraints and has no triads. Hence $N^*$ has no triads. The result for $N$ holds by duality. □

**Corollary 3.5.3.** *Let $N$ be a matroid in $\mathcal{N}_{3,q}$. Then $\delta(N) = 1$.*

*Proof.* The result follows immediately by combining Lemmas 3.4.4 and 3.5.2. □

This corollary implies that for any matroid $N \in \mathcal{N}_{3,q}$, there is a positive integer $k$ such that $r(N) = k + 1$ and $r^*(N) = k$. The next three lemmas provide information about the number of and interaction between triads of $N$.

**Lemma 3.5.4.** *Let $N$ be a matroid in $\mathcal{N}_{3,q}$. Then $N$ has more than one triad.*

*Proof.* Assume $N$ has only one triad $T^* = \{a, b, c\}$, and let $H$ be the complementary hyperplane. If $N$ has a four-circuit $C$, then create a matroid $M$ by adding an element $e$ to the plane spanned by $C$ such that $e$ belongs to at least two triangles. This implies that $N^*$ must have two triangles, a contradiction. Therefore $N$ has no four-circuits. As $N^*$ has a triangle, anywhere we add $e$ to $\mathrm{cl}_{PG}(H)$, this addition must create either a four-circuit that uses two elements from $\{a, b, c\}$ and the element $e$, or a triangle that uses the element $e$.

Since $|H| = 2k - 2$, the number of ways to create a triangle in $N$ by adding $e$ to $\mathrm{cl}_{PG}(H)$ is at most $(q-1)\binom{2k-2}{2} + 3 = (q-1)(2k^2 - 5k + 3) + 3$, and the number of ways to create a four-circuit using two elements of $T^*$ is at most $\binom{3}{2}(2k-2)(q-1) = (6k-6)(q-1)$. Thus, there are at most $(q-1)(2k^2 - 5k + 3) + (6k-6)(q-1) + 3 = (q-1)(2k^2 + k - 3) + 3$ ways to do either of these. Viewing $H$ as a restriction of $PG(k-1, q)$, we see that there are $\frac{q^k-1}{q-1}$ possible elements in $H$, only $2k-2$ of which are present. Therefore, if $k \geq 7$, then for all $q$ we can create a matroid $M$ by adding $e$ to $\mathrm{cl}_{PG}(H)$ such that $e$ is not in any triangles and not in any four-circuits that use two of $\{a, b, c\}$. This is a contradiction since $N^*$ has a triangle and no triads.

The same contradiction holds for $k = 4$ when $q > 4$ and for $k \in \{5, 6\}$ when $q > 2$. It remains only to consider the cases where $k = 3$ and $q$ is arbitrary; where $k = 4$ and $q \in \{2, 3, 4\}$; and where $k \in \{5, 6\}$ and $N$ is binary.

First we consider $k = 3$. Then $|E(N)| = 7$ and $r(N) = 4$, and we know that $N$ has neither triangles nor four-circuits. This implies that $N \cong U_{4,7}$, which contradicts the fact that $N$ has a triad.

In each of the remaining cases, we consider constructing a representation for $N|H$. We let a basis for $N|H$ be represented by a rank-$k$ identity matrix and consider adding another $k-2$ columns to build the representation. Note that if we add columns with one, two, or three non-zero entries, we have circuits of size two, three, or four respectively. Therefore, in each case below, the added columns must have at least four non-zero entries.

Suppose $k = 4$. If $q = 2$, then there is only one column we can add, namely, the all-ones column, but we need two columns, a contradiction. It is not hard to check that, when $q \in \{3, 4\}$, it is impossible to add two columns that will not create either a triangle or a four-circuit.

For the remainder of the proof, we assume $N$ is binary. Suppose $k = 5$. Then we need to add three columns to build a representation of $N|H$. As there is only one column with five non-zero entries, we must have at least two columns with exactly four non-zero entries. However, taking these two columns with the appropriate two basis vectors forms a four-circuit, a contradiction.

Finally, assume $k = 6$. Note that if we have three columns with exactly four non-zero entries, we must have either a triangle or a four-circuit, so we have at most two such columns. Hence, we must have at least one column with exactly five non-zero entries, which implies we cannot have the all-ones column, otherwise we create a triangle. Thus, we must have two columns with exactly five non-zero entries. But these two columns taken with the appropriate two basis vectors form a four-circuit, a contradiction. $\square$

Recall that $r(N) = k + 1$.

**Lemma 3.5.5.** *Let $N$ be a matroid in $\mathcal{N}_{3,q}$ such that $|E(N)| > 5$. Then $N^*$ has no four-point lines.*

*Proof.* Assume $N$ has a four-point set $\{a, b, c, d\}$ such that every three-element subset forms a triad and let $H_1 = E(N) - \{a, b, c\}$ and $H_2 = E(N) - \{b, c, d\}$. Then $r(H_1 \cap H_2) = r(N) - 2 = k - 1$ and $|H_1 \cap H_2| = 2k - 3$. Thinking of $N$ as a restriction of $PG(k, q)$, we can view $N|(H_1 \cap H_2)$ as a restriction of $PG(k-2, q)$. Thus we see that there are $\frac{q^{k-1}-1}{q-1} - (2k-3)$ possible ways to add $e$ to $\mathrm{cl}_{PG}(H_1 \cap H_2)$ to create a 3-connected $GF(q)$-representable matroid $M$ that satisfies the obligatory rank constraints. It is not hard to check that if $r(H_1 \cap H_2) \geq 3$, then we can add $e$ in this way for all $q \geq 2$. However, in this construction, we cannot destroy all triads of $M$ with a single contraction, a contradiction. Thus $r(H_1 \cap H_2) \leq 2$.

If $r(H_1 \cap H_2) \leq 1$, then $|E(N)| \leq 5$, a contradiction. By Lemma 3.5.2, if $r(H_1 \cap H_2) = 2$, then $|H_1 \cap H_2| = 2$ so $|E(N)|$ is even which contradicts the fact that $|E(N)| = 2k + 1$. The result holds by duality. $\square$

**Lemma 3.5.6.** *Let $N$ be a matroid in $\mathcal{N}_{3,q}$. Then $N$ does not have disjoint triads.*

*Proof.* Assume $N$ has two triads $T_1^*$ and $T_2^*$ that are disjoint, and let $H_1$ and $H_2$ be the complementary hyperplanes. We know $r(H_1 \cap H_2) = r(H_1 - T_2^*) \geq r(H_1) - |T_2^*| = r(N) - 1 - 3 = r(N) - 4 = k - 3$, and $|H_1 \cap H_2| = |E(N)| - 6 = 2k - 5$. As in the previous lemma, we consider $r(H_1 \cap H_2)$. If $r(H_1 \cap H_2) \geq 4$, then we can add an element $e$ to $\mathrm{cl}_{PG}(H_1 \cap H_2)$ to create a matroid $M$, but no single-element contraction of $M$ destroys both triads, a contradiction. If $r(H_1 \cap H_2) = 3$, the same contradiction holds immediately for $q > 2$. Moreover, it is easy to see it also holds when $q = 2$, since $N$ has no triangles. Thus $r(H_1 \cap H_2) \leq 2$.

If $r(H_1 \cap H_2) \in \{0, 2\}$, then $|E(N)|$ is even, a contradiction. Therefore $r(H_1 \cap H_2) = 1$ and $|E(N)| = 7$. This implies $r^*(N) = 3$ and $N^*$ has disjoint triangles. In other words, $N^*$ is a 3-connected single-element extension of two disjoint triangles in rank three that, by Lemma 3.5.5, has no four-point lines. There are a total of four such extensions and these are depicted in Figure 3.14. Note that none of these matroids is binary, so the result holds for $N$ in $\mathcal{N}_{3,2}$. Furthermore, the only matroid in Figure 3.14 that is representable over $GF(3)$ is $P_7$, which is one of the two ternary spikes. Consider the matrix in Figure 3.13 viewed over $GF(3)$.

The matroid $M[A]$ is 3-connected and satisfies the obligatory rank constraints for $P_7$. We can obtain a $P_7^*$-minor from $M[A]$ by deleting 8, but $M[A]$ contains no $P_7$-minor. Therefore, for $N \in \mathcal{N}_{3,3}$, the result holds.

$$
A = \begin{array}{c}
\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 2 \\
0 & 1 & 0 & 0 & 1 & 2 & 0 & 2 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 2 \\
0 & 0 & 0 & 1 & 0 & 1 & 2 & 0
\end{bmatrix}
\end{array}
$$

Figure 3.13. A matroid over $GF(3)$.

It is not hard to check that none of the matroids in Figure 3.14 is in $\mathcal{N}_{3,q}$, for $q \geq 4$. This is shown explicitly in Appendix A. □

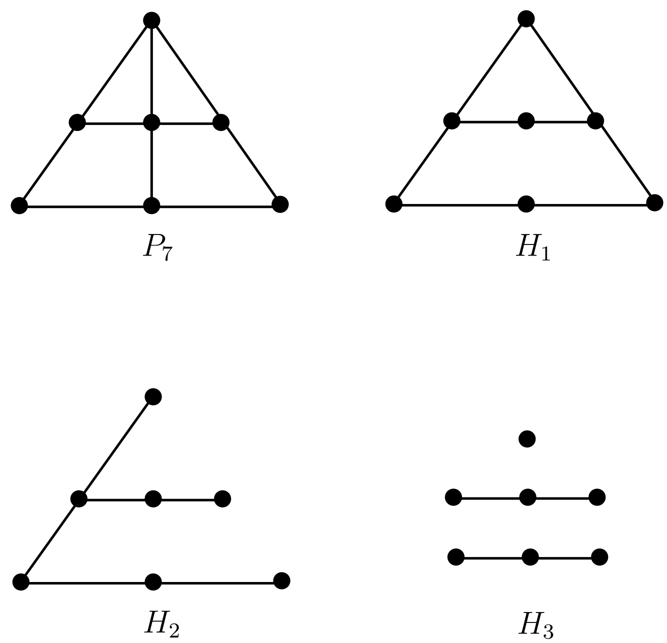



Figure 3.14. Geometric representations of all possibilities for $N^*$ in Lemma 3.5.6.

Combining Lemmas 3.5.4 and 3.5.6 yields the following result.

**Corollary 3.5.7.** *Let $N$ be a matroid in $\mathcal{N}_{3,q}$. Then $N$ has two triads that intersect in a single element.*

The next two lemmas provide additional structure to members of $\mathcal{N}_{3,q}$, and determine some members of $\mathcal{N}_{3,q}$, respectively. The first is elementary; the second follows from [11, Theorem 1.6].

**Lemma 3.5.8.** *Let $N$ be a matroid in $\mathcal{N}_{3,q}$, and let $T_1^*$ and $T_2^*$ be triads that meet in a single element $x$. Adding an element $e$ to $\mathrm{cl}_{PG}(H_1 \cap H_2)$, where $H_i$ is the complementary hyperplane of $T_i^*$, does not produce a triangle containing $x$.*

**Lemma 3.5.9.** *The matroids $U_{2,5}$ and $U_{3,5}$ are in $\mathcal{N}_{3,q}$ if and only if $q \geq 4$.*

For the remainder of the chapter, for any $N \in \mathcal{N}_{3,q}$, we let $T_1^* = \{a, b, x\}$ and $T_2^* = \{c, d, x\}$ be the triads guaranteed by Corollary 3.5.7, and we let $H_1$ and $H_2$ be their complementary hyperplanes. Lemma 3.5.5 implies that $\{a, b, c, d\}$ is a four-cocircuit, which means $(H_1 \cap H_2) \cup \{x\}$ is a hyperplane. Thus we see that

$$r(H_1 \cap H_2) = k - 1 \text{ and } |H_1 \cap H_2| = |E(N)| - 5 = 2k - 4. \tag{3.5.1}$$

Throughout the next few proofs, we employ the strategy of adding an element $e$ to $\mathrm{cl}_{PG}(H_1 \cap H_2)$ to obtain $M$. This construction forces the contraction of $x$ to obtain an $N^*$-minor, since we must destroy both triads. We use the term *new triangles* to refer to triangles in $N^*$ that use the added element $e$. Note that these can arise from triangles of $M$ that use $e$ and from four-circuits of $M$ that use both $e$ and $x$. The following two lemmas will be useful.

**Lemma 3.5.10.** *Let $N$ be a matroid in $\mathcal{N}_{3,q}$ such that $r(N) = k + 1$. If $k = 4$, then $N|(H_1 \cap H_2)$ is a four-circuit.*

*Proof.* Assume $k = 4$. Then $r(N) = 5$ and $|E(N)| = 9$. It follows that $H_1 \cap H_2$ has rank three and contains four elements. By Lemma 3.5.2, $N$ has no triangles, so the result holds. $\square$

**Lemma 3.5.11.** *Let $N$ be a matroid in $\mathcal{N}_{3,q}$, let $T_1^*$ and $T_2^*$ be triads of $N$ such that $T_1^* \cap T_2^* = \{x\}$, and let $H_1$ and $H_2$ denote the hyperplanes that are complementary to $T_1^*$ and $T_2^*$, respectively. There are at most two ways to add an element $e$ to $\mathrm{cl}_{PG}(H_1 \cap H_2)$ such that $e$ forms a triangle that uses elements not in $H_1 \cap H_2$, and there are at most four ways to add $e$ to $\mathrm{cl}_{PG}(H_1 \cap H_2)$ such that $e$ forms a four-circuit that uses $x$.*

*Proof.* Let $T_1^* = \{a, b, x\}$ and $T_2^* = \{c, d, x\}$. Consider adding an element $e$ to $\mathrm{cl}_{PG}(H_1 \cap H_2)$ to create a matroid $M$. By Lemma 3.5.8, we know that $M$ does not have a triangle using both $e$ and $x$. Therefore, the only possible way that $e$ can create a triangle in $M$ that uses elements outside of $H_1 \cap H_2$ is if $e$ is on the line through $\{a, b\}$ or the line through $\{c, d\}$, for a total of at most two ways to create a triangle that uses elements outside of $H_1 \cap H_2$.

Now consider creating a four-circuit that uses $x$. By orthogonality, every four-circuit of the constructed matroid $M$ that uses $x$ must also use an element from each of $\{a, b\}$ and $\{c, d\}$. The planes spanned by $\{x, a, c\}$, $\{x, a, d\}$, $\{x, b, c\}$, and $\{x, b, d\}$ each intersect $\mathrm{cl}_{PG}(H_1 \cap H_2)$ in exactly one point. Thus there are at most four possible spots in $\mathrm{cl}_{PG}(H_1 \cap H_2)$ such that adding $e$ creates a four-circuit using $x$. □

The theorem that follows restricts the rank of matroids in $\mathcal{N}_{3,2}$ and determines that, for $q \geq 3$, all members of $\mathcal{N}_{3,q}$ have five or fewer elements.

