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SCS 89: Continuity Concepts for Partially Ordered Sets

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SEMINAR ON CONTINUITY IN SEMILATTICES

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TOPIC: Continuity concepts for partially ordered sets

1. *Continuity and upper continuity*

Recently, several successful attempts have been made to extend the theory of continuous lattices to partially ordered sets ("posets"); see, for example, [7],[15],[18],[20]. A similar generalization of meet-continuous lattices (that is, upper continuous lattices in the sense of [5]) will be discussed in the subsequent paragraphs. Moreover, we shall see that both notions of continuity are covered by a common general concept presented in Section 2.

The following notations will be convenient. Given a poset P and an element $y \in P$, define

$\downarrow y := \{x \in P : x \leq y\}$ (*principal ideal* generated by y),

$\uparrow y := \{x \in P : y \leq x\}$ (*principal dual ideal* generated by y),

and for $Y \subseteq P$,

$\downarrow Y := \bigcup\{\downarrow y : y \in Y\}$ (*lower set* generated by Y),

$\uparrow Y := \bigcup\{\uparrow y : y \in Y\}$ (*upper set* generated by Y).

Both the lower sets and the upper sets form a topology on P, called the *lower A(lexandroff)-topology* $\theta(P)$ and the *upper A-topology* $\theta^*(P)$, respectively. Notice that the $\theta(P)$-closed sets are precisely the $\theta^*(P)$-open sets. Hence $\theta(P)$ and $\theta^*(P)$ are also topological closure systems.

A subset D of P is *directed* if every finite subset of D has an upper bound in P (whence $D \neq \emptyset$). The system of all directed lower sets ("ideals" in the sense of [12]) is denoted by $\mathfrak{i}(P)$.

Henceforth, let P be an *up-complete* poset; that is, every directed subset D of P has a join (denoted by $\bigvee D$). Using the so-called "way-below" sets

$$\Downarrow x = \bigcap\{D \in \mathfrak{i}(P) : x \leq \bigvee D\} \quad (x \in P),$$

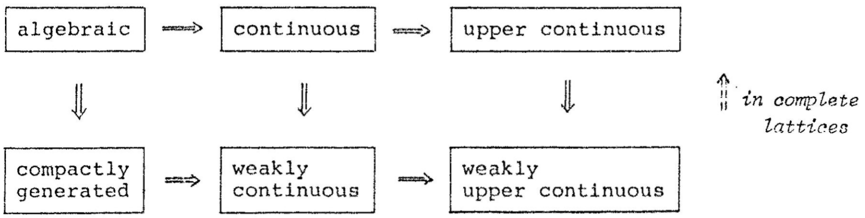
we say P to be *weakly continuous* if $x = \bigvee \Downarrow x$ for each $x \in P$. If, in addition, each way-below set $\Downarrow x$ is directed then P is called a *continuous poset* (cf. R.-E. HOFFMANN [15],[16], J. D. LAWSON [18]). Apparently, these two properties may also be described as follows. An up - complete poset P is weakly continuous (resp., continuous) iff for each $x \in P$, there is a (directed) set E such that $x = \bigvee E$ and $E \subseteq D$ for all $D \in \dot{\iota}(P)$ with $x \leq \bigvee D$. Now we only change the position of the quantifiers in this definition and call an up - complete poset P *weakly upper continuous* (resp., *upper continuous*) provided that for each $x \in P$ and each $D \in \dot{\iota}(P)$ with $x \leq \bigvee D$, there exists a (directed) set $E \subseteq D$ with $x = \bigvee E$. Hence in this definition the set E may depend not only on the choice of x but also on that of D.

REMARK. Replacing directed sets with finite sets in the preceding definitions one arrives at the notion of *distributive* ∨ - *semilattices* in the sense of KATRIŇÁK [17] (see also GRÄTZER [13,p.99] and ERNÉ [9] for slightly modified definitions).

An element $x \in P$ with $x \in \twoheaddownarrow x$ is called *compact*, and P is said to be *compactly generated* if every element of P is a join of compact elements. If, moreover, for each $x \in P$ there exists a *directed* set D of compact elements such that $x = \bigvee D$ then P is an *algebraic poset*. For more details on compactly generated and algebraic posets, respectively, see [11] (cf.also [14],[15]). At the moment, we only mention the important fact that *the systems* $\iota(P)$ *are, up to isomorphism, precisely the algebraic posets*.

The following implications between the various kinds of compact generation and continuity are obvious:

DIAGRAM 1.



In general, none of these implications can be inverted.

EXAMPLE 1. (cf. [11]). Let X be an uncountable set and S the system of all subsets of X which have either at most one element or a countable complement. Then S is a compactly generated (hence weakly continuous) and upper continuous $\wedge$-semilattice, but S is not continuous since for $y \in S$, the way-below set $\Downarrow y = \{\{x\} : x \in y\} \cup \{\emptyset\}$ is not directed unless y is a compact element of S (i.e., a singleton or empty).

Many examples of continuous complete lattices which are not algebraic can be found in the Compendium [12]. The standard example is the unit interval [0,1]. Every complete Boolean lattice is upper continuous, but only the atomic ones are (weakly) continuous (cf.[12,I-4.18]). An example of a compactly generated poset which is not even upper continuous is obtained as follows:

EXAMPLE 2. For $n \in \mathbb{N} = \{1,2,3,\ldots\}$ let

$$\underline{n} := \{1,\ldots,n\}.$$

Consider the following system of subsets of $\mathbb{N}$:

$$P = \{\{n\} : n \in \mathbb{N}\} \cup \{\underline{n} : n \in \mathbb{N}\} \cup \{\mathbb{N} \setminus \{n\} : n \in \mathbb{N}\} \cup \{\emptyset, \mathbb{N}\}.$$

P is partially ordered by inclusion and closed under directed unions, hence up-complete. The compact members of P are exactly the finite ones. Thus P is compactly generated; however, it is not upper continuous because for $m \in \mathbb{N}$, we have

$$\mathbb{N} \setminus \{m\} \leq \bigvee \{\underline{n} : n \in \mathbb{N}\} = \mathbb{N},$$

but there is no directed subset D of P such that $\mathbb{N} \setminus \{m\} = \bigvee D$ and $\mathbb{N} \setminus \{m\} \notin D$. (In other words, the co-atoms $\mathbb{N} \setminus \{m\}$ are inaccessible but not compact). By the way, we notice that the atoms of a weakly upper continuous poset are necessarily compact.

In contrast to Example 2, a compactly generated $\wedge$-*semilattice* must always be upper continuous, as our first proposition shows.

PROPOSITION 1. *For an up-complete $\wedge$-semilattice S, the following conditions are equivalent:*

(a) S *is meet-continuous; that is,* $x \wedge \bigvee D = \bigvee \{x \wedge d : d \in D\}$ *for all* $x \in S$ *and each directed subset* D *of* S.

(b) $\bigvee D_1 \wedge \bigvee D_2 = \bigvee \{d_1 \wedge d_2 : d_1 \in D_1, d_2 \in D_2\}$ *for any two directed subsets* D_1, D_2 *of* S.

(c) *The join map* $\bigvee : \downarrow(S) \to S,\ D \mapsto \bigvee D$ *preserves finite meets (and di-*

rected joins).

(d) S *is upper continuous.*

(e) S *is weakly upper continous.*

PROOF. (a) <=> (b) <=> (c): Straightforward.
(a) => (d): If $x \leq \bigvee D$ for some directed set D then the set $E = \{ x \wedge d : d \in D \}$ is directed, contained in $\downarrow D$, and has join x.
(d) => (e): Trivial.
(e) => (a): For $y := x \wedge \bigvee D \leq \bigvee D$, we find a set $E \subseteq \downarrow D$ with $y = \bigvee E$. Hence for each $e \in E$, we have $e \leq x \wedge d$ for some $d \in D$. Since y is an upper bound for the set $F = \{ x \wedge d : d \in D \}$ and $E \subseteq \downarrow F$, it follows that $y = \bigvee F$, as desired. □

REMARK. In the preceding statements, "directed" may be replaced with "non-empty totally ordered" or "non-empty well-ordered". This follows from a well-known (but non-trivial) set-theoretical fact (see, for example, MAYER-KALKSCHMIDT and STEINER [19]): *Suppose $\mathfrak{X}$ is a system of sets such that for every non-empty subsystem $\mathfrak{Y}$ which is well-ordered by inclusion, the union $\bigcup \mathfrak{Y}$ belongs to $\mathfrak{X}$. Then so does the union of every directed subsystem of $\mathfrak{X}$ (i.e. $\mathfrak{X}$ is "inductive").* Applying this principle to the system of all subsets of a fixed poset which have a join, one concludes that it suffices to postulate the existence of joins for all non-empty well-ordered subsets in order to guarantee up-completeness. Similarly, if

$$(*) \quad x \wedge \bigvee D = \bigvee \{ x \wedge y : y \in D \}$$

holds for all elements x and all non-empty well-ordered subsets D of an up-complete $\wedge$-semilattice, then (*) must also be true for every directed subset D of S. To see this, consider the system $\mathfrak{X}$ of all $D \subseteq S$ possessing a join and satisfying (*) for all $x \in S$. If $\mathfrak{Y} \subseteq \mathfrak{X}$ is non-empty and well-ordered by inclusion then $W := \{\bigvee Y : Y \in \mathfrak{Y}\}$ is a non-empty well-ordered chain in S, whence $x \wedge \bigvee \bigcup \mathfrak{Y} = x \wedge \bigvee W = \bigvee \{ x \wedge \bigvee Y : Y \in \mathfrak{Y} \} = \bigvee \{ \bigvee \{ x \wedge y : y \in Y \} : Y \in \mathfrak{Y} \} = \bigvee \{ x \wedge y : y \in \bigcup \mathfrak{Y} \}$, and so $\bigcup \mathfrak{Y} \in \mathfrak{X}$. Now, if D is a directed set then $\downarrow D = \bigcup \{ \downarrow y : y \in D \}$ belongs to $\mathfrak{X}$ since $\downarrow y \in \mathfrak{X}$ for all $y \in S$, and it follows that $x \wedge \bigvee D = x \wedge \bigvee \downarrow D = \bigvee \{ x \wedge y : y \in \downarrow D \} = \bigvee \{ x \wedge y : y \in D \}$. □

These considerations show that our definition of upper continuity in fact generalizes the classical definition of upper continuous lattices (cf. CRAWLEY and DILWORTH [5 , p.15]).

An immediate consequence of Proposition 1 is

COROLLARY 1. *Every upper continuous* $\wedge$ *- semilattice* S *is the image of an algebraic* $\wedge$ *- semilattice (namely* $\iota(S)$*) under a map which preserves binary meets and directed joins.*

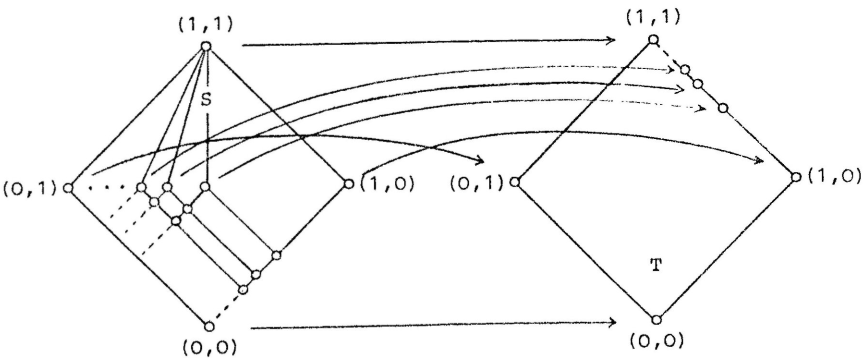
However, the image of an arbitrary algebraic $\wedge$ - semilattice under a map of this kind need not be upper continuous in general.

EXAMPLE 3. The following subsets of the unit square $[0,1]^2$ are partially ordered componentwise:

$$A := \{ (0,0), (0,1), (1,1) \},$$

$$S := \{ (\tfrac{1}{n}, 1 - \tfrac{1}{m}) \ : \ 1 \leq m \leq n;\ m,n \in \mathbb{N} \} \cup A,$$

$$T := \{ (1, 1 - \tfrac{1}{m}) \ : \ m \in \mathbb{N} \} \cup A.$$



The following facts are easily checked:

(1) S and T are complete lattices.

(2) S satisfies the ascending chain condition and is therefore algebraic.

(3) T is not upper continuous.

(4) The map $f : S \to T$, $f(x,y) = \begin{cases} (x,y) & \text{if } (x,y) \in A \\ (1,y) & \text{otherwise} \end{cases}$

is onto and preserves finite meets (and of course directed joins).