**Theorem 3.5.12.** *Let $N$ be a matroid in $\mathcal{N}_{3,q}$.*

*(i) If $q = 2$, then $r(N) \leq 7$.*

*(ii) If $q \geq 3$, then $|E(N)| \leq 5$.*

*Proof.* Suppose $r(N) \geq 8$, so $k \geq 7$. Let $T_1^* = \{a, b, x\}$ and $T_2^* = \{c, d, x\}$. Observe that, by Lemma 3.5.8, the lines through $x$ and any other element in $(T_1^* \cup T_2^*) - \{x\}$ do not intersect $H_1 \cap H_2$. Thus, viewing $N$ as a restriction of $PG(k, q)$, we observe using orthogonality and statement 3.5.1 that there are at most $(q-1)\binom{2k-4}{2} + 2 = (q-1)(2k^2 - 9k + 10) + 2$ possible places to add $e$ to $\mathrm{cl}_{PG}(H_1 \cap H_2)$ that create a triangle in our constructed matroid $M$. The extra two spots in the last expression are the spots mentioned in Lemma 3.5.11.

By Lemma 3.5.11, there are at most 4 possible spots in $\mathrm{cl}_{PG}(H_1 \cap H_2)$ such that adding $e$ creates a four-circuit using $x$. Hence, the number of ways to create a new triangle in $N^*$ by adding an element to $\mathrm{cl}_{PG}(H_1 \cap H_2)$ is at most $(q-1)(2k^2 - 9k + 10) + 6$. There are $\frac{q^{k-1}-1}{q-1}$

elements in $\mathrm{cl}_{PG}(H_1 \cap H_2)$, only $2k-4$ of which are present in $N$. Therefore, since $k \geq 7$, we can construct $M$ by adding $e$ to $\mathrm{cl}_{PG}(H_1 \cap H_2)$ such that $e$ does not form any new triangles in $N^*$. We can also construct a new matroid $M_0$ by adding $e$ to $\mathrm{cl}_{PG}(H_1 \cap H_2)$ such that it creates a new triangle in $N^*$. Thus we have two constructions of $N^*$ that contain different numbers of triangles. This contradiction completes the proof of (i).

We obtain a similar contradiction where $k = 4$ and $q \geq 9$, where $k = 5$ and $q \geq 4$, and where $k = 6$ and $q \geq 3$. Therefore, to finish the proof of (ii), we must check the cases where $k = 3$ for $q \geq 3$, where $k = 4$ for $3 \leq q \leq 8$, and where $k = 5$ for $q = 3$. First we prove the following.

**3.5.12.1.** *If $q = 3$, then $k \notin \{4, 5\}$.*

Assume $N$ is ternary. We consider each $k$ separately. First assume $k = 5$; so $r(N) = 6$ and $|E(N)| = 11$. Then $r(H_1 \cap H_2) = 4$ and $|H_1 \cap H_2| = 6$. There are $\frac{3^4-1}{2} = 40$ elements in $\mathrm{cl}_{PG}(H_1 \cap H_2)$, only six of which are present in $N$. Thus, there are 34 possible places to add $e$ to $\mathrm{cl}_{PG}(H_1 \cap H_2)$. By Lemma 3.5.11, there are at most two spots that create a triangle outside of $\mathrm{cl}_{PG}(H_1 \cap H_2)$ and at most four spots that create a four-circuit using $x$. Thus we have a total of at most six places in $\mathrm{cl}_{PG}(H_1 \cap H_2)$ such that adding $e$ creates new triangles in $N^*$ that use elements outside of $H_1 \cap H_2$. Hence, there are at least 28 places to add $e$ to $\mathrm{cl}_{PG}(H_1 \cap H_2)$ such that this addition creates no new triangles in $N^*$ that use elements outside of $H_1 \cap H_2$. If some of elements in $H_1 \cap H_2$ occupy any of the mentioned six spots, we have more than 28 places to add $e$ such that no new triangles are created.

Consider the number of ways to add $e$ to $\mathrm{cl}_{PG}(H_1 \cap H_2)$ such that we create a triangle in $\mathrm{cl}_{PG}(H_1 \cap H_2)$. Since $|H_1 \cap H_2| = 6$, there are $\binom{6}{2} = 15$ lines spanned by two elements in $H_1 \cap H_2$, and there are two open spots on each line, for a total of at most 30 places in $\mathrm{cl}_{PG}(H_1 \cap H_2)$ where the addition of $e$ will create a triangle in $\mathrm{cl}_{PG}(H_1 \cap H_2)$. We now show that

**3.5.12.2.** *$H_1 \cap H_2$ contains at least one four-circuit.*

Consider possible representations of $N|(H_1 \cap H_2)$. We may assume that $N|(H_1 \cap H_2) = [I_4|A]$, where $A$ is a $4 \times 2$ matrix. Let $\{v_1, \ldots, v_6\}$ label the columns of $[I_4|A]$. Note that if either of $v_5$ or $v_6$ contains exactly four, three, or two zero entries, we get a contradiction since $N$ is simple with no triangles. If either of $v_5$ or $v_6$ has exactly one zero entry, then $H_1 \cap H_2$ contains a four-circuit. Thus we may assume that every entry of $A$ is non-zero. We may further assume that the first row and column of $A$ are ones, and we let $v_6 = (1, a, b, c)$ for some $a, b, c \in \{1, -1\}$. If $a = b = c$, then either $N$ is not simple or $N$ has a triangle, both of which are contradictions. Therefore we may assume that $a = b \neq c$. Then $\{v_1, v_4, v_5, v_6\}$ is a four-circuit, otherwise $\{v_4, v_5, v_6\}$ is a triangle. Thus 3.5.12.2 holds.

Let $C$ be a four-circuit contained in $H_1 \cap H_2$. Then $\mathrm{cl}_{PG}(C) \cong PG(2,3)$. Consider partitioning the elements of $C$ into two sets of two. Since we are in a projective geometry, the lines through the elements in each part of the partition must intersect. There are three ways to choose such a partition. This implies, since $C \subseteq H_1 \cap H_2$, that we over-counted the number of open spots in $\mathrm{cl}_{PG}(H_1 \cap H_2)$ on lines created by pairs of elements in $H_1 \cap H_2$ by at least three. This leaves us with 27 places that create a triangle in $\mathrm{cl}_{PG}(H_1 \cap H_2)$. As we have at least 28 places available, we can add $e$ such that it creates no new triangles in $N^*$. This allows us to construct two copies of $N^*$ with different numbers of triangles to obtain a contradiction.

Now assume $k = 4$. Again we consider $H_1 \cap H_2$, which is a four-circuit by Lemma 3.5.10. Considering $N$ as a subset of $PG(4,3)$, since $r(H_1 \cap H_2) = 3$, we see that $\mathrm{cl}_{PG}(H_1 \cap H_2) \cong PG(2,3)$, so $|\mathrm{cl}_{PG}(H_1 \cap H_2)| = 13$. As $|H_1 \cap H_2| = 4$, there are nine possible ways to add $e$ to $\mathrm{cl}_{PG}(H_1 \cap H_2)$. By considering the structure of $PG(2,3)$, since $H_1 \cap H_2$ is a four-circuit, we see that three of the nine spots correspond to forming two triangles in $\mathrm{cl}_{PG}(H_1 \cap H_2)$, while the other six correspond to forming a single triangle in $\mathrm{cl}_{PG}(H_1 \cap H_2)$. There are also, by Lemma 3.5.11, at most six ways to add $e$ to $\mathrm{cl}_{PG}(H_1 \cap H_2)$ that create new triangles in $N^*$ that use elements outside of $H_1 \cap H_2$. Note there must be overlap between the spots that create triangles inside $\mathrm{cl}_{PG}(H_1 \cap H_2)$ and the spots that create new triangles in $N^*$ that use elements outside of $H_1 \cap H_2$.

Assume at least one of the four elements that form a four-circuit with $x$ is already present in $H_1 \cap H_2$. Then there are at most 5 available spots in $\mathrm{cl}_{PG}(H_1 \cap H_2)$ such that adding $e$ creates a new triangle in $N^*$ that uses elements outside of $H_1 \cap H_2$. Since there are six spots that correspond to forming a single triangle in $\mathrm{cl}_{PG}(H_1 \cap H_2)$, there must be an element $e$ we can add to $\mathrm{cl}_{PG}(H_1 \cap H_2)$ such that $e$ forms only a single new triangle in $N^*$. However, this is a contradiction since we can construct $N^*$ by adding $e$ to one of the three spots in $\mathrm{cl}_{PG}(H_1 \cap H_2)$ that creates two new triangles in $N^*$.

Similarly, assume one of the available spots in $\mathrm{cl}_{PG}(H_1 \cap H_2)$ that corresponds to forming two triangles in $\mathrm{cl}_{PG}(H_1 \cap H_2)$ also corresponds to forming a new triangle in $N^*$ that uses elements outside of $H_1 \cap H_2$. Then by adding $e$ we can create three new triangles in $N^*$. However, we can instead add $e$ to one of the six spots in $\mathrm{cl}_{PG}(H_1 \cap H_2)$ that corresponds to forming a single triangle in $\mathrm{cl}_{PG}(H_1 \cap H_2)$. In this construction, we are guaranteed to create fewer than three new triangles in $N^*$, a contradiction. Therefore, we may assume that there are no four-circuits that use $x$ in $N$, and each available spot in $\mathrm{cl}_{PG}(H_1 \cap H_2)$ corresponds to forming exactly two new triangles in $N^*$.

More specifically, for every possible $e$ we can add to $\mathrm{cl}_{PG}(H_1 \cap H_2)$, exactly one of the following holds:

1. $e$ forms two triangles in $\mathrm{cl}_{PG}(H_1 \cap H_2)$ and no other triangles, and $e$ forms no four-circuits using $x$; or

2. $e$ forms a single triangle in $\mathrm{cl}_{PG}(H_1 \cap H_2)$ and a single triangle that uses elements outside of $H_1 \cap H_2$, and $e$ forms no four-circuits using $x$; or

3. $e$ forms a single triangle in $\mathrm{cl}_{PG}(H_1 \cap H_2)$ and no other triangles, and $e$ forms one four-circuit using $x$.

However, the reader can check that this arrangement is impossible since the matroid is ternary. This completes the proof of 3.5.12.1.

Next we consider the case where $k = 4$ and $q \geq 4$. Then $N|(H_1 \cap H_2)$ is a four-circuit by 3.5.10. Note that since $|H_1 \cap H_2| = 4$, there are $\binom{4}{2} = 6$ lines through pairs of elements in $H_1 \cap H_2$. Thus there are exactly three ways to add $e$ to $\mathrm{cl}_{PG}(H_1 \cap H_2)$ to create two triangles in $\mathrm{cl}_{PG}(H_1 \cap H_2)$, one for each partition of the four elements of $H_1 \cap H_2$ into two sets of two. Also, since $\mathrm{cl}_{PG}(H_1 \cap H_2) \cong PG(3, q)$, there are $6(q-2)$ ways to add $e$ to $\mathrm{cl}_{PG}(H_1 \cap H_2)$ to create a single triangle in $\mathrm{cl}_{PG}(H_1 \cap H_2)$. However, by Lemma 3.5.11, there are at most six ways to add $e$ to $\mathrm{cl}_{PG}(H_1 \cap H_2)$ to create a new triangle in $N^*$ that uses elements outside of $H_1 \cap H_2$. Therefore, as above, for all $q \geq 4$, we can add $e$ to $\mathrm{cl}_{PG}(H_1 \cap H_2)$ to obtain multiple constructions of $N^*$ with differing numbers of triangles, a contradiction.

We conclude that $k = 3$. This implies $r(N) = 4$ and $r^*(N) = 3$, so $|E(N)| = 7$. In the ternary case, there are only two 3-connected matroids with the required rank and corank, namely, $(F_7^-)^*$ and $(P_7)^*$, the duals of the two rank-three ternary spikes. However, it is routine to check that neither of these is in $\mathcal{N}_{3,3}$. It is straightforward, if a bit tedious, to check that there are no seven-element matroids in $\mathcal{N}_{3,q}$ when $q \geq 4$, which finishes the proof of (ii). This is shown explicitly in Appendix A. □

**Theorem 3.5.13.** *Let $N$ be a 3-connected ternary matroid that is not self-dual. Let $M$ be a 3-connected ternary matroid for which*

$$\min\{r(M), r^*(M)\} \geq \max\{r(N), r^*(N)\}.$$

*The following are equivalent:*

*(i) $M$ has an $N$-minor if and only if $M$ has an $N^*$-minor.*

*(ii) The matroid $N$ is $U_{0,1}$, $U_{1,1}$, $U_{1,3}$, or $U_{2,3}$.*

**Theorem 3.5.14.** *For $q \geq 4$, let $N$ be a 3-connected $GF(q)$-representable matroid that is not self-dual such that $|E(N)| \geq 4$. Let $M$ be a 3-connected $GF(q)$-representable matroid for which*

$$\min\{r(M), r^*(M)\} \geq \max\{r(N), r^*(N)\}.$$

*The following are equivalent:*

*(i) $M$ has an $N$-minor if and only if $M$ has an $N^*$-minor.*

*(ii) The matroid $N$ is $U_{2,5}$ or $U_{3,5}$.*

*Proofs of Theorems 3.5.13 and 3.5.14.* Combining Lemma 3.5.9 and Theorem 3.5.12 gives the results immediately. □

The next proposition further restricts the rank of matroids in $\mathcal{N}_{3,2}$.

**Proposition 3.5.15.** *Let $N$ be a matroid in $\mathcal{N}_{3,2}$. Then $r(N) \leq 5$.*

*Proof.* By Theorem 3.5.12, we know $r(N) \leq 7$. Assume $r(N) \in \{6, 7\}$. First we show that

**3.5.15.1.** *up to symmetry, $N|(H_1 \cap H_2)$ is represented by one of the matrices in Figure 3.15.*

Consider possible representations of $N|(H_1 \cap H_2)$. Suppose $r(N) = 6$. Then, by statement 3.5.1, $r(H_1 \cap H_2) = 4$ and $|H_1 \cap H_2| = 6$. Let $[I_4|A]$ be a matrix representation of $N|(H_1 \cap H_2)$, with column labels $\{v_1, \ldots, v_6\}$. If $v_5$ or $v_6$ has two or more zero entries, then either we contradict the simplicity of $N$, or $N$ contains a triangle, which contradicts Lemma 3.5.2. Thus both $v_5$ and $v_6$ have at most one zero entry. If both $v_5$ and $v_6$ have no zero entries, then, as $N$ is binary, this means $N$ has a two-circuit; a contradiction. Thus we may assume $v_5$ has exactly one zero entry. If $v_6$ has no zero entries, then $N$ contains a triangle, a contradiction. Therefore, up to symmetry, the matrix $V$ in Figure 3.15 is the only representation of $N|(H_1 \cap H_2)$.

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \quad A_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Figure 3.15. Possible representations for $N|(H_1 \cap H_2)$.

Now suppose $r(N) = 7$. Then $r(H_1 \cap H_2) = 5$ and $|H_1 \cap H_2| = 8$. Let $[I_5|A]$ be a matrix representation of $N|(H_1 \cap H_2)$, where $\{v_1, \ldots, v_8\}$ are the columns labels. If $v_6$, $v_7$, or $v_8$ has more than two zero entries, then either $N$ is not simple or $N$ contains a triangle, both of which are contradictions. Thus each of $v_6$, $v_7$, and $v_8$ has two or fewer zero entries. Note that there are many different possible combinations for the columns $v_6$, $v_7$, and $v_8$. A routine, if tedious, check shows that the possible different representations of $N|(H_1 \cap H_2)$, up to symmetry, are the matrices $A_1$, $A_2$, $A_3$, and $A_4$ in Figure 3.15, where many of the possible representations end up being symmetric. Thus 3.5.15.1 holds.