It is evident that the ascending chain condition implies compact generation and (upper) continuity. A less trivial result is

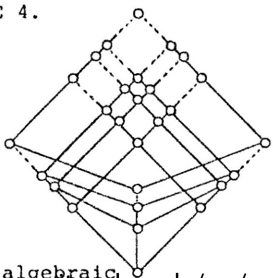
PROPOSITION 2. *A poset satisfying the descending chain condition is upper continuous iff it is algebraic.*

PROOF. Let P be upper continuous and satisfy the descending chain condition. Assuming that P be non - algebraic, we find a minimal $x \in P$ for which there is no directed set of compact elements whose join is x. In particular, x itself cannot be compact, and we find a directed lower set D with $x \leq \bigvee D$ but $x \notin D$. By upper continuity, we may choose a directed set $E \subseteq D$ with $x = \bigvee E$ and $x \notin E$. Hence, by minimality of x, each $y \in E$ is the join of the set K_y of all compact elements dominated by y. As the assignment $y \mapsto K_y$ is isotone, it follows that the union $K := \bigcup\{K_y : y \in E\}$ is a directed set of compact elements with $\bigvee K = \bigvee E = x$, contradicting the assumption on x. □

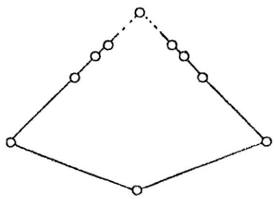
Example 2 shows that there exist weakly upper continuous (moreover, compactly generated) posets satisfying the descending chain condition which are not algebraic.
the descending chain condition need not be upper continuous (in particular, not compactly generated), as the lattice T in Example 3 demonstrates.

In [11] there has been given an example of an algebraic poset containing a complete interval which is not even (weakly) upper continuous:

EXAMPLE 4.



algebraic



not upper continuous

This example shows that none of the six properties depicted in Diagram 1 is inherited by intervals (or principal dual ideals). This cannot happen if we are concerned with complete lattices.

2. *(i,m) - continuous posets*

A common generalization of the previous continuity concepts is obtained by the following definition. Let m be any cardinal number greater than 1, and write $Y \subseteq_m X$ if Y is a subset of X with less than m elements (i.e. $|Y| < m$). An up-complete poset P is called *weakly (i,m)-continuous* if for each system $\mathfrak{Y} \subseteq_m \mathfrak{i}(P)$ and each $x \in P$ with $x \leq \bigvee Y$ for all $Y \in \mathfrak{Y}$, there exists a set $E \subseteq \bigcap \mathfrak{Y}$ such that $x = \bigvee E$. Similarly, we say P to be *(i,m)-continuous* if in the preceding definition E can be chosen in $\dot{\mathfrak{i}}(P)$. Obviously, "(weakly) upper continuous" means "(weakly) (i,m)-continuous", and "(weakly) continuous" means "(weakly) (i,m)-continuous for all m > 1". The following observation is almost evident for ∧-semilattices but non-trivial for arbitrary posets:

PROPOSITION 3. *Let* $1 < m < \omega$. *Then a poset is weakly (i,m)-continuous iff it is weakly* (i,ω)*-continuous.*

This follows from a more general result in [9, Satz 3.2]. Later on, we shall see that an analogous statement on (i,m)-continuity is valid only if m > 2.

For any poset P, the *cut operator* $\Delta : \mathfrak{P}(P) \to \mathfrak{P}(P)$ is defined by

$$\Delta(Y) := \bigcap \{ \downarrow x : Y \subseteq \downarrow x \} \qquad (Y \subseteq P).$$

The fixed points of Δ are called *cuts*; they form a closure system $\delta(P)$, called the *Dedekind-MacNeille completion* of P (cf. [3], [6]-[9], [21]).

A straightforward computation (involving Proposition 3) yields

PROPOSITION 4. *Let* m > 2. *Then an up-complete poset P is weakly (i,m)-continuous if and only if*

$$\bigcap \Delta[\mathfrak{Y}] = \Delta(\bigcap \mathfrak{Y}) \text{ for all } \mathfrak{Y} \subseteq_m \mathfrak{i}(P).$$

Using the fact that in a complete lattice L

$\bigcap \Delta[\mathfrak{Y}] = \downarrow \bigwedge\{\bigvee Y : Y \in \mathfrak{Y}\}$ and

$\bigcap \mathfrak{Y} = \downarrow \{\bigwedge \psi[Y] : \psi \in \prod_{Y \in \mathfrak{Y}} Y\} \quad (\mathfrak{Y} \subseteq \theta(L))$

we arrive at

PROPOSITION 5. *Let* L *be a complete lattice and* m *a cardinal greater than* 2. *Then the following statements are equivalent:*

(a) *The distributive law*

$$\bigwedge\{\bigvee Y : Y \in \mathfrak{Y}\} = \bigvee\{\bigwedge \psi[\mathfrak{Y}] : \psi \in \prod_{Y \in \mathfrak{Y}} Y\}$$

holds for every system $\mathfrak{Y}$ *of directed (lower) sets of* L *with* $|\mathfrak{Y}| < m$.

(b) *The identity*

$$\bigwedge\{\bigvee Y : Y \in \mathfrak{Y}\} = \bigvee\{\bigwedge\{\bigvee \psi(Y) : Y \in \mathfrak{Y}\} : \psi \in \prod_{Y \in \mathfrak{Y}} \{Z : Z \subseteq_\omega Y\}\}$$

holds for every system $\mathfrak{Y}$ *of subsets of* L *with* $|\mathfrak{Y}| < m$.

(c) *The join map* $\vee : \dot{\imath}(L) \to L$, $Y \mapsto \bigvee Y$ *preserves meets of systems with less than* m *elements:* $\mathfrak{Y} \subseteq_m \dot{\imath}(L)$ *implies* $\vee(\bigcap \mathfrak{Y}) = \bigwedge \vee[\mathfrak{Y}]$.

(d) L *is* $(\dot{\imath}, m)$ *- continuous.*

(e) L *is weakly* $(\dot{\imath}, m)$ *- continuous.*

PROOF. (a) => (b): Let $\tilde{Y} := \downarrow\{\bigvee Z : Z \subseteq_\omega Y\}$ $(Y \in \mathfrak{Y})$. Then $(\tilde{Y} : Y \in \mathfrak{Y})$ is a family of directed lower sets whence $\bigwedge\{\bigvee Y : Y \in \mathfrak{Y}\} =$
$\bigwedge\{\bigvee \tilde{Y} : Y \in \mathfrak{Y}\} = \bigvee\{\bigwedge \psi[\mathfrak{Y}] : \psi \in \prod_{Y \in \mathfrak{Y}} \tilde{Y}\} =$
$\bigvee\{\bigwedge\{\bigvee \psi(Y) : Y \in \mathfrak{Y}\} : \psi \in \prod_{Y \in \mathfrak{Y}} \{Z : Z \subseteq_\omega Y\}\}$.