We deduce, from looking at the possible representations of $N|(H_1 \cap H_2)$, that $H_1 \cap H_2$ contains a four-circuit $C$. We now show that

**3.5.15.2.** *there is at least one element $e$ that can be added to $\mathrm{cl}_{PG}(H_1 \cap H_2)$ such that $e$ creates no triangles in $\mathrm{cl}_{PG}(H_1 \cap H_2)$.*

To do this, we look at the possible representations of $N|(H_1 \cap H_2)$ from Figure 3.15. If $r(N) = 6$, then $N|(H_1 \cap H_2)$ can be represented by the matrix $V$. By adding the column $v_e = (1, 1, 0, 1)$, we create no triangles in $\mathrm{cl}_{PG}(H_1 \cap H_2)$. Now suppose that $r(N) = 7$. By adding the columns $v_1 = (1, 1, 1, 0, 1)$, $v_2 = (1, 0, 1, 0, 1)$, $v_3 = v_4 = (1, 1, 1, 1, 1)$ to each of $A_1$, $A_2$, $A_3$, and $A_4$, respectively, we create no triangles in $\mathrm{cl}_{PG}(H_1 \cap H_2)$. Thus 3.5.15.2 holds.

Now, if we can add $e$ to $\mathrm{cl}_{PG}(H_1 \cap H_2)$ without creating any new triangles in $N^*$, then we can build two constructions of $N^*$ that contain different numbers of triangles, a contradiction. Similarly, if we can add $e$ such that it creates a single new triangle in $N^*$, the same type of contradiction holds since we can add $e$ to $\mathrm{cl}_{PG}(H_1 \cap H_2)$ in such a way as to create two new triangles using elements of $C$. Therefore, we assume that if we extend $N$ by adding $e$ to $\mathrm{cl}_{PG}(H_1 \cap H_2)$ without creating any triangles in $H_1 \cap H_2$, then we create two new triangles in $N^*$ that use elements outside of $H_1 \cap H_2$.

Assume adding $e$ creates one triangle in $M$ using elements in $E(M) - (H_1 \cap H_2)$ and one four-circuit containing $x$. We know, by Lemma 3.5.8, that $e$ does not form a triangle using $x$. Without loss of generality, assume $\{a, b, e\}$ is the triangle and $\{a, c, e, x\}$ is the four-circuit. Then $b \in \mathrm{cl}_M(\{a, c, e, x\})$ and $r(\mathrm{cl}_M(\{a, c, e, x\})) = 3$. As $M$ is binary, there are two triangles in $\mathrm{cl}_M(\{a, c, e, x\})$, neither of which can use $x$, a contradiction.

There is only a single way to add $e$ to $\mathrm{cl}_{PG}(H_1 \cap H_2)$ such that it creates two triangles in $M$ and no triangles in $H_1 \cap H_2$. Also, there are only two distinct ways to add $e$ to $\mathrm{cl}_{PG}(H_1 \cap H_2)$ such that it creates no triangles in $M$ and two four-circuits that use $x$. Thus, as before, there are at most three distinct ways that $e$ can be added to $\mathrm{cl}_{PG}(H_1 \cap H_2)$ such that $e$ is in no triangles in $H_1 \cap H_2$ and we do not get contradictory constructions of $N^*$, as before.

Assume $r(N) = 6$, so $r(H_1 \cap H_2) = 4$ and $|H_1 \cap H_2| = 6$. Since $N$ is triangle-free, there is only one matroid, up to isomorphism, for $N|(H_1 \cap H_2)$, namely, a deletion of $F_7^*$. We can add $e$ to $\mathrm{cl}_{PG}(H_1 \cap H_2)$ without creating triangles in $\mathrm{cl}_{PG}(H_1 \cap H_2)$, so we must assume that, whenever we add $e$ in such a way, it creates two new triangles in $N^*$. However, if instead we add $e$ such that it creates three triangles in $\mathrm{cl}_{PG}(H_1 \cap H_2)$, then we still must contract $x$ to obtain $N^*$. But now we have three new triangles, a contradiction.

To complete the proof, assume $r(N) = 7$, so $r(H_1 \cap H_2) = 5$ and $|H_1 \cap H_2| = 8$. Looking at possible choices of $N|(H_1 \cap H_2)$ tells us that either there are more than three distinct ways to add $e$ to $\mathrm{cl}_{PG}(H_1 \cap H_2)$ without creating a triangle in $\mathrm{cl}_{PG}(H_1 \cap H_2)$, or there are exactly three such ways to add $e$ to $\mathrm{cl}_{PG}(H_1 \cap H_2)$. The former gives a contradiction so the latter holds. Then each of those three distinct extensions must correspond to adding two new triangles in $N^*$. It is not hard to see that, instead, we can add $e$ to $\mathrm{cl}_{PG}(H_1 \cap H_2)$ in such a way as to produce a single triangle in $\mathrm{cl}_{PG}(H_1 \cap H_2)$, and we are guaranteed to create no other new triangles in $N^*$, which is a contradiction. $\square$

For each $r \geq 3$, the rank-$r$ binary spike with tip $t$ is the matroid $Z_r$ that is represented by the binary matrix in Figure 3.16. Evidently, $Z_3 \cong F_7$, the Fano matroid. The following result [10] will be useful in our final argument in the 3-connected binary case, the result for which is stated in Theorem 3.1.2.

$$\begin{array}{c}
\begin{array}{cccc|ccccc|c} x_1 & x_2 & \cdots & x_r & y_1 & y_2 & y_3 & \cdots & y_r & t \end{array} \\
\left[\begin{array}{c|ccccc|c}
 & 0 & 1 & 1 & \cdots & 1 & 1 \\
 & 1 & 0 & 1 & \cdots & 1 & 1 \\
I_r & 1 & 1 & 0 & \cdots & 1 & 1 \\
 & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 & 1 & 1 & 1 & \cdots & 0 & 1
\end{array}\right]
\end{array}$$

Figure 3.16. A binary representation for $Z_r$.

**Theorem 3.5.16.** *Let $M$ be a binary matroid with $r(M) \geq 3$. Then $M$ is 3-connected and has no $M(\mathcal{W}_4)$-minor if and only if $M \cong Z_r$, $Z_r^*$, $Z_r \backslash y_r$, or $Z_r \backslash t$ for some $r \geq 3$.*

*Proof of Theorem 3.1.2.* Take $N \in \mathcal{N}_{3,2}$ and assume $r(N) > r^*(N)$. Proposition 3.5.15 tells us that $r(N) \leq 5$, and clearly $r(N) \geq 4$. If $r(N) = 4$, then $|E(N)| = 7$ and, by Theorem 3.5.16, we know $N \cong Z_3^* \cong F_7^*$. Now suppose $r(N) = 5$. By Theorem 3.5.16, either $N$ has an $M(\mathcal{W}_4)$-minor, or $N \cong Z_4^*$. Assume the former. Then $N$ is a single-element coextension of $M(\mathcal{W}_4)$, so $N^*$ is a single-element extension of $M(\mathcal{W}_4)$. Consider building a binary representation for $N^*$ by adding a column vector to the binary matrix $A$ in Figure 3.17 that represents $M(\mathcal{W}_4)$.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Figure 3.17. A representation for $M(\mathcal{W}_4)$.

If we add a column with any zero entries, then $N^*$ will contain a triad, a contradiction to Lemma 3.5.2. Thus, we must add the all-ones column, that is, $N^* \cong M^*(K_{3,3})$. Thus $N \cong M(K_{3,3})$. To see that this matroid is not in $\mathcal{N}_{3,2}$, consider adding an edge between two non-adjacent vertices of $K_{3,3}$. Let $M$ be the 3-connected graphic matroid that corresponds to the resulting graph. Then $M$ meets the obligatory rank constraints but, as $N^*$ is not graphic, $M$ has no $N^*$-minor. This contradiction implies that $N$ has no $M(\mathcal{W}_4)$-minor. Therefore we may assume $N \cong Z_4^*$.

We now build a 3-connected binary matroid $M$ that has a $Z_4$-minor and no $Z_4^*$-minor, thus showing $N \ncong Z_4^*$. Consider the binary matrix $B$ in Figure 3.18 that represents $Z_4$ extended by two elements.

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Figure 3.18. A representation for $Z_4$ extended by two elements.

To obtain our example, we coextend the matroid $M[B]$. Let $M$ be the matroid represented by the matrix in Figure 3.19.

$$C = \begin{array}{c} \begin{array}{cccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & t & 9 & 10 \end{array} \\ \left[\begin{array}{cccccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \end{array}$$

Figure 3.19. A coextension of the matrix in Figure 3.18.

Essentially, $M$ is obtained from $Z_4$ by adding a triad in a new rank. To obtain a $Z_4^*$-minor from $M$, we must delete three elements. As $Z_4^*$ has no triangles, but $M[C]$ has $\{t, 8, 3\}$, $\{t, 7, 2\}$, $\{t, 6, 1\}$, and $\{t, 5, 0\}$ as triangles, we know we must delete $t$. We also know that we cannot delete any elements from the triad $\{4, 9, 10\}$, otherwise we create a cocircuit of size less than three. To see that $M$ does not contain a $Z_4^*$-minor, we consider spanning circuits. It is easily checked that $Z_4^*$ has no spanning circuits. We argue that it is impossible to destroy all spanning circuits of $M$ by deleting $t$ and two elements from $\{0, 1, 2, 3, 5, 6, 7, 8\}$.

First note that the following sets are spanning circuits of $M$: $\{0, 1, 5, 7, 9, 10\}$, $\{2, 3, 6, 8, 9, 10\}$, $\{0, 2, 5, 6, 9, 10\}$, $\{1, 3, 7, 8, 9, 10\}$, $\{0, 2, 3, 7, 9, 10\}$, $\{1, 5, 6, 8, 9, 10\}$, $\{2, 5, 7, 8, 9, 10\}$, $\{0, 1, 3, 6, 9, 10\}$. Observe that these circuits are listed in pairs that contain $\{9, 10\}$ and partition $\{0, 1, 2, 3, 5, 6, 7, 8\}$. As we cannot delete 9 or 10, we must delete one of $\{0, 8\}$, $\{1, 2\}$, $\{6, 7\}$, or $\{3, 5\}$ to obtain $Z_4^*$. However, in each case, the resulting matroid contains the spanning circuit $\{1, 2, 3, 4, 6, 9\}$, $\{0, 3, 4, 6, 8, 9\}$, $\{1, 3, 4, 5, 8, 9\}$, or $\{1, 4, 6, 7, 8, 9\}$, respectively. Thus $M$ has no $Z_4^*$-minor. Hence $r(N) \neq 5$, and the result holds by duality. □

### 3.6 Graphic Case

In this section, we revisit the 3-connected problem with the additional assumption that the matroids are graphic. Specifically, we determine the set $\mathcal{N}_G$ of 3-connected graphic matroids $N$ such that if a 3-connected graphic matroid $M$ satisfies the obligatory rank constraints, then $M$ has an $N$-minor if and only if $M$ has an $N^*$-minor. Recall from Section 3.3 that all $N \in \mathcal{N}_G$ are planar graphic. We will continue our assumptions that $|E(N)| \geq 4$ and $\delta(N) > 0$ for $N \in \mathcal{N}_G$. As in previous sections, we will construct matroids $M$ from $N$ by adding elements that do not change the rank to obtain matroids that satisfy the obligatory rank constraints. In this construction, each contraction from $M$ to obtain an $N^*$-minor must drop the rank of the matroid. In other words, we must contract an independent set of $M$ to obtain $N^*$.

For the remainder of this section, we will primarily use graphs, rather than matroids, in our arguments, noting that we can always assume that a graph corresponding to a graphic matroid is connected. Further note that graphs $G$ that are 3-connected, simple, and planar have unique planar embeddings. Thus the faces of $G$ are well-defined. We let $V_H$, $E_H$, and $F_H$ denote the numbers of vertices, edges, and faces, respectively, of a planar graph $H$. Throughout the figures in this section, it will be useful to highlight vertices of degree three, that is, vertices whose incident edges form triads in the underlying matroid. This property will be indicated using a circled vertex. We will call the set of edges incident with a degree-three vertex a *vertex-triad*.

Let $G$ be a plane graph that contains a face $F$ of size at least four. We can add an edge $e$ as a *chord*, or *facial chord*, of $F$ by adding $e$ such that it is incident to two non-adjacent vertices on the boundary of $F$. Note that adding $e$ in this way yields a graph that is planar.

This section will rely heavily on the notion of matroid and graph fans. In a simple, cosimple matroid $M$, a subset $\Phi$ of $E(M)$ having at least three elements is a *fan* of $M$ if there is an ordering $(f_1, f_2, \ldots, f_n)$ of $\Phi$ such that, for all $i$ in $\{1, 2, \ldots, n-2\}$, the set $\{f_i, f_{i+1}, f_{i+2}\}$ is a triangle or a triad and, when $\{f_i, f_{i+1}, f_{i+2}\}$ is a triangle, $\{f_{i+1}, f_{i+2}, f_{i+3}\}$ is a triad and vice versa. Note that, for a fan $\Phi = (f_1, f_2, \ldots, f_n)$, if $n$ is even then there is exactly one triangle and exactly one triad in $\{\{f_1, f_2, f_3\}, \{f_{n-2}, f_{n-1}, f_n\}\}$. If $n$ is odd, then either both $\{f_1, f_2, f_3\}$ and $\{f_{n-2}, f_{n-1}, f_n\}$ are triangles or they are both triads. Consider the fan

$\Phi = (r_1, s_1, r_2, s_2, \ldots, r_5, s_5)$ depicted in Figure 3.20. We call the edges labeled with $s_i$ *spoke elements* of $\Phi$, and the edges labeled with $r_i$ *rim elements* of $\Phi$. For any edge $e$ in $\Phi$, we call a vertex incident to $e$ a *vertex of* $\Phi$. We call a fan that contains $k$ elements a *$k$-fan.*

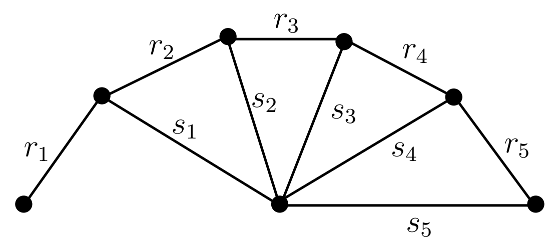



Figure 3.20. A ten-element fan.

Recall that $P$ denotes the six-vertex triangular prism graph shown in Figure 3.1. We now prove the following.