(b) => (a): If Y is a directed lower set then we have $\bigvee Z \in Y$ for all $Z \subseteq_\omega Y$, so the identity in (b) reduces to that in (a).

(a) <=> (c): Straightforward.

(a) => (d): We may assume that m is infinite. If $x \leq \bigvee Y$ for all $Y \in \mathfrak{Y}$ where $\mathfrak{Y} \subseteq_m \dot{\imath}(L)$ then
$x = \bigvee \downarrow x \wedge \bigwedge\{\bigvee Y : Y \in \mathfrak{Y}\} = \bigvee\{\bigwedge \psi[\mathfrak{Y}] : \psi \in \prod_{Y \in \mathfrak{Y}'} Y\}$
where $\mathfrak{Y}' = \mathfrak{Y} \cup \{\downarrow x\}$, and $|\mathfrak{Y}'| = |\mathfrak{Y}| < m$ since m is infinite.
Hence $E = \downarrow\{\bigwedge \psi[\mathfrak{Y}'] : \psi \in \prod_{Y \in \mathfrak{Y}'} Y\} = \downarrow x \cap \bigcap \mathfrak{Y} \in \dot{\imath}(L)$, $x = \bigvee E$, and $E \subseteq \bigcap \mathfrak{Y}$.

(d) => (e): Clear.

(e) => (a): Apply Proposition 4. □

Of course, the statements in Propositions 4 and 5 fail to be equivalent for $m = 2$. In this case, (a), (b) and (c) are trivially fulfilled for *every* complete lattice.

For $m > |L|$ condition (b) is the "equational characterization" of continuous lattices (cf. [12, I-2.3]).

COROLLARY 2. *The class of $(\dot{\imath},m)$ - continuous complete lattices is closed under the formation of direct products and subcomplete lattices. Furthermore, the image of an $(\dot{\imath},m)$ - continuous complete lattice under a map which preserves meets of sets with less than m elements and arbitrary joins is again an $(\dot{\imath},m)$ - continuous complete lattice.*

However, Example 3 shows that the image of an upper (i.e. $(\dot{\imath},\omega)$ -) continuous complete lattice under a map which preserves finite meets and directed joins need not be upper continuous.

As was pointed out in [2], [10] and [20], it appears reasonable to take as morphisms between continuous posets those upper adjoint maps (see [12, Ch.O.3]) which preserve directed joins. The next proposition justifies this choice within the theory of $(\dot{\imath},m)$ - continuous posets:

PROPOSITION 6. *The image of an $(\dot{\imath},m)$ - continuous poset P under an upper adjoint map f which preserves directed joins is again $(\dot{\imath},m)$ - continuous.*

PROOF. (The case $m > |P|$ has been treated in the more general framework of so-called Z - *continuous posets*; see [2]).
Let $Q = f[P]$, and let $g : Q \to P$ denote the lower adjoint of f. Notice that the surjectivity of f ensures that $f \circ g$ is the identity on Q. Given a directed subset D of Q, we know that $g[D]$ is a directed subset of P (because g is isotone); consequently, $g[D]$ has a join x, and $f(x) = f(\bigvee g[D]) = \bigvee f[g[D]] = \bigvee D$. Hence Q is up - complete.
Now consider a system $\mathfrak{Y} \subseteq_m \dot{\imath}(Q)$, and define

$$\mathfrak{Z} := \{ \downarrow g[Y] : Y \in \mathfrak{Y} \} \subseteq_m \dot{\imath}(P).$$

Suppose $x \leq \bigvee Y$ for all $Y \in \mathfrak{Y}$. Then $x \leq \bigvee f[g[Y]] = f(\bigvee g[Y])$, $g(x) \leq \bigvee g[Y] = \bigvee \downarrow g[Y]$, that is, $g(x) \leq \bigvee Z$ for all $Z \in \mathfrak{Z}$. Hence we find a directed set $E \subseteq \bigcap \mathfrak{Z}$ such that $g(x) = \bigvee E$, whence $x = f(g(x)) = \bigvee f[E]$. Furthermore, $f[E]$ is a directed set with $f[E] \subseteq Y$ for all $Y \in \mathfrak{Y}$. Indeed, $E \subseteq \downarrow g[Y]$ implies $f[E] \subseteq f[\downarrow g[Y]] \subseteq \downarrow f[g[Y]] = \downarrow Y = Y$. □

Seminar on Continuity in Semilattices, Vol. 1, Iss. 1 [2023], Art. 90

COROLLARY 3. *The image of an $(\dot{\imath}, m)$-continuous complete lattice under a map which preserves arbitrary meets and directed joins is again $(\dot{\imath}, m)$-continuous.*

(But see Example 3!)

COROLLARY 4. *For a poset P, the following conditions are equivalent:*

(a) *P is continuous.*

(b) *The join map $\vee : \dot{\imath}(P) \to P$ is well-defined and has a lower adjoint (namely the way-below map $\Downarrow : P \to \dot{\imath}(P)$).*

(c) *P is the image of an algebraic poset under an upper adjoint map which preserves directed joins.*

In the light of this characterization, continuous posets appear as a very natural generalization of algebraic posets.

3. *The Scott operator and Scott-closed sets*

Given an arbitrary poset P, we define the *Scott operator* $\Sigma : \mathfrak{P}(P) \to \mathfrak{P}(P)$ by

$$\Sigma(Y) := \{x \in P : x \in \Delta(D) \text{ for some directed } D \subseteq {\downarrow}Y\} \quad (Y \subseteq P).$$

Whenever D has a join then $x \in \Delta(D)$ means $x \leq \bigvee D$. A set Y with $\Sigma(Y) = Y$ is called *Scott closed*. Special Scott-closed sets are the *finitely generated* lower sets, i.e. the sets ${\downarrow}Z$ with $Z \subseteq_\omega P$. But clearly not all Scott-closed sets must be of this kind. For example, if P satisfies the ascending chain condition then *every* lower set is Scott-closed. The Scott-closed subsets of a power set $\mathfrak{P}(X)$ (considered as a complete lattice) are precisely the systems of finite character contained in $\mathfrak{P}(X)$.

Let us return to the general case of an arbitrary poset P. As Σ is always extensive and preserves finite unions, it is clear that the Scott-closed sets form a topological closure system $\sigma(P)^c$. The corresponding system of open sets, denoted by $\sigma(P)$, is the so-called *Scott topology* of P (cf. [8], [12], [22]).