**Lemma 3.6.1.** *The matroids $M(P)$ and $M(K_5 \backslash e)$ are both in $\mathcal{N}_G$.*

*Proof.* Up to isomorphism, there is only one 3-connected planar graphic single-element extension of $M(P)$. The associated graph is depicted in Figure 3.21, where $g$ represents the new edge. However, note that contracting $f$ gives $K_5 \backslash e$. Further note that we cannot coextend $M(P)$ by a single element and remain 3-connected and planar graphic because every vertex of $H$ has degree three. Thus the result holds. $\square$

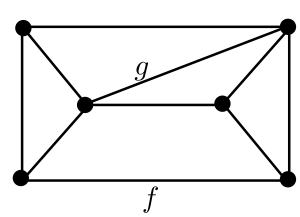



Figure 3.21. The only 3-connected planar extension of $P$.

The next theorem is Euler's well-known formula for planar graphs, which will be useful in proving the lemma that follows it; the proof of Euler's formula is omitted.

**Theorem 3.6.2.** *Let $G$ be a connected planar graph. Then $V_G - E_G + F_G = 2$.*

**Lemma 3.6.3.** *Let $N$ be a matroid in $\mathcal{N}_G$ such that $|E(N)| \geq 4$ and $\delta(N) > 0$. It is possible to construct a 3-connected, planar graphic matroid $M$ from $N$ by adding $\delta(N)$ elements to $N$ such that $r(M) = r(N)$ and $r^*(M) = r^*(N) + \delta(N)$. Moreover, letting $\Phi$ be a maximum-sized fan in $N$, we can construct such an $M$ in which $\Phi$ remains a fan.*

*Proof.* Recall $\delta(N) = r(N) - r^*(N)$. Since $\delta(N) > 0$, we know that $r(N) > r^*(N)$. Let $H$ be a graph such that $M(H) = N$. Then $r(N) = V_H - 1$ and $r^*(N) = E_H - V_H + 1$. Therefore $\delta(N) = V_H - 1 - E_H + V_H - 1 = 2V_H - E_H - 2$. We begin by proving the following.

**3.6.3.1.** *There is a 3-connected planar graphic matroid $M$ that can be built from $N$ by adding $\delta(N)$ elements to $N$ such that $r(M) = r(N)$ and $r^*(M) = r^*(N) + \delta(N)$.*

We show that we can construct a 3-connected, simple, planar graph $H'$ from $H$ by adding $\delta(N)$ edges. By Lemma 3.3.6, if $E_H + \delta(N) = E_H + 2V_H - E_H - 2 \leq 3V_H - 6$, then we are guaranteed to be able to construct $H'$. Thus, by the last inequality, when $V_H \geq 4$, the result holds. If $V_H < 4$, then $|E(N)| \leq 4$, a contradiction. Thus 3.6.3.1 holds.

To show that we can construct a matroid $M$ in which $\Phi$ remains a fan we must show that we can add $\delta(N)$ edges to $H$ such that none is incident to a degree-three vertex of $\Phi$. Consider the set $V_3$ of degree-three vertices of $\Phi$ and assume $\Phi$ is a $k$-fan. If every face that contains a vertex in $V_3$ on its boundary is a triangle, then $k = 5$ and $\Phi$ has triangles on both ends. In this case, since the only degree-three vertex of $\Phi$ is surrounded by triangular faces, we can use 3.6.3.1 to see that the result holds. Thus we may assume, in $H$, every degree-three vertex of $\Phi$ is on the boundary of a common non-triangular face $A$.

Consider $\Phi$ as pictured in Figure 3.22, where the dotted edges may, or may not, be present in $H$. Construct a graph $H_0$ by adding the dotted edges $v_1v_2$ and $v_1v_3$, if they are not already present in $H$. Then we need to add an additional $n$ edges to $H_0$ to construct $H'$, where $n \in \{\delta(N), \delta(N) - 1, \delta(N) - 2\}$, depending on how many of the dotted edges were already present in $H$. Let $F_A$ denote the set of all faces of $H_0$ except $A$. Finish constructing

$H'$ from $H_0$ by adding $n$ edges as chords to faces in $F_A$ such that as many faces as possible become triangular faces, and no edges are added as chords in $A$. If we can add $n$ edges in this way, then the result holds.

Suppose we cannot add $n$ edges in this way. Then construct $H'$ by adding $j < n$ edges to make every face of $H_0$ a triangle except $A$, and then adding $n - j$ edges to $A$ such that each edge added to $A$ creates a triangular face. Then, letting $M(H') = M$, we have a 3-connected planar graphic matroid $M$ that satisfies the obligatory ranks constraints and so must have an $N^*$-minor. However, every element of $M$ is in a triangle, so we cannot contract any element and remain 3-connected, a contradiction. Thus we must be able to add $n$ edges without adding any as chords in $A$, and the result holds. □

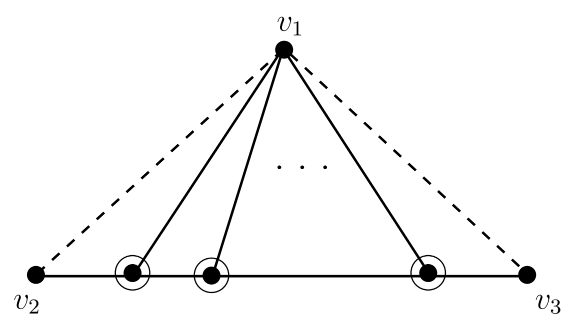



Figure 3.22. A maximal fan $\Phi$ in $H$.

The next two lemmas restrict the structure of $N$ and size of $\delta(N)$ for $N \in \mathcal{N}_G$, and give us information regarding maximal fans in $N$.

**Lemma 3.6.4.** *Take a matroid $N \in \mathcal{N}_G$ with $\delta(N) > 0$. Then $N$ has a vertex-triad and $N^*$ has a triangular face.*

*Proof.* Let $H$ be a graph such that $M(H) = N$. Create a graph $H'$ from $H$ by adding $\delta(N)$ edges such that at least one triangular face is created and the graph $H$ is 3-connected, simple, and planar. We know such a construction exists by Lemma 3.6.3. Let $M = M(H')$. Then $M$ meets the obligatory rank constraints and thus must have an $N^*$-minor. We cannot contract any elements in the triangle to obtain $N^*$, so $N^*$ has a triangular face which, by duality, implies that $N$ has a vertex-triad. $\square$

The last lemma guarantees that every $N$ in $\mathcal{N}_G$ has a fan.

**Lemma 3.6.5.** *Let $N$ be a matroid in $\mathcal{N}_G$ and assume that a maximum-sized fan in $N$ has $k$ elements. Then*

*(i) $k$ is odd; and*

*(ii) $\delta(N) = 1$; and*

*(iii) $N$ has a fan $(f_1, f_2, \ldots, f_k)$ such that both of the sets $\{f_1, f_2, f_3\}$ and $\{f_{k-2}, f_{k-1}, f_k\}$ are triads.*

*Proof.* First suppose that $k$ is even and let $\Phi = (f_1, f_2, \ldots, f_k)$ be a fan in $N$. Then one of $\{f_1, f_2, f_3\}$ and $\{f_{k-2}, f_{k-1}, f_k\}$ is a triangle and the other is a triad. Without loss of generality, assume $\{f_1, f_2, f_3\}$ is a triad. Because $\{f_2, f_3\}$ is contained in a triangle, the triad $\{f_1, f_2, f_3\}$ must be a vertex-triad. Construct a planar graphic matroid $M$ from $N$ by adding $e$ such that $\{e, f_1, f_2\}$ is a triangle and adding the remaining $\delta(N) - 1$ elements such that $M$ is planar graphic and 3-connected, and $\Phi$ is still a fan in $M$. Then $M$ has a $(k+1)$-fan with triangles on both ends, which means that $N^*$ has the same $(k+1)$-fan, contradicting the maximality of $k$. Thus (i) holds.

Suppose $\delta(N) \geq 2$. If $N$ has a $k$-fan $\Phi$ that ends in triads, then construct a new matroid $M$ by using two elements to create a $(k+2)$-fan from $\Phi$, and add the remaining $\delta(N) - 2$ elements such that $M$ is planar graphic, 3-connected, and contains $\Phi$. This construction is guaranteed by Lemma 3.6.3. Then $M$, and hence $N^*$, contains a $(k+2)$-fan, contradicting the maximality of $k$. Therefore, all $k$-fans in $N$ end in triangles. However, as long as the $\delta(N)$ elements are added such that $M$ is 3-connected, $M$, and hence $N^*$, will contain the same

$k$-fan with triangle ends. This implies that $N$ has a $k$-fan with triad ends, a contradiction. Thus (ii) holds.

Now suppose that every fan of size $k$ in $N$ has triangles on both ends and let $H$ be a graph such that $M(H) = N$. Let $A$ denote the face of $H$ that has all the rim elements of $\Phi$ on its boundary. Construct a new 3-connected, simple, planar graph $H'$ by adding an edge $e$ as a chord in any face but $A$. If this construction is possible, then $H^*$ contains the $k$-fan $\Phi$ which has triangles on both ends, so $H$ contains a $k$-fan with triads on both ends. Thus we assume that we cannot add $e$ anywhere but in $A$. This implies that every face except $A$ is a triangle.

If $A$ is a triangular face, then either $\delta(N) \leq 0$ or $|E(N)| < 4$, both of which are contradictions. Thus $A$ is not a triangle. Construct a new $H'$ by adding $e$ as a chord in $A$ such that it creates a triangle using two consecutive edges on the boundary of $A$. This gives an $H'$ in which we cannot contract any edge and retain 3-connectivity, a contradiction. Thus (iii) holds. □

**Lemma 3.6.6.** *Let $H$ be a graph such that $M(H) = N \in \mathcal{N}_G$ and $\delta(N) = 1$. Then $H$ has at least two triangular faces and at least six vertices of degree three. Furthermore, $H$ has at least two vertices of degree three that are adjacent.*

*Proof.* Since $\delta(N) = 1$, we know $r(N) = r^*(N) + 1$. This implies $V_H = F_H + 1$ for a graph $H$ such that $M(H) = N$. Using Euler's formula, and the fact that the sum of the vertex degrees in $H$ is equal to twice the number of edges, we see that $2 = V_H - E_H + F_H = V_H - E_H + V_H - 1$, so $2E_H = 4V_H - 6$. Since $H$ is 3-connected, every vertex has degree at least three. The last equation implies that $H$ has at least six vertices of degree three.

Similarly, the sum of the number of boundary edges over every face is twice the number of edges. Thus, using Euler's formula, we see that $2 = V_H - E_H + F_H = F_H + 1 - E_H + F_H$, so $2E_H = 4E_H - 2$. Thus $H$ must have at least two triangular faces.

If $H$ has two triangular faces that share an edge, then $H^*$ also has two such faces, so $H$ has two vertex-triads with non-empty intersection. In other words, $H$ has two adjacent degree-three vertices. Assume not and let $T$ be a triangular face of $H$. Construct a graph $H'$ from $H$ by adding an edge $e$ such that $e$ creates a triangular face adjacent to $T$. Letting $M(H') = M$, we see that the matroid $M$ satisfies the obligatory rank constraints and must contain $N^*$ as a minor. This implies that $H'$ must contain $H^*$ as a minor. However, upon contracting an edge $f$ to obtain $H^*$ from $H'$, we must still have two triangular faces that share an edge in $H^*$. By duality, $H$ has two vertex-triads whose intersection is non-empty and so has two adjacent vertices of degree three. □

The next lemma gives us more information regarding the fans present in a matroid $N \in \mathcal{N}_G$.

**Lemma 3.6.7.** *Let $N$ be a matroid in $\mathcal{N}_G$. Then $N$ has a fan that contains at least five elements and has triads on both ends.*

*Proof.* Let $H$ be a graph such that $M(H) = N$. By Lemma 3.6.6, we know that $H$ contains at least six vertex-triads. We show first that

**3.6.7.1.** *$H$ has a fan of size at least four.*

Suppose that $H$ has no fans of size larger than three. Let $\{a, b, c\}$ be a vertex-triad. Construct a new graph $H'$ from $H$ by adding $e$ such that $(a, b, c, e)$ is a 4-fan. Then, since $N^*$ cannot contain a 4-fan, we must contract $a$ to obtain $H^*$. Note that adding $e$ and contracting $a$ in this way amounts to performing a $Y$-$\Delta$ exchange on $\{a, b, c\}$ in $H$. Similarly, performing a $Y$-$\Delta$ exchange on any of our vertex-triads must result in $H^*$.

Now consider two vertex-triads whose intersection is non-empty. The existence of these triads is guaranteed by Lemma 3.6.6. Performing a $Y$-$\Delta$ exchange on one of them results in a degree-four vertex in $H^*$. This implies that there is a 4-face in $H$. Construct a new graph $H'$ from $H$ by adding $e$ as a chord of this 4-face, creating two triangular faces. Both of these faces must be present in $H^*$, so $H^*$ has at least two more triangular faces than $H$. More specifically, when we perform a $Y$-$\Delta$ exchange on a vertex-triad, we must create two new triangular faces. This implies that each degree-three vertex of $H$ is on the boundary of a 4-face.

Now consider one of the triangular faces $T$ in $H$ guaranteed by Lemma 3.6.6. As $H$ has no fans of size greater than three, each vertex on the boundary of $T$ must have degree greater than or equal to four. Suppose that each vertex has degree greater than or equal to five. This implies that $H^*$ also has a triangular face in which each boundary vertex has degree greater than or equal to five. However, this means that $H$ has a triad such that each of the three faces for which it is on the boundary has size at least five, a contradiction. Thus we assume that at least one of the vertices $v$ on the boundary of $T$ has degree exactly four.

Construct a new graph $H'$ by adding $e$ to $H$ such that it forms a triangle with the two edges incident to $v$ that are not in $T$. Now each edge incident to $v$ is in a triangle, so we must have a degree-four vertex in $H^*$ such that each edge is in a triangular face. By duality, this implies that $H$ has a 4-face in which two non-adjacent vertices on the boundary are degree-three vertices.

Construct a new $H'$ by adding $e$ to this 4-face such that it is adjacent to neither of the degree-three vertices, as depicted in Figure 3.23. This gives us a structure such that it is impossible to destroy all the fans in $H'$ with a single contraction. This contradiction tells us that $H$ must contain a fan of size at least four, so 3.6.7.1 holds.

Let $\Phi = (a, b, c, d)$ be a fan of size four in $H$, where $\{a, b, c\}$ is a triangle and $\{b, c, d\}$ is a triad. Construct a new $H'$ by adding $e$ such that $\{c, d, e\}$ is a triangle. Then $H'$ contains a 5-fan with triangles on both ends and so $H^*$ does as well. The result holds by duality. $\square$

The focus in the remaining arguments will center on graphs $H$ such that $M(H) = N \in \mathcal{N}_G$. Recall that $\delta(N) = 1$. Now consider a largest fan $\Phi = (f_1, f_2, \ldots, f_k)$ with triads on both ends. By Lemma 3.6.5, such a fan has at least five elements. Consider constructing a graph

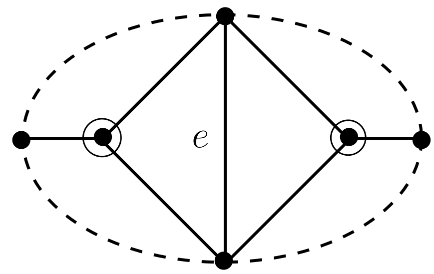



Figure 3.23. The substructure in $H'$ representing the extension that guarantees a fan in $H^*$.