We denote by $\hat{\Sigma}$ the corresponding closure operator, i.e.

$$\hat{\Sigma}(Y) = \bigcap\{A \in \sigma(P)^c : Y \subseteq A\} \quad (Y \subseteq P).$$

Obviously, Σ is idempotent if and only if $\Sigma = \hat{\Sigma}$. But unfortunately Σ may fail to be idempotent even if P is an upper continuous complete lattice.

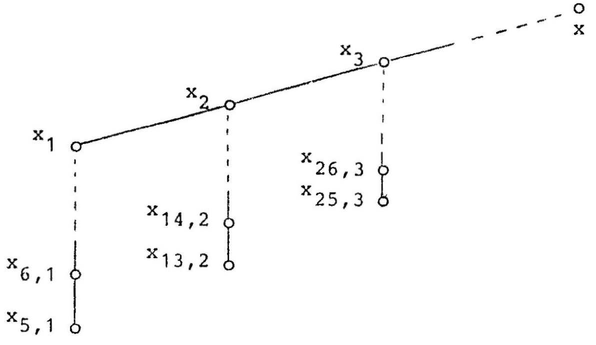
EXAMPLE 5. Let L denote the complete Boolean lattice of all regular open subsets of $\mathbb{R}$ (with respect to the Euclidean topology). Consider the following elements of L:

$$x_{n,k} := \bigcup \{]\tfrac{r}{k}+\tfrac{1}{n}, \tfrac{r+1}{k}-\tfrac{1}{n}[\; : r = 0,\ldots,k-1 \} \cup \;]1,k[$$

$$(n,k \in \mathbb{N},\; n > 2k(k+1)),$$

$x_k := \,]0,k[\quad (k \in \mathbb{N})$,

$x := \,]0,\infty[$, (where $]a,b[\; = \{ x \in \mathbb{R} : a < x < b \}$).



Obviously,

$$x_{n,k} \subseteq x_{m,k} \text{ for } n \leq m,$$

while for $k < \ell$, $x_{n,k}$ and $x_{m,\ell}$ are incomparable. Indeed, $x_{n,k}$ includes an open interval $]\frac{1}{n}, \frac{1}{k} - \frac{1}{n}[$ of length $\frac{1}{k} - \frac{1}{n} - \frac{1}{n} > \frac{1}{k} - \frac{1}{k(k+1)} = \frac{1}{k+1} \geq \frac{1}{\ell}$, and this interval contains a number $\frac{r}{\ell}$ with $r \in \{0,\ldots,\ell-1\}$, while $\frac{r}{\ell} \notin x_{m,\ell}$. On the other hand, $\ell - \frac{1}{2} \in x_{m,\ell} \setminus x_{n,k}$.
Now in L we compute

$$\bigvee \{ x_{n,k} : n > 2k(k+1) \} = x_k \quad (k \in \mathbb{N}),$$

$$\bigvee \{ x_k : k \in \mathbb{N} \} = x,$$

and for $Y = \{ x_{n,k} : n,k \in \mathbb{N},\; n > 2k(k+1) \}$ it follows that $x \in \Sigma(\Sigma(Y))$ but $x \notin \Sigma(Y)$. Hence L is an upper continuous complete lat-

In contrast to this situation in the "finitary" case, one can prove:

PROPOSITION 7. *The Scott operator* Σ *of a continuous poset* P *is idempotent.*

PROOF. This follows from results in [8] and [20], but for the sake of convenience, we give a direct proof.
Let $x \in \Sigma(\Sigma(Y))$. Then there exists a directed set $D \subseteq \Sigma(Y)$ with $x \leq \bigvee D$. Each way-below set $⇓y$ is a directed lower set, and the system $\{⇓y : y \in D\}$ is directed by inclusion since the map $y \mapsto ⇓y$ is isotone ; hence $E = \bigcup\{⇓y : y \in D\}$ is a directed lower set. For $y \in D \subseteq \Sigma(Y)$ we find a directed set D_y with $D \subseteq \downarrow Y$ and $y \leq \bigvee D_y$, whence $⇓y \subseteq \downarrow D_y \subseteq \downarrow Y$. Thus we obtain $E \subseteq \downarrow D$, and finally $x \leq \bigvee D = \bigvee\{\bigvee ⇓y : y \in D\} = \bigvee E$, which proves $x \in \Sigma(Y)$. □

In terms of the operators Σ resp. $\hat{\Sigma}$, upper continuity can be characterized as follows:

PROPOSITION 8. *Let* P *be an up-complete poset. Then*

(1) P *is upper continuous iff*

$$\downarrow x \cap \Sigma(Y) = \Sigma(\downarrow x \cap \downarrow Y) \text{ for all } x \in P,\ Y \subseteq P.$$

(2) *The lattice* $\sigma(P)^c$ *of Scott-closed sets is upper continuous iff*

$$\downarrow x \cap \hat{\Sigma}(Y) = \hat{\Sigma}(\downarrow x \cap \downarrow Y) \text{ for all } x \in P,\ Y \subseteq P.$$

(3) *If* P *is upper continuous then so is* $\sigma(P)^c$. *The converse holds if* $\iota(P)$ *is closed under binary intersections (e.g., if* P *is a* $\wedge$*-semilattice) or if* Σ *is idempotent.*

More general results have been proved in [9, Satz 3.6], where the system of directed sets is replaced with an arbitrary system of subsets. For complete lattices, Proposition 8(3) has been established in [12, II-4.15].

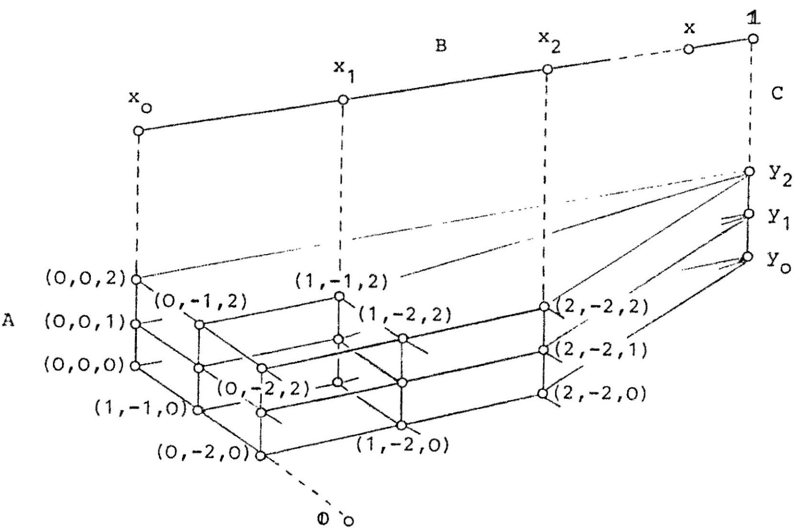
The following example shows that (in contrast to the complete case) an up-complete poset P with upper continuous $\sigma(P)^c$ need not be upper continuous in its own right.