$H'$ from $H$ by adding $e$ such that $\{e, f_1, f_2\}$ is a triangle. Then $\Phi \cup e$ is a $(k+1)$-fan in $H'$ and we must contract $f_k$ from $H'$ to obtain $H^*$. Note that this sequence of moves, which is depicted in Figure 3.24, has replaced the $k$-fan $\Phi$ that ends in triads in $H$ with a $k$-fan that ends in triangles, and has left the rest of $H$ unchanged. We will call this sequence of moves a *fan-exchange*. We refer to the vertices $x$, $y$, and $z$ and the faces $F_1$, $F_2$, and $F_3$ in Figure 3.24 as *perimeter-vertices* and *perimeter-faces* of the fan. Note that performing a fan-exchange on a maximum-sized fan that ends in triads in $H$ must always give us $H^*$. It is worth drawing attention to the fact that a fan-exchange is a local set of moves which leads to global duality. This idea will be used frequently throughout the rest of the chapter.

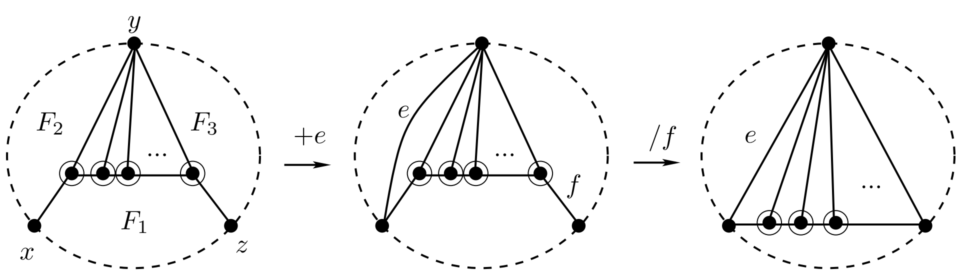



Figure 3.24. Performing a fan-exchange.

We define $t_H$ to be the difference between the number of vertex-triads in $H$ and the number of triangular faces in $H$. Evidently $t_{H^*} = -t_H$. The next two lemmas provides more information regarding $t_H$.

**Lemma 3.6.8.** *Let $H$ be a graph such that $M(H) = N \in \mathcal{N}_G$ and $\delta(N) = 1$. Then $t_H \geq 1$.*

*Proof.* Assume that $t_H \leq 0$, so $H$ has at least as many triangular faces as vertex-triads and $t_{H^*} \geq 0$. Construct a graph $H'$ by adding an edge $e$ to $H$ such that $e$ is a chord of some face of $H$, and $e$ creates a new triangular face. This construction is guaranteed by Lemma 3.6.3. Then $t_{H'} < 0$. However, we can neither destroy triangles nor create triads when contracting to obtain $H^*$. Hence $t_{H^*} < 0$, which is a contradiction. Thus $t_H \geq 1$. □

**Lemma 3.6.9.** *Let $H$ be a graph such that $M(H) = N \in \mathcal{N}_G$ and $\delta(N) = 1$. If $t_H = k$, then, in obtaining $H^*$ from $H$ by adding $\delta(N)$ edges and contracting an independent set of size $\delta(N)$, we must destroy exactly $k$ vertex-triads and create exactly $k$ triangular faces.*

*Proof.* Since $t_H = k$, let $m$ and $m - k$, respectively, denote the number of vertex-triads and triangular faces of $H$, for $m > k$. Then $H^*$ has $m - k$ vertex triads and $m$ triangular faces. By Lemma 3.6.3, we know we can obtain $H^*$ from $H$ by adding an edge $e$ to $H$ to obtain a new graph $H'$, and then contracting an edge $f$ from $H'$. Thus, by adding $e$ and contracting $f$, we must destroy $m - (m - k) = k$ vertex-triads, and we must create $m - (m - k) = k$ triangular faces. □

We will use the following explicit consequences of Lemma 3.6.9.

(a) If $t_H = 2$, then, to obtain $H^*$ from $H$, we must destroy exactly two vertex-triads and create exactly two triangular faces.

(b) If $t_H = 3$, then, to obtain $H^*$ from $H$, we must destroy exactly three vertex-triads and create exactly three triangular faces.

(c) If $t_H = 4$, then, to obtain $H^*$ from $H$, we must destroy exactly four vertex-triads and create exactly four triangular faces.

The next lemma states an important relationship between maximum-sized fans and $t_H$.

**Lemma 3.6.10.** *Let $H$ be a graph such that $M(H) = N \in \mathcal{N}_G$ and $\delta(N) = 1$. Let $t_H = k$ and let $\Phi$ be a maximum-sized fan of $H$ that ends in triads. Exactly $k-1$ of the perimeter-vertices of $\Phi$ have degree-three and exactly $k-1$ of the perimeter-faces of $\Phi$ are 4-faces.*

*Proof.* Recall that, since $\delta(N) = 1$, performing a fan-exchange on $\Phi$ in $H$ gives us $H^*$. By Lemma 3.6.9, we know that performing the fan-exchange must have created exactly $k$ triangular faces and destroyed exactly $k$ vertex-triads. First consider the triangular faces. Adding the edge $e$ in Figure 3.24 creates a triangular face and decreases the size of the face $F_2$ by one. Contracting the edge $f$ decreases the size of the faces $F_1$ and $F_3$ by one. Thus, the only faces that are affected by the fan-exchange are the new triangular face obtained by adding $e$ and the perimeter-faces of $\Phi$ in $H$. Since the sizes of each of $F_1$, $F_2$, and $F_3$ are lowered by one, and since we must create $k$ triangular faces, this implies that $k-1$ of the perimeter faces have size four. A similar argument holds for the vertex-triads. □

The next three lemmas restrict the size of $t_H$.

**Lemma 3.6.11.** *For a graph $H$ such that $M(H) = N \in \mathcal{N}_G$ and $\delta(N) = 1$,*

*(i)* $2 \leq t_H \leq 4$*; and*

*(ii) if* $t_H = 4$*, then* $H \in \{P, K_5 \backslash e\}$.

*Proof.* We know from Lemma 3.6.8 that $t_H \geq 1$. By Lemma 3.6.5, we know that $H$ has a largest fan $\Phi$ of size $k$ such that $k$ is odd and $\Phi$ has triads on both ends. Construct a new graph $H'$ by adding an edge $e$ to $H$ such that $e$ is incident to a degree-three vertex $v$ on one end of $\Phi$, the edges of $\Phi$ still form a fan in $H'$, and $e$ creates a triangular face such that every edge incident to $v$ is on the boundary of a triangular face in $H'$. Then $H^*$ has a degree-four vertex, which implies $H$ has a 4-face.

Construct a new graph $H'$ by adding $e$ as a chord in a 4-face of $H$. Then $H'$, and hence $H^*$, has at least two more triangular faces than $H$. Thus, by Lemma 3.6.9, we know $t_H \geq 2$.

This proves the first part of the inequality in (i). To see that the other part holds, note that $\Phi$ has three perimeter-faces and three perimeter-vertices. Since performing a fan-exchange on $\Phi$ in $H$ yields $H^*$, by Lemmas 3.6.9 and 3.6.10, the second part of the inequality holds. Hence (i) holds.

Now assume $t_H = 4$. Consider performing a fan-exchange on $\Phi$ to obtain $H^*$. Since $t_H = 4$, by Lemma 3.6.10, we know that all of the perimeter-vertices of $\Phi$ are distinct and the edge sets incident to them are vertex-triads in $H$. We also know that all of the perimeter-faces of $\Phi$ are 4-faces in $H$. These facts imply that $\Phi$ is a 5-fan and that $H = P$. By duality, (ii) holds. ☐

**Lemma 3.6.12.** *Let $H$ be a graph such that $M(H) = N \in \mathcal{N}_G$. Then $t_H \neq 3$.*

*Proof.* Assume $t_H = 3$. By adding an edge $e$ and contracting an edge $f$ to obtain $H^*$ from $H$, Lemma 3.6.9 tells us we must create exactly three new triangular faces and destroy exactly three vertex-triads. If $H$ has a face of size six or more, then we can construct a graph $H'$ by adding $e$ such that it creates no new triangular faces. However, we must be able to contract a single edge to obtain $H^*$, which has three more triangular faces than $H$, a contradiction. Thus all faces in $H$ have size less than or equal to five and, by duality, all vertices in $H^*$ must have degree less than or equal to five. We now show that

**3.6.12.1.** *the largest fan in $H$ has size five.*

By Lemma 3.6.5, we know that $H$ has a maximum-sized fan $\Phi$ with triads on both ends. Consider performing a fan-exchange on $\Phi$ to obtain $H^*$. By Lemma 3.6.10, exactly two of the perimeter-vertices of $\Phi$ have degree three and exactly two of the perimeter-faces of $\Phi$ are 4-faces. Note that the perimeter-face $F_\alpha$ of $\Phi$ that is not a 4-face must be a 5-face. Moreover, since $H^*$ cannot have any vertices of degree greater than five and the fan-exchange increases the degree of each perimeter-vertex, we know that the perimeter-vertex $v$ of $\Phi$ that does not have degree three must have degree four.

As the perimeter-vertices can have degree at most four and the perimeter-faces of $\Phi$ must be distinct, we know that $\Phi$ has size at most seven. Suppose $\Phi$ is a 7-fan. Then $v = y$ as labelled in Figure 3.24. Since all the perimeter-faces must have size four or five, we know that $F_\alpha = F_1$. Thus the two perimeter-vertices of degree three are adjacent. As the other two perimeter-faces have size four, we know that each of the perimeter-vertices is adjacent to a common vertex, as depicted in Figure 3.25. However, this implies that either $H$ is not 3-connected, or all of $H$ is pictured in Figure 3.25. The former is a contradiction, so we may assume the latter holds. Then construct a graph $H'$ by adding an edge $e$ to $H$ as shown in Figure 3.26. Every edge of $H'$ is in a triangle except $f$. This implies that we must contract $f$ to obtain $H^*$. However, this addition of $e$ and contraction of $f$ destroys four vertex-triads, a contradiction. Thus, using Lemmas 3.6.5 and 3.6.7, we deduce the largest fan in $H$ has size 5, so 3.6.12.1 holds. Next we show that

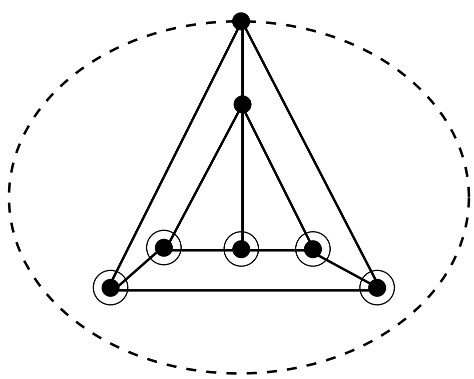


Figure 3.25. The structure surrounding a 7-fan if $t_H = 3$.

**3.6.12.2.** *$H$ cannot have a triangular face $T$ such that every vertex on the boundary of $T$ has degree three.*

Suppose $H$ does have such a formation and consider it as labeled in Figure 3.27. Note that all of $(a, b, c, d, j)$, $(a, c, b, d, k)$, and $(j, c, d, b, k)$ are 5-fans in $H$. As such, since each maximum-sized fan has exactly two perimeter-vertices of degree three, exactly two vertices

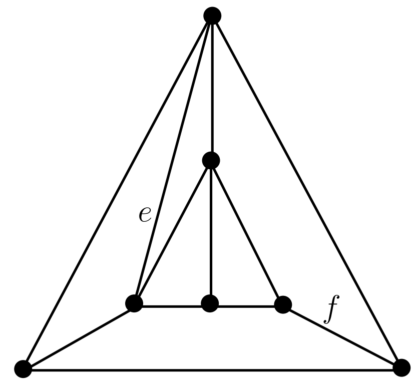



Figure 3.26. The extension that guarantees that $H$ does not contain a 7-fan.

are triads from each of the sets $\{1, 2, 6\}$, $\{1, 3, 5\}$, and $\{2, 3, 4\}$, which is a contradiction. Thus 3.6.12.2 holds. Note that 3.6.12.2 implies that

**3.6.12.3.** *$H^*$ has no $K_4$-subgraph.*

By 3.6.12.2, we may assume that, in each 5-fan with triad ends in $H$, the perimeter-vertices to which the end edges of the fan are incident have degree three. Let $F_0$ be the face of $H$ that contains the rim elements of $\Phi$. Assume the two perimeter-vertices of degree three of $\Phi$ are adjacent to each other. Then $F_0$ is a 4-face. Performing a fan-exchange on $\Phi$ gives us a $K_4$-subgraph in $H^*$, a contradiction. Therefore, the two perimeter-vertices of degree three of $\Phi$ are not adjacent, and $F_0$ is a 5-face. Thus

**3.6.12.4.** *for each 5-fan that ends in triads in $H$, the surrounding structure is as in Figure 3.28.*

We now show that

**3.6.12.5.** *$H$ has only one fan of size five.*

Suppose $H$ has a 5-fan that ends in triangles. It is not hard to see that we can construct a new graph $H'$ from $H$ by adding an edge $e$ such that it forms a $K_4$-subgraph in $H'$. But this implies $H^*$ has a $K_4$-subgraph, a contradiction. Thus every 5-fan in $H$ ends in triads. Assume we have at least two such fans. Obtain $H^*$ from $H$ by performing a fan-exchange

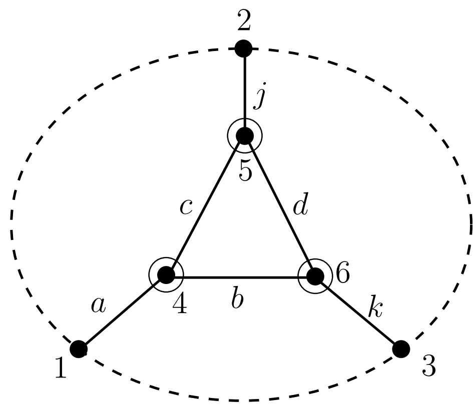



Figure 3.27. A forbidden sub-structure in $H$ when $t_H = 3$.

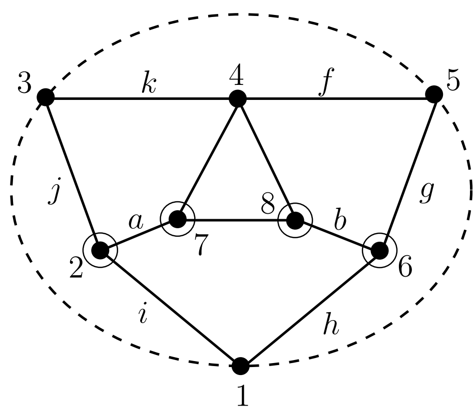



Figure 3.28. The structure surrounding every 5-fan with triads at both ends when $t_H = 3$.

on one of the 5-fans. Then either there is a 5-fan that ends in triads that is still intact in $H^*$, or $H$ has a 5-fan $\Phi_1$ such that a vertex-triad at one end of $\Phi_1$ corresponds to one of the perimeter-vertices of degree three of $\Phi$. The former does not hold, otherwise $H^*$ has a 5-fan that ends in triads and so $H$ has a 5-fan that ends in triangles, a contradiction. Thus, the latter holds and, by symmetry, we may assume that the vertex-triad at one end of $\Phi_1$ corresponds to the edges incident to 2 from Figure 3.28. Then the vertices 1 and 3 must be adjacent, and one of 1 and 3 must have degree three. However, this implies that either $H$ is not 3-connected, or every vertex of $H$ is pictured in Figure 3.28. The former is a contradiction, so we assume the latter holds. Then, as $H$ is 3-connected and one of 1 and 3 has degree three, exactly one of the edges 35 and 15 must be present in $H$. Construct a graph $H'$ by adding whichever of 35 and 15 is not already present, as pictured in Figure 3.29.

Then every edge of $H'$ is in a triangle except $a$ and $b$. However, contracting either of $a$ or $b$ will not give us a vertex with degree 5, which we know is present in $N^*$, a contradiction. Therefore 3.6.12.5 holds, and structure surrounding the unique 5-fan is as in Figure 3.28.

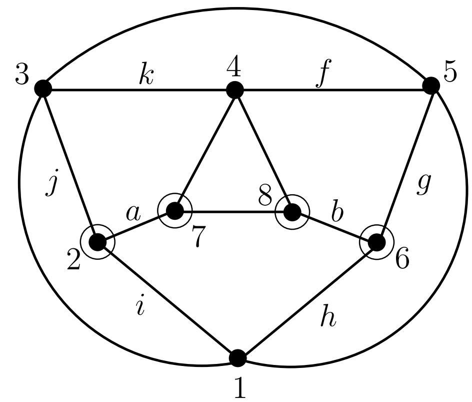



Figure 3.29. A graph $H'$.

To complete the proof of the lemma, we create a new graph $H'$ by adding an edge $e$ incident to 1 and 8 in Figure 3.28 and show that this leads to a contradiction. Clearly adding $e$ in this way gives us a graph $H'$ which must contain $H^*$ as a minor. First we show that

**3.6.12.6.** *we must contract one of the edges shown in Figure 3.28 to obtain $H^*$ from $H'$.*

Assume that we do not contract any of the edges shown in Figure 3.28 to obtain $H^*$. Then, by dualizing the structure we get from adding the edge 18 to the portion of the graph $H$ shown in Figure 3.28, we see that $H$ contains the sub-structure depicted in Figure 3.30. Note that in both Figures 3.28 and 3.30, there are no other edges of $H$ that meet the vertices in the figures except for the three degree-two vertices in each figure.

Create a new graph $H_1$ by adding an edge $x$ to $H$ such that it is incident to both $v$ and $w$ in Figure 3.30. This creates two new 5-fans in $H_1$. Moreover, since every edge of the fans is in a triangle, both must be present in $H^*$, a contradiction. Therefore 3.6.12.6 holds.

Note that the only possible edges we can contract from $H'$ are $a$, $f$, $g$, $i$, $j$, and $k$, since the rest are in triangles. If we contract any of $f$, $i$, $j$, or $k$, we obtain, in $H$, the substructure in Figure 3.30 with the edge $f'$, $i'$, $j'$, or $k'$ deleted, respectively. In each case, we can arrive

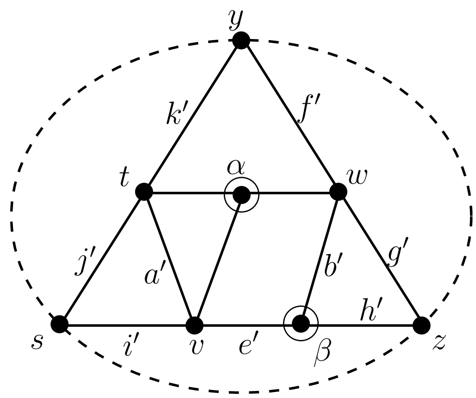



Figure 3.30. A structure in $H$ if we do not contract edges surrounding the 5-fan.

at the contradiction above with two 5-fans in $H^*$ by creating a graph via adding an edge $e$ between the vertices that correspond to those labeled by $v$ and $w$ in Figure 3.30. Therefore, it remains to check the situations that arise from contracting each of $a$ and $g$. We now show that

**3.6.12.7.** *contracting the edge $g$ provides a contradiction.*

If we contract $g$, then $H$ has the substructure depicted in Figure 3.30 with the edge $g'$ deleted. Note that the degree-three vertices $\alpha$ and $\beta$ cannot be contained in the 5-fan $\Phi$ of $H$, since the faces for which $\alpha$ is on the boundary have sizes 3, 4, and 4, while all the faces for which $\beta$ is on the boundary have sizes of at least 4. Construct a graph $H_g$ from $H$ by adding $e$ incident to $\alpha$ and $\beta$. This extension destroys two vertex-triads. The fan $\Phi$ is present in $H_g$, but must not be present in $H^*$. Therefore we must contract either the edge labeled $a$ or $b$ from Figure 3.28. However, each contraction destroys an additional two vertex-triads, contradicting the fact that $t_H = 3$. Thus 3.6.12.7 holds.

To complete the proof of the lemma we show that

**3.6.12.8.** *contracting the edge $a$ provides a contradiction.*

If we contract $a$, then $H$ has the substructure depicted in Figure 3.30 with the edge $a'$ deleted. Construct a new graph $H_a$ from $H$ by adding an edge $e$ incident to $\alpha$ and $\beta$. This extension does not create a 5-fan ending in triangles in $H_a$. Note that it is impossible for $\alpha$ to be one of the vertices of degree-three contained in $\Phi$. If the edge set incident to $\beta$ is not a vertex-triad of $\Phi$, then we must destroy $\Phi$ when contracting to obtain $H^*$. This implies we must contract either $a$ or $b$ from Figure 3.28. However, as above, this sequence of extension and contraction destroys four vertex-triads, a contradiction.

Thus we assume that the edge set incident to $\beta$ is one of the vertex-triads of $\Phi$. Then, combining the structures and labels from Figures 3.28 and 3.30, we see that $H$ contains the substructure depicted in Figure 3.31. Clearly, $H$ is not 3-connected unless every vertex of $H$ is depicted in Figure 3.31. The only possible edge missing from Figure 3.31 is the edge incident to $(5, y)$ and $(1, s)$. This edge must be present in $H$, otherwise $\delta(N) \neq 1$, a contradiction. Therefore, $H$ must be the graph pictured in Figure 3.32.

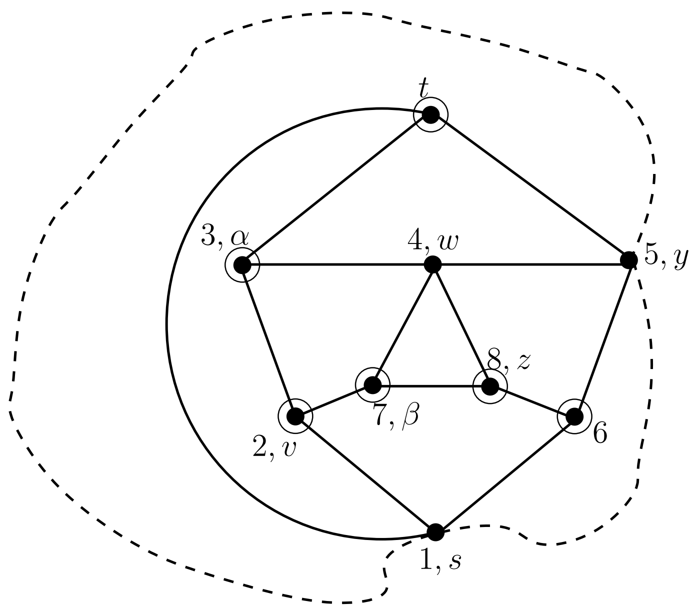



Figure 3.31. The structure in $H$ if we contract $a$.

To see that $M(H)$ is not in $\mathcal{N}_G$, construct a graph $H'$ from $H$ by adding $e$ as pictured in Figure 3.33. Note that every unlabeled edge in Figure 3.33 is contained in a triangle of $H'$, and so we cannot contract any of the unlabeled edges to obtain an $H^*$-minor. Since adding $e$ destroyed two vertex-triads, the edge we contract in $H'$ must be adjacent to exactly one

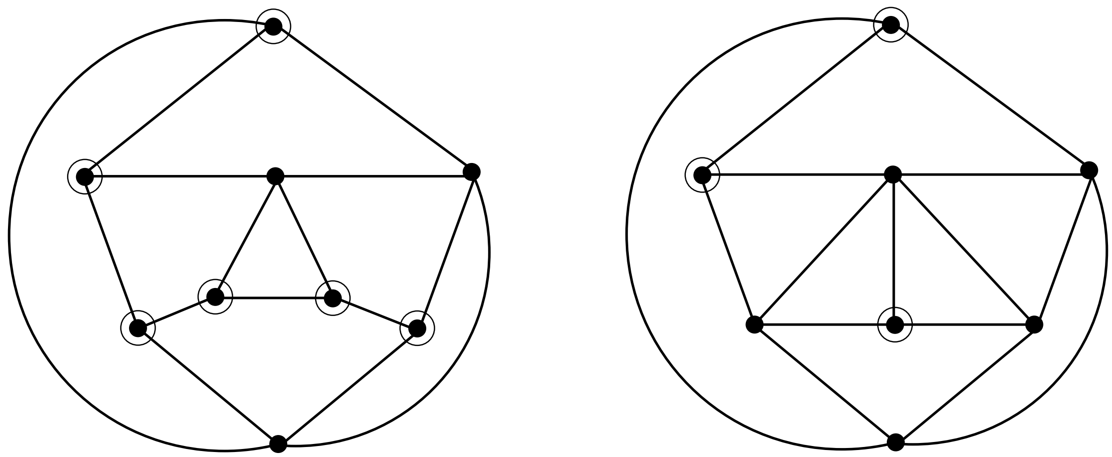


Figure 3.32. A possible graph $H$ and its dual, such that $M(H)$ is in $\mathcal{N}_G$.

degree-three vertex. This implies we cannot contract $x$, $z$, or $s$. Thus it remains to check the contraction of $y$, $w$, and $t$.

If we contract $t$, then the degree-five vertex in the resulting graph is adjacent to a single degree-three vertex. This implies that $H'/t$ has no $H^*$-minor since, in $H^*$, the degree-five vertex is adjacent to two vertices of degree three. If we contract either of $y$ or $w$, then the resulting graph does not have two adjacent degree-three vertices. This implies that we do not have an $H^*$-minor in either case. Thus 3.6.12.8 holds, and the result holds.

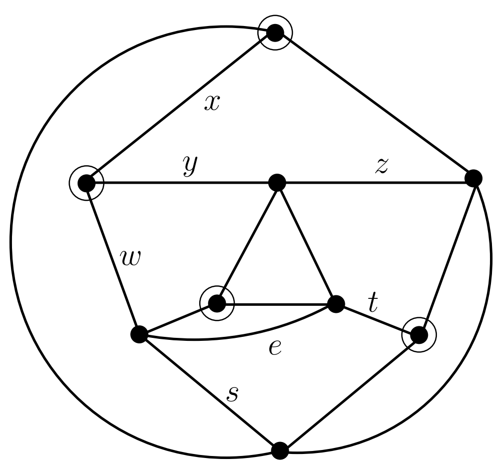



Figure 3.33. An extension that shows the graph in Figure 3.32 is not in $\mathcal{N}_G$.

□

**Lemma 3.6.13.** *Let $H$ be a graph such that $M(H) = N \in \mathcal{N}_G$. Then $t_H \neq 2$.*

*Proof.* Assume $t_H = 2$. By adding an edge $e$ and contracting an edge $f$ to obtain $H^*$ from $H$, Lemma 3.6.9 tells us we must create exactly two new triangular faces and destroy exactly two vertex-triads. We begin by showing that

**3.6.13.1.** *there is a fan of size greater than five in $H$.*

By Lemma 3.6.5, we know that $H$ has a 5-fan with triads on both ends. Let $\Phi$ denote such a fan, and assume that $H$ has no larger fans. Performing a fan-exchange on $\Phi$ gives us $H^*$, which implies, by Lemma 3.6.10, that exactly one perimeter-vertex of $\Phi$ is a triad and exactly one perimeter-face of $\Phi$ is a 4-face. Assume in every 5-fan with triad ends that the perimeter-vertex of degree three is the vertex that lies on the boundary of the triangle in $\Phi$. Then $H$ contains the configuration in Figure 3.34 and, without loss of generality, $F_1$ is a 4-face.

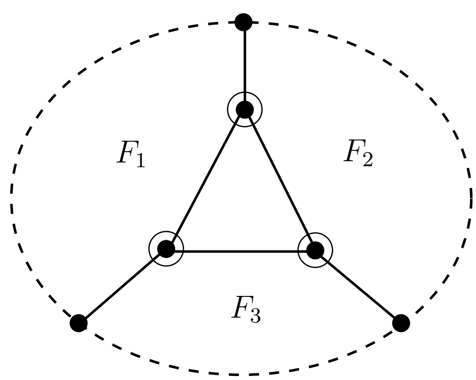



Figure 3.34. A possible configuration surrounding a 5-fan.

Construct a graph $H'$ by adding $e$ to $F_1$, as in Figure 3.35. Then $H'$ has a 6-fan and we must contract $f$ to obtain $H^*$. This gives a 5-fan $\Phi_T$ in $H^*$ that ends in triangles, where $\alpha$ is the degree-three vertex of $\Phi_T$. Using duality, the fan $\Phi_T$ gives rise to a 5-fan $\Phi'$ in $H$ such that the perimeter-vertex of degree three of $\Phi'$ is not on the boundary of the triangle in $\Phi'$, a contradiction. Therefore, there is a 5-fan such that the perimeter-vertex of degree three is not on the boundary of the triangular face in the fan. Assume $\Phi$ is such a fan. We now prove the following.

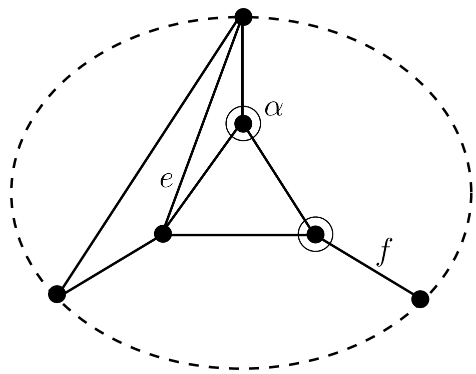



Figure 3.35. Adding an edge to a 5-fan.

**3.6.13.2.** *The perimeter-4-face of* $\Phi$ *is the face shared by the two vertices that correspond to vertex-triads of* $\Phi$.

Suppose not. Add $e$ to the perimeter-4-face such that it is incident to a degree-three vertex of $\Phi$. This extension destroys $\Phi$ and creates two new triangular faces. This implies, by Lemma 3.6.9, that, when we contract to obtain $H^*$, we cannot create any new triangular faces. Adding $e$ in this way either creates a new 5-fan with triangles on both ends or it does not. However, the latter is a contradiction since the number of 5-fans in $H$ and $H^*$ is the same, adding $e$ destroys $\Phi$, and we cannot create a 5-fan with triangle ends via contraction. The last statement holds because we created two new triangular faces by adding $e$ and, by Lemma 3.6.10, since $t_H = 2$, we cannot create further new triangular faces. Thus we may assume that adding $e$ creates a new 5-fan.