EXAMPLE 6. The set $\mathbb{Z} \cup \{\pm\omega, \omega+1\}$ is linearly ordered as usual:

$$-\omega < n < \omega < \omega+1 \quad (n \in \mathbb{Z}) .$$

Let

$$\mathbb{0} := (0,-\omega,0), \; x := (\omega,0,\omega), \; \mathbb{1} := (\omega,0,\omega+1),$$
$$A := \{ (k,-m,n) : k,m,n \in \omega, \; k \leq m \} \cup \{\mathbb{0}\},$$
$$B := \{ x_k : k \in \omega \}, \; x_k = (k,0,\omega),$$
$$C := \{ y_n : n \in \omega \}, \; y_n = (\omega+1,0,n),$$
$$P := A \cup B \cup C \cup \{x,\mathbb{1}\}.$$



P is an up-complete poset (partially ordered componentwise). In this poset we always have

$$\downarrow y \cap \Sigma(Y) = \Sigma(\downarrow y \cap \downarrow Y)$$

unless $y = x$, while

$$\downarrow x \cap \Sigma(C) = \downarrow x \neq \downarrow x \setminus \{x\} = \Sigma(A) = \Sigma(\downarrow x \cap \downarrow C).$$

But since $\hat{\Sigma}(Y) = \Sigma^2(Y) = \downarrow x$ if $\Sigma(Y) = \downarrow B$ and $\Sigma(Y) = \hat{\Sigma}(Y)$ otherwise, it is easy to see that

$$\downarrow y \cap \hat{\Sigma}(Y) = \hat{\Sigma}(\downarrow y \cap \downarrow Y) \text{ for all } y \in P \text{ and all } Y \subseteq P.$$

Hence, by Proposition 8, $\sigma(P)^c$ is upper continuous while P is not. But see Proposition 10!

In analogy to Propositions 4 and 8, we have the following characterization of $(\dot{\imath},m)$-continuity:

PROPOSITION 9. *Let* $m > 2$ *and* P *an up-complete poset. Then*

(1) P *is* $(\dot{\imath},m)$*-continuous iff*

$$\Sigma(\bigcap\mathfrak{Y}) = \bigcap\Sigma[\mathfrak{Y}] \text{ for all } \mathfrak{Y} \subseteq_m \theta(P) \text{ (resp. } \mathfrak{Y} \subseteq_m \dot{\imath}(P)).$$

(2) $\sigma(P)^c$ *is* $(\dot{\imath},m)$*-continuous iff*

$$\hat{\Sigma}(\bigcap\mathfrak{Y}) = \bigcap\hat{\Sigma}[\mathfrak{Y}] \text{ for all } \mathfrak{Y} \subseteq_m \theta(P).$$

PROOF. (1) Suppose P is $(\dot{\imath},m)$-continuous, and $\mathfrak{Y} \subseteq_m \theta(P)$. Of course, we have $\Sigma(\bigcap\mathfrak{Y}) \subseteq \bigcap\Sigma[\mathfrak{Y}]$. Conversely, consider an element $x \in \bigcap\Sigma[\mathfrak{Y}]$. Then, by definition of Σ, we find directed sets $D_Y \subseteq Y$ with $x \leq \bigvee D_Y$ $(Y \in \mathfrak{Y})$. As $|\{D_Y : Y \in \mathfrak{Y}\}| < m$, there exists a directed set D such that $x = \bigvee D$ and $D \subseteq \bigcap\{\downarrow D_Y : Y \in \mathfrak{Y}\} \subseteq \bigcap\mathfrak{Y}$, whence $x \in \Sigma(\bigcap\mathfrak{Y})$.
Now assume $\Sigma(\bigcap\mathfrak{Y}) = \bigcap\Sigma[\mathfrak{Y}]$ for all systems $\mathfrak{Y}$ consisting of less than m directed lower sets. Let $\mathfrak{Y}$ be one such system and suppose $x \leq \bigvee Y$ for all $Y \in \mathfrak{Y}$. Then $x \in \bigcap\Sigma[\mathfrak{Y}] = \Sigma(\bigcap\mathfrak{Y})$. Accordingly there is a directed set $E \subseteq \bigcap\mathfrak{Y}$ with $x \leq \bigvee E$, and it follows that $x \in \downarrow x \cap \Sigma(E) = \Sigma(\downarrow x \cap \downarrow E)$. (Use the fact that $m > 2$). Thus we find a directed $D \subseteq \downarrow x \cap \downarrow E \subseteq \bigcap\mathfrak{Y}$ with $x \leq \bigvee D$, and $D \subseteq \downarrow x$ yields $x = \bigvee D$.
(2) Suppose $\hat{\Sigma}(\bigcap\mathfrak{Y}) = \bigcap\hat{\Sigma}[\mathfrak{Y}]$ for all $\mathfrak{Y} \subseteq_m \theta(P)$, and let be given a family $(\mathfrak{Y}_j : j \in J)$ of subsets of $\sigma(P)^c$ with $|J| < m$. Then each union $\bigcup\mathfrak{Y}_j$ is a lower set, and we compute in the complete lattice $\sigma(P)^c$:
$\bigwedge\{\bigvee\mathfrak{Y}_j : j \in J\} = \bigcap\{\hat{\Sigma}(\bigcup\mathfrak{Y}_j) : j \in J\} = \hat{\Sigma}(\bigcap\{\bigcup\mathfrak{Y}_j : j \in J\}) =$
$\hat{\Sigma}(\bigcup\{\bigcap\psi[J] : \psi \in \prod_{j\in J}\mathfrak{Y}_j\}) = \bigvee\{\bigwedge\psi[J] : \psi \in \prod_{j\in J}\mathfrak{Y}_j\}$. By Proposition 5, this implies $(\dot{\imath},m)$-continuity of the complete lattice $\sigma(P)^c$. (Notice that we did not need the hypothesis that the systems $\mathfrak{Y}_j$ be directed, so we have shown in fact more, namely " (θ,m)-continuity " of $\sigma(P)^c$).
Conversely, assume $\sigma(P)^c$ is $(\dot{\imath},m)$-continuous, and let $\mathfrak{Y}$ be a system of lower sets with $|\mathfrak{Y}| < m$. Then for each $Y \in \mathfrak{Y}$ the system $\mathfrak{Z}_Y = \{\downarrow Z : Z \subseteq_\omega Y\}$ is a directed subsystem of $\sigma(P)^c$, and we compute
$\bigcap\{\hat{\Sigma}(Y) : Y \in \mathfrak{Y}\} = \bigwedge\{\bigvee\mathfrak{Z}_Y : Y \in \mathfrak{Y}\} = \bigvee\{\bigwedge\psi[\mathfrak{Y}] : \psi \in \prod_{Y\in\mathfrak{Y}}\mathfrak{Z}_Y\} =$
$\hat{\Sigma}(\bigcup\{\bigcap\psi[\mathfrak{Y}] : \psi \in \prod_{Y\in\mathfrak{Y}}\mathfrak{Z}_Y\}) = \hat{\Sigma}(\bigcap\{\bigcup\mathfrak{Z}_Y : Y \in \mathfrak{Y}\}) = \hat{\Sigma}(\bigcap\mathfrak{Y})$. □