Then $H$ contains the configuration depicted in Figure 3.36. Create a new graph $H'$ by adding $e$ to $H$ such that $e$ is incident with $x$ and $y$, as pictured in Figure 3.37. Then $H'$ contains a 6-fan and, to obtain $H^*$, we must contract the edge labeled $f$ in Figure 3.37. This structure implies, by duality, that we have a 5-fan $\Phi_1$ in $H$ such that the perimeter-4-face is not the face shared by the two vertices that correspond to vertex-triads of $\Phi_1$, and three vertices on the boundary of the perimeter-4-face have degree three. Thus $H$ contains the substructure shown in Figure 3.38.

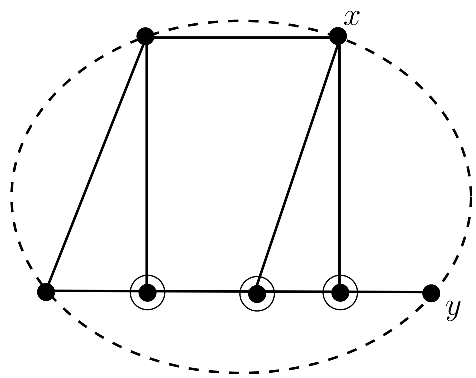



Figure 3.36. A configuration surrounding $\Phi$ in $H$.

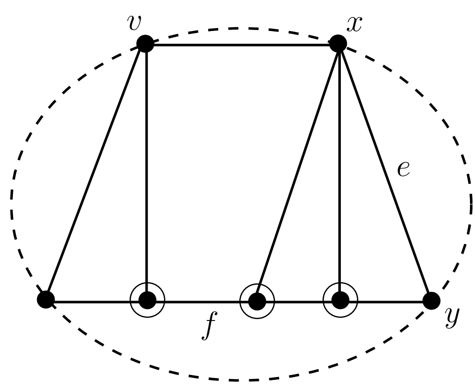



Figure 3.37. Adding $e$ to create a 6-fan.

Create a new graph $H'$ by adding the edge $e = vw$. Adding $e$ in this way destroys two 5-fans, destroys two vertex-triads, and creates two new triangular faces. By Lemma 3.6.10, we cannot create any more triangular faces via contraction. Furthermore, adding $e$ in this way clearly does not create two new 5-fans. Thus $H'$ has no $H^*$-minor, a contradiction. Therefore 3.6.13.2 holds.

Consider performing a fan-exchange on $\Phi$. This gives a $K_4$-subgraph in $H^*$ that is not present in $H$, which implies that $H$ has a triangular face $T$ such that all vertices on the boundary of $T$ are vertices of degree three. Moreover, since there are multiple 5-fans within the structure surrounding $T$, and each 5-fan has a single perimeter-4-face, we know that

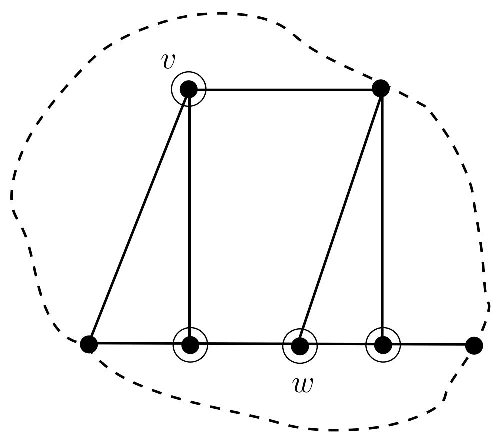



Figure 3.38. A configuration surrounding $\Phi_1$ in $H$.

exactly one face that shares an edge with $T$ is a 4-face. Consider performing a fan-exchange as depicted in Figure 3.39 to obtain $H^*$. Because each 5-fan has exactly one perimeter-vertex of degree three, we know that none of the vertices labeled $x$, $y$, and $z$ is a degree-three vertex. Since each 5-fan also has exactly one perimeter-4-face, we know that $y$ and $z$ are not adjacent and that $x$ and $z$ are not adjacent. However, $H^*$ has a $K_4$-subgraph that is not present in $H$, so we must have created a $K_4$-subgraph upon performing the fan-exchange. This implies that at least one of the faces labeled $F_j$ and $F_k$ in Figure 3.39 must be part of the $K_4$-subgraph, since they are the two new triangular faces. However, we clearly see that neither of these faces is part of a $K_4$-subgraph in $H^*$, a contradiction. Therefore, $H$ has a $k$-fan such that $k > 5$, so 3.6.13.1 holds.

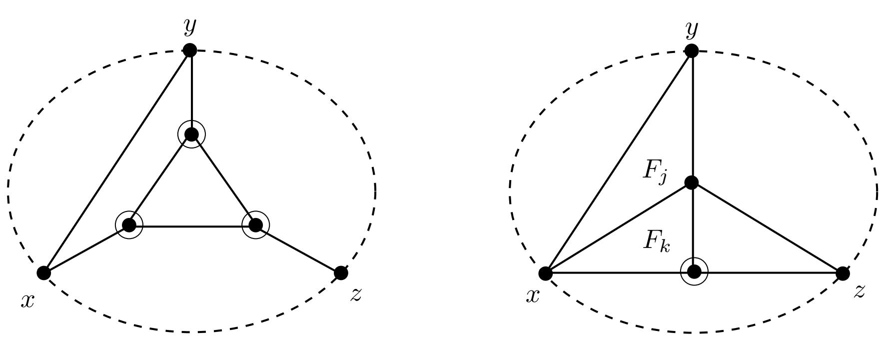



Figure 3.39. Performing a fan-exchange.

By Lemma 3.6.5, we know that $H$ has a $k$-fan that ends in triads such that $k$ is odd and maximal. Let $\Phi$ be such a fan. By 3.6.13.1, we know $k \geq 7$. Since $k \geq 7$, we know that the perimeter-4-face of $\Phi$ cannot be the face for which every degree-three vertex of $\Phi$ is on the boundary. Create a new graph $H'$ by adding an edge $e$ in the perimeter-4-face of $\Phi$ such that it is incident to a vertex that corresponds to a vertex-triad of $\Phi$. Adding $e$ in this way destroys the maximum-sized fan and creates both new triangular faces. This is a contradiction unless $e$ also creates a new $k$-fan. As in the case where $k = 5$, add $e$ as pictured in Figure 3.40. This creates a $(k+1)$-fan in $H'$ so we must contract the edge labeled $f$. This implies we have the structure depicted in Figure 3.41 in $H^*$. However, the $k$-fan that ends in the two edges incident to $w$ on the right-hand side of Figure 3.41 corresponds with a $k$-fan $\Phi_m$ of $H$ that ends in triads such that the perimeter-4-face of $\Phi_m$ has three vertices of degree three on its boundary.

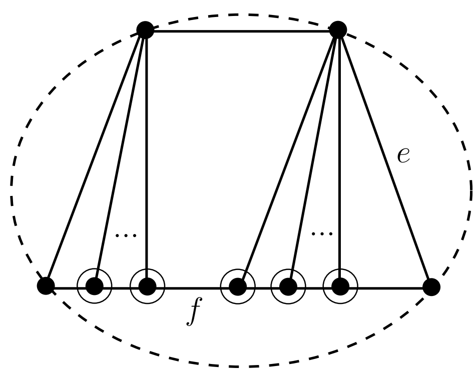



Figure 3.40. Adding $e$ to create a $(k+1)$-fan.

As in the case where $k = 5$, construct a new graph $H'$ by adding $e$ to the perimeter-4-face of $\Phi_m$ such that $e$ is incident to a degree-three vertex of $\Phi_m$. Adding $e$ in this way destroys the maximum-sized fan, creates both new triangular faces, and destroys both vertex-triads. Moreover, we know that adding $e$ in this way does not create a new $k$-fan in $H'$. Since we cannot create new triangular faces when contracting to obtain $H^*$, we know that $H^*$ has fewer $k$-fans that $H$, a contradiction.

□

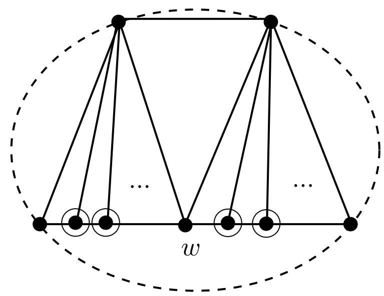



Figure 3.41. Two large fans that share a perimeter-vertex.

*Proof of Theorem 3.1.3.* The result follows immediately by combining Lemmas 3.6.11, 3.6.12, and 3.6.13. $\square$

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# Appendix A: Seven-element Matroids

In this appendix, we show explicitly that there are no seven-element matroids in $\mathcal{N}_{3,q}$ when $q \geq 4$. Note that, for every matrix in this section, we suppress the leading identity matrix. The elements of $GF(4)$ are $\{0, 1, \omega, \omega + 1\}$.

Let $N$ be a matroid in $\mathcal{N}_{3,q}$, for some $q \geq 4$, such that $r(N) > r^*(N)$. Consider $N^*$ and note that, since $|E(N)| = 7$ and by Lemma 3.5.3, we know $r(N^*) = 3$. By Lemma 3.5.5, $N^*$ cannot contain any 4-point lines. We begin by showing that, for $q \geq 4$, none of the four matroids from Figure A.1 is in $\mathcal{N}_{3,q}$. This will complete the proof of Lemma 3.5.6.

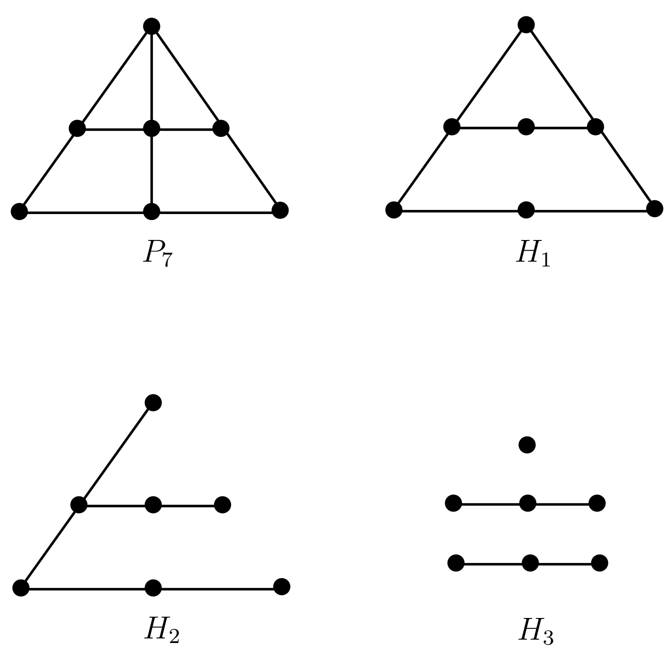



Figure A.1. All possible matroids in $\mathcal{N}_{3,q}$ that contain two disjoint triangles.

Note that $H_2$ has a $P_6$-minor and is thus not in $\mathcal{N}_{3,4}$. Also, for $q \leq 5$, the matroid $H_3$ is not $GF(q)$-representable [5, Lemma 3.1]. Therefore $H_3$ is in neither $\mathcal{N}_{3,4}$ nor $\mathcal{N}_{3,5}$.

To see that none of these matroids in is $\mathcal{N}_{3,q}$ for $q \geq 4$, consider the matroids in Figure A.2. For $i$ in $\{1, 2, 3\}$, each $J_i$ satisfies the obligatory rank constraints and has $H_i^*$, but not $H_i$, as a minor. Similarly, $J_P$ contains $P_7^*$, but not $P_7$, as a minor. To determine the representability of each matroid in Figure A.2, consider the matroids in Figure A.3 obtained from the matroids in Figure A.2 via a $Y$-$\Delta$ exchange on the triad represented by the three elements that are furthest right in each representation. Though not directly stated in their paper, Akkari and Oxley [1, pp. 381-382] showed that performing a $Y$-$\Delta$ exchange on a matroid $M$ produces a matroid $M'$ that is representable over precisely the same fields as $M$. Thus, if a matroid in Figure A.3 is representable over a field $\mathbb{F}$, then its corresponding matroid in Figure A.2 is also $\mathbb{F}$-representable.

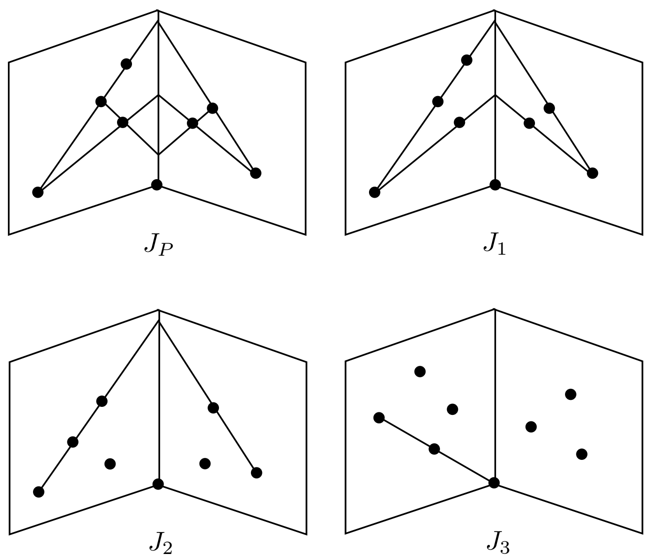
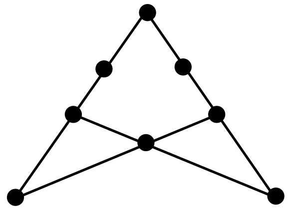


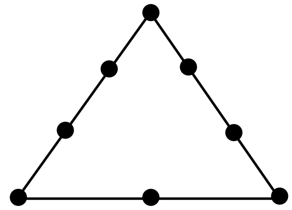

Figure A.2. Examples showing that none of the matroids in Figure A.1 are in $\mathcal{N}_{3,q}$.

It is not hard to check that for $q \geq 7$, each of $L_P$, $L_1$, $L_2$, and $L_3$ is $GF(q)$-representable. Therefore, to finish the proof of Lemma 3.5.6, we need to verify that $P_7$ and $H_1$ are not in $\mathcal{N}_{3,4}$, and that none of $P_7$, $H_1$, and $H_2$ is in $\mathcal{N}_{3,5}$. Consider the matrices in Figure A.4 viewed over $GF(4)$.

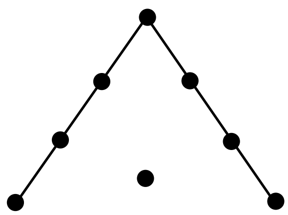
The matroids $M[U]$ and $M[V]$ are 3-connected, satisfy the obligatory rank constraints, and contain $P_7^*$ and $H_1^*$ but not $P_7$ and $H_1$, respectively. Thus there are no matroids with two disjoint triangles in $\mathcal{N}_{3,4}$. Now consider the matrices in Figure A.5 viewed over $GF(5)$.



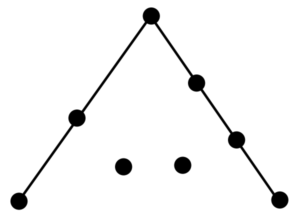
$L_P$



$L_1$



$L_2$



$L_3$

Figure A.3. Matroids obtained from each matroid in Figure A.2 via a $Y - \Delta$ exchange.

$$U = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & \omega+1 \\ 1 & \omega & 0 & \omega+1 \\ 1 & 0 & 1 & \omega+1 \\ 0 & 1 & \omega & 1 \end{bmatrix} \end{array} \qquad V = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & \omega \\ 1 & \omega & 0 & 1 \\ 1 & 0 & \omega & 1 \\ 0 & 1 & \omega & 1 \end{bmatrix} \end{array}$$

Figure A.4. Matrices over $GF(4)$.

The matroids $M[Q]$, $M[R]$, and $M[S]$ are 3-connected matroids that satisfy the obligatory rank constraints and contain $P_7^*$, $H_1^*$, and $H_2^*$, but not $P_7$, $H_1$, and $H_2$, respectively. Thus Lemma 3.5.6 holds for all $q$.

We now know, by Lemma 3.5.6 and Corollary 3.5.7, that $N^*$ does not have two disjoint triangles and does have a pair of intersecting triangles. Since $N^*$ has no 4-point lines, there are a total of eight such 3-connected seven-element matroids and these are depicted in Figure A.6.

Note that the matroids $N_3$, $N_4$, and $N_6$ all contain $U_{2,6}$ as a minor, and the matroid $N_2$ has a $P_6$-minor, so none are in $\mathcal{N}_{3,4}$.

$$
Q = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 1 & 0 & 2 \\ 1 & 0 & 3 & 2 \\ 0 & 1 & 3 & 2 \end{bmatrix} \end{array} \qquad R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & 3 & 0 & 3 \\ 1 & 0 & 3 & 3 \\ 0 & 1 & 3 & 4 \end{bmatrix} \end{array}
$$

$$
S = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 0 & 1 \\ 1 & 2 & 3 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} \end{array}
$$

Figure A.5. Matrices over $GF(5)$.

To see that none of these seven-element matroids is in $\mathcal{N}_{3,q}$ for $q \geq 4$, consider the matroids in Figure A.7. For $i$ in $\{1, 2, \ldots, 6\}$, each $M_i$ in Figure A.7 satisfies the obligatory rank constraints and has $N_i^*$, but not $N_i$, as a minor. Similarly, $M_F$ and $M_{F^-}$ have $F_7^*$ and $(F_7^-)^*$, but not $F_7$ and $F_7^-$, as minors, respectively. To determine the representability of each matroid in Figure A.7, consider the matroids in Figure A.8 obtained from the matroids in Figure A.7 via a $Y$-$\Delta$ exchange on the triad represented by the three elements that are furthest right in each representation.

It easy to see, by considering $Y_7$ as an extension of the Fano plane, that $Y_7$ is representable over $GF(q)$ if and only if the field has characteristic two and $q \neq 2$. Similarly, $Y_{7^-}$ is $GF(q)$-representable if and only $GF(q)$ has characteristic not equal to two and $q \geq 5$. Thus, neither $F_7$ nor $F_7^-$ is in $\mathcal{N}_{3,q}$ for $q \geq 4$. It is not hard to check that, for each $q \geq 7$, each of $Y_2$, $Y_3$, $Y_4$, $Y_5$, and $Y_6$ is $GF(q)$-representable. It is also not hard to check that $Y_1$ is $GF(q)$-representable for all $q \geq 5$. Thus, to complete our argument, we need to verify that $N_1$ and $N_5$ are not in $\mathcal{N}_{3,4}$, and that $N_1$, $N_2$, $N_3$, $N_4$, $N_5$, and $N_6$ are not in $\mathcal{N}_{3,5}$. Consider the matrices $A$ and $B$ in Figure A.9 viewed over $GF(4)$.

The matroids $M[A]$ and $M[B]$ are 3-connected, satisfy the obligatory rank constraints, and contain $N_1^*$ and $N_5^*$ but not $N_1$ and $N_5$, respectively. Thus there are no seven-element

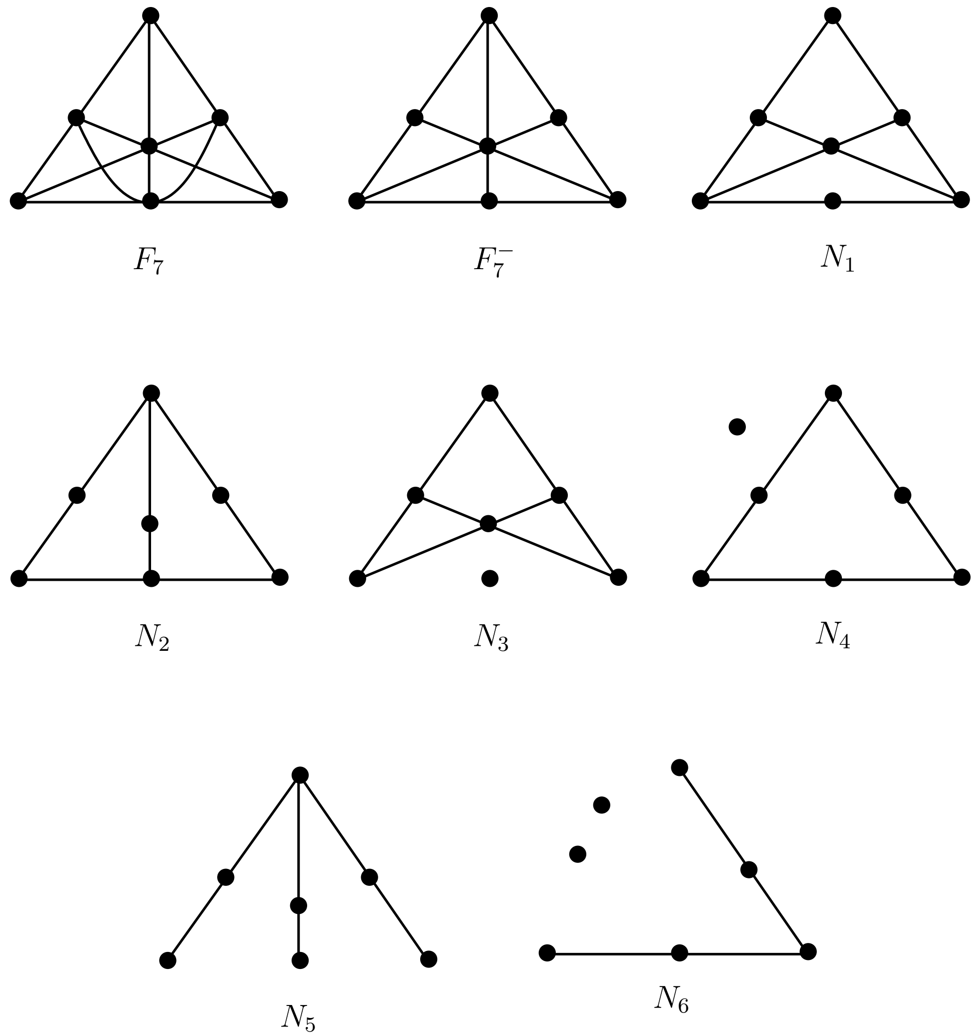



Figure A.6. All possible 3-connected, rank-3, seven-element matroids in $\mathcal{N}_{3,q}$ for $q \geq 4$.

matroids in $\mathcal{N}_{3,4}$. Now consider the matrices $D$, $E$, $F$, $G$, $H$, and $J$ in Figure A.9 viewed over $GF(5)$.

The matroids $M[D]$, $M[E]$, $M[F]$, $M[G]$, $M[H]$, and $M[J]$ are 3-connected matroids that satisfy the obligatory rank constraints and contain $N_1^*$, $N_2^*$, $N_3^*$, $N_4^*$, $N_5^*$, and $N_6^*$, but not $N_1$, $N_2$, $N_3$, $N_4$, $N_5$, and $N_6$, respectively. Thus, there are no seven-element matroids in $\mathcal{N}_{3,5}$. Hence, there are no seven-element matroids in $\mathcal{N}_{3,q}$ for $q \geq 4$.

Several of the explicit matroid representations in this appendix were found using the matroids package for Sage [18] developed by Rudi Pendavingh, Stefan van Zwam, and others.

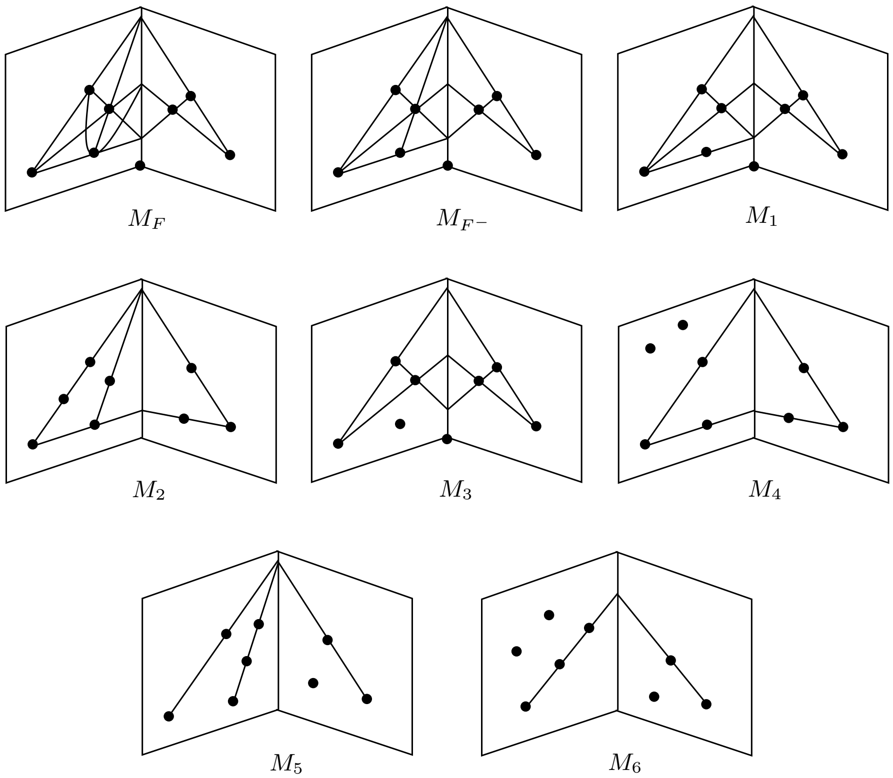



Figure A.7. Matroids that are examples of why no matroid in Figure A.6 is in $\mathcal{N}_{3,q}$ for $q \geq 6$.

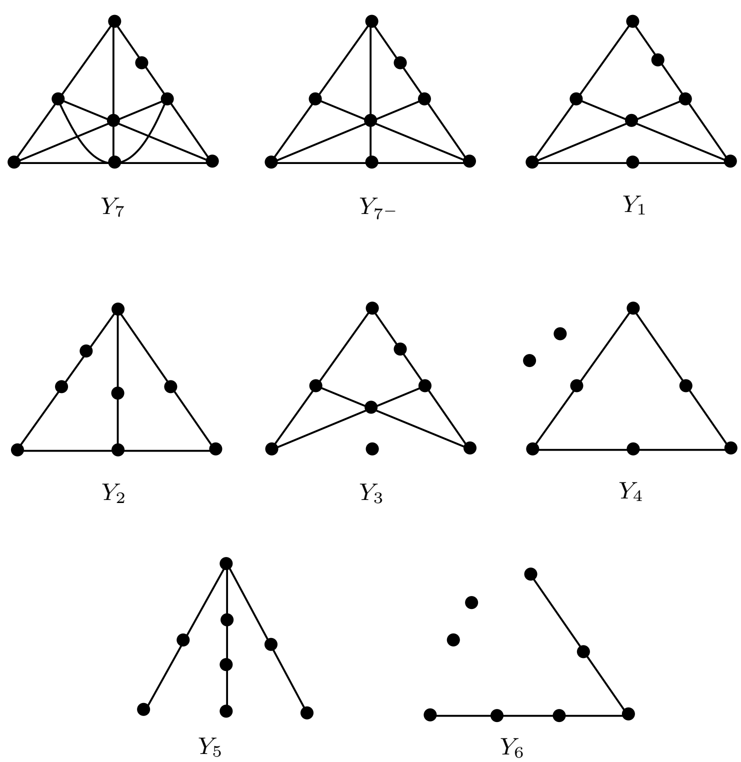



Figure A.8. Matroids obtained from each matroid in Figure A.7 via a $Y$-$\Delta$ exchange.

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & \omega+1 \\ 0 & 1 & \omega & \omega+1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & \omega+1 \end{bmatrix} \end{array} \qquad B = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & \omega \\ 1 & 1 & \omega & \omega+1 \\ 1 & 0 & \omega+1 & 1 \\ 0 & 1 & \omega+1 & \omega \end{bmatrix} \end{array}$$

$$D = \begin{array}{c|cccc} & 5 & 6 & 7 & 8 \\ \hline 1 & 1 & 1 & 0 & 4 \\ 2 & 0 & 1 & 2 & 1 \\ 3 & 1 & 0 & 1 & 1 \\ 4 & 1 & 1 & 1 & 1 \end{array} \qquad E = \begin{array}{c|cccc} & 5 & 6 & 7 & 8 \\ \hline 1 & 1 & 2 & 0 & 4 \\ 2 & 0 & 1 & 3 & 1 \\ 3 & 1 & 0 & 1 & 1 \\ 4 & 1 & 1 & 1 & 1 \end{array} \qquad F = \begin{array}{c|cccc} & 5 & 6 & 7 & 8 \\ \hline 1 & 1 & 4 & 0 & 1 \\ 2 & 0 & 1 & 2 & 1 \\ 3 & 1 & 0 & 2 & 1 \\ 4 & 1 & 1 & 1 & 1 \end{array}$$

$$G = \begin{array}{c|cccc} & 5 & 6 & 7 & 8 \\ \hline 1 & 1 & 2 & 0 & 0 \\ 2 & 0 & 1 & 2 & 4 \\ 3 & 1 & 0 & 3 & 3 \\ 4 & 1 & 1 & 1 & 1 \end{array} \qquad H = \begin{array}{c|cccc} & 5 & 6 & 7 & 8 \\ \hline 1 & 1 & 0 & 1 & 3 \\ 2 & 1 & 1 & 0 & 3 \\ 3 & 1 & 4 & 4 & 3 \\ 4 & 1 & 2 & 2 & 2 \end{array} \qquad J = \begin{array}{c|cccc} & 5 & 6 & 7 & 8 \\ \hline 1 & 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 & 1 \\ 3 & 1 & 2 & 3 & 1 \\ 4 & 1 & 4 & 2 & 1 \end{array}$$

Figure A.9. Matrices viewed over $GF(4)$ and $GF(5)$.

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# Vita

Jesse Taylor was born in Mississippi, although he spent most of his life in Tennessee. He finished his undergraduate degree in mathematics at Middle Tennessee State University in 2008. In August 2008, he came to Louisiana State University to pursue graduate studies in mathematics. He earned a Master of Science degree in mathematics from Louisiana State University in 2010. He is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in August 2014.