Combining the last two propositions we arrive at

COROLLARY 5. *Let* P *be an up-complete poset with idempotent Scott operator, and let* m *be any cardinal greater than* 1. *Then* P *is* $(\mathfrak{i},m)$*-continuous iff* $\sigma(P)^c$ *is* $(\mathfrak{i},m)$*-continuous.*

For large m, this result can be strengthened as follows:

PROPOSITION 10. *For an up-complete poset* P, *the following conditions are equivalent:*

(a) P *is continuous.*

(b) $\sigma(P)^c$ *is continuous.*

(c) $\sigma(P)^c$ *(resp.* $\sigma(P)$*) is completely distributive.*

(d) *The Scott operator* Σ *of* P *preserves arbitrary intersections of lower sets.*

Each of these conditions implies that Σ *is idempotent.*

PROOF. By Propositions 7 and 9, it only remains to show that complete distributivity of $\sigma(P)^c$ implies continuity of P. For the proof of this fact, see [8],[16] or [18].
Notice that a continuous topological closure system must already be completely distributive.$_\square$

Obviously Proposition 9 cannot be extended to m = 2. Unfortunately, an $(\mathfrak{i},2)$-continuous poset need not be $(\mathfrak{i},3)$- (i.e. $(\mathfrak{i},\omega)$-) continuous, although being weakly $(\mathfrak{i},\omega)$-continuous (see Proposition 3).

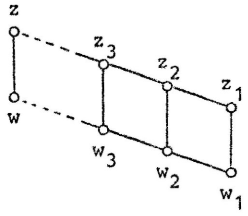
EXAMPLE 7. Define inductively up-complete posets P_k as follows. Choose pairwise distinct elements x, x_n, y_n $(n \in \mathbb{N})$. Then $P_1 := \{x\} \cup \{x_n : n \in \mathbb{N}\} \cup \{y_n : n \in \mathbb{N}\}$ is partially ordered by setting

$$x_n < x_m < x, \quad y_n < y_m < x \qquad (n < m)$$

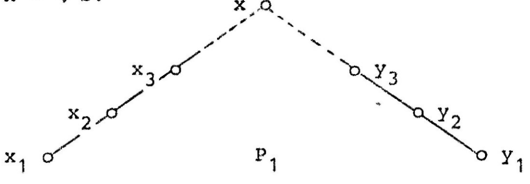
and no further relations.
We call the sequences (x_n) and (y_n) generating chains of level 1.
Now suppose up-complete posets $P_1 \subset \ldots \subset P_{k-1}$ have been defined together with certain generating chains of level 1,...,k-1. Then, for every generating chain (z_n) of level < k and all $w \in P_{k-1}$ with $w \leq z = \bigvee\{z_n : n \in \mathbb{N}\}$, choose a sequence (w_n) of new elements (not in P_{k-1}), call it a generating chain of level k, and set

$w_n < w_m < w,$ $\quad (n < m).$

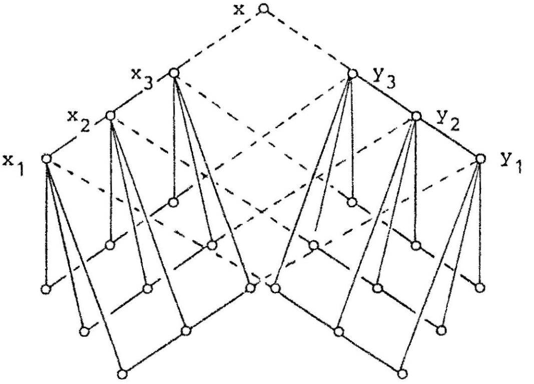
$w_n < z_n$



By adjoining the elements of all generating chains of level k, the poset P_{k-1} is enlarged to an up - complete poset P_k.
It is then easy to see that the directed union $P = \bigcup\{P_k : k \in \mathbb{N}\}$ is an $(i,2)$ - continuous poset, but P is not $(i,3)$ - continuous because there is no directed set $D \subseteq \downarrow\{x_n : n \in \mathbb{N}\} \cap \downarrow\{y_n : n \in \mathbb{N}\}$ with $x = \bigvee D$.



P_1



P_2

4. *Irreducible elements*

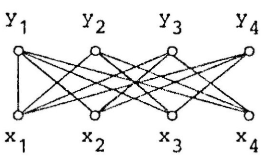
In this section, we derive a common generalization of the following two important theorems of lattice theory:

(I) Every element of an algebraic complete lattice is a meet of completely irreducible elements (cf. [4],[5, 6.1], or [12, I-4.23]).

(II) Every element of a continuous complete lattice is a meet of irreducible elements (cf. [12, I-3.10]).

Let m be any cardinal number >1. An element q of a poset P is called m-*irreducible* (or, to be more precise, m-*meet-irreducible*) if q cannot be the greatest lower bound of a set Y unless $|Y| \geq m$ or $q \in Y$. Hence "completely irreducible" means "m-irreducible for one (resp. all) $m > |P|$". If P is a $\wedge$-semilattice then "irreducible" means "m-irreducible for one (resp. all) m with $2 < m \leq \omega$". However, in arbitrary posets there can exist (m-1)-irreducible elements which are not m-irreducible ($m \in \mathbb{N}$, $m \geq 2$).

EXAMPLE 8. Consider a 2m-element set $P = \{x_1,...,x_m\} \cup \{y_1,...,y_m\}$, partially ordered by $x_k < y_n \Leftrightarrow k \neq n$ or $k = 1$ ($1 \leq k, n \leq m$).



m = 4

Here x_1 is (m-1)-irreducible but not m-irreducible.

Recall that a subset U of an up-complete poset P is Scott open (i.e. $U \in \sigma(P)$) iff for each directed set $D \subseteq P$, $\bigvee D \in \uparrow U$ implies $D \cap U \neq \emptyset$ (in particular, U must be an upper set). Now we say U is m-$\bigwedge$-*closed* if for all $Y \subseteq_m U$, $x = \bigwedge Y$ implies $x \in U$. Finally, we call P m-*generated* if for all $x,y \in P$ with $x \not\leq y$ there exists an m-$\bigwedge$-closed open subset U with $x \in U$ and $y \notin U$. This is certainly fulfilled if $\sigma(P)$ has a base of m-$\bigwedge$-closed sets.

The following fact is well-known in the case of complete lattices (cf. [12, I-3.7 and I-4.22]).

PROPOSITION 11. (1) *Every compactly generated poset is* m *- generated for all* m > 1.

(2) *Every weakly continuous poset is* ω *- generated.*

PROOF. (1) If $x \not\leq y$ in a compactly generated poset then there exist a compact element z with $z \leq x$ and $z \not\leq y$, whence $U = \uparrow z$ is an open set with $x \in U$ and $y \notin U$, and clearly U is closed under arbitrary meets.
(2) Given $x \not\leq y$, we may choose inductively elements z_n such that $z_o = x$, $z_{n+1} \in \Downarrow z_n$ and $z_n \not\leq y$. Then $U = \uparrow \{ z_n : n \in \omega \}$ is an ω - $\bigwedge$ - closed open set with $x \in U$ and $y \notin U$. □

It is clear that a *complete* lattice is compactly generated (i.e. algebraic) if and only if it is m - generated for all m > 1. However, there are up - complete posets which are m - generated for all m > 1 but not compactly generated, as the poset P_1 in Example 7 shows. An ω - generated complete lattice which is not even (weakly) upper continuous will be presented in Example 8.

Now Theorems (I) and (II) admit the following common generalization:

PROPOSITION 12. *Every element of an* m *- generated up - complete poset* P *is a meet of* m *- irreducible elements.*

PROOF. We have to show that for $x,y \in P$ with $x \not\leq y$, there exist an m - irreducible q with $y \leq q$ but $x \not\leq q$. Choose an m - $\bigwedge$ - closed open set U with $x \in U$ and $y \notin U$. Then $P \setminus U$ is closed under directed joins and has therefore a maximal element $q \geq y$. Clearly $q \not\geq x$, and q must be m - irreducible. (For the case m = ω, see [12, I - 3.7]). □

COROLLARY 6. (1) *Every element of a compactly generated poset is a meet of completely irreducible elements.*

(2) *Every element of a weakly continuous poset is a meet of* ω *- irreducible elements.*

We conclude this note with several applications arising in order theory, algebra and geometry. (For several interesting applications of continuous and algebraic posets to topology, see [1], [12] and [18]).

EXAMPLE 8. The convex *closed* subsets of $\mathbb{R}^n$ form a closure system $\overline{K_n}$.

Since each singleton is a convex closed set, $\overline{K_n}$ is a complete lattice in which every element is a join of atoms. But none of these atoms is compact, so $\overline{K_n}$ cannot even be upper continuous (all the less algebraic). Moreover, $\overline{K_n}$ contains no completely irreducible elements at all. However, a special version of the Hahn - Banach Theorem states that every closed convex set is an intersection of "closed halfspaces"

$$L_{a,c} = \{ x \in \mathbb{R}^n : \langle x,a\rangle \leq c \} \qquad (a \in \mathbb{R}^n \setminus \{0\},\ c \in \mathbb{R}) ,$$

and it is easy to see that these halfspaces are precisely the irreducible elements of $\overline{K_n}$.

The above mentioned intersection theorem can be reduced to Corollary 6(2) as follows.

EXAMPLE 9. Consider the system K_n^o of all nonempty convex open subsets of $\mathbb{R}^n$. It is not hard to see that the closure map $Y \mapsto \overline{Y}$ is an isomorphism between K_n^o and the system $\overline{K_n^o}$ of all convex closed subsets of $\mathbb{R}^n$ with nonempty interior. The inverse isomorphism is induced by the interior map $Y \mapsto \overset{o}{Y}$. In particular, each $Y \in K_n^o$ is a regular open set. Obviously, K_n^o is closed under finite intersections and directed unions, so it is an up - complete $\wedge$ - semilattice. (Moreover, $K_n^o \cup \{\emptyset\}$ is a complete lattice). For $X,Y \in K_n^o$, $X \in \Downarrow Y$ means that $\overline{X}$ is compact in the topological sense and contained in Y. Hence K_n^o has no compact elements at all, but it is a continuous poset (use the fact that $\mathbb{R}^n$ is a locally compact space). Hence every convex open set is an intersection of irreducible ones, and as before it is easy to see that the irreducible members of K_n^o are precisely the open halfspaces $L_{a,c}^o$ (they are *not* completely irreducible, although they cannot be represented as intersections of larger convex open sets!).
By passage from K_n^o to $\overline{K_n^o}$, we see that every closed convex set with nonempty interior is an intersection of closed halfspaces. But a nonempty convex closed set K with empty interior can be represented in the form

$$K = (x + V) \cap (K + V^{\perp})$$

where $x \in K$, V is a subspace of $\mathbb{R}^n$ and $V^{\perp}$ its orthogonal space. Now $K + V$ is a convex open set with nonempty interior, and as $x + V$ is trivially an intersection of closed halfspaces, the same is true for K.

EXAMPLE 10. Let U be a fixed nonempty bounded convex open subset of

$\mathbb{R}^n$ and $\mathfrak{U}_U$ the system of all "similar" open sets contained in U, i.e.

$$\mathfrak{U}_U = \{ x + cU : x \in \mathbb{R}^n , c > 0, x + cU \subseteq U \} .$$

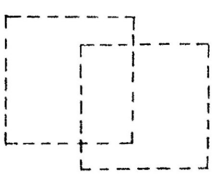
It is a non-trivial exercise to show that $\mathfrak{U}_U$ is closed under directed unions. (Notice that, for example, the system $\{ x + cU : c \in \mathbb{R}, c > 0 \}$ is *not* inductive). However, in general, $\mathfrak{U}_U$ is neither a $\vee$- nor a $\wedge$-semilattice. For $V,W \in \mathfrak{U}_U$, one finds that $V \in \twoheaddownarrow W$ means $\overline{V} \subseteq W$ (observe that $\overline{V}$ is compact). From this characterization one can deduce by some further computations that $\mathfrak{U}_U$ is a continuous poset without completely irreducible elements. However, by Corollary 6(2), each element of $\mathfrak{U}_U$ is a meet (not: intersection!) of irreducible ones. The feature of these irreducible elements heavily depends on the form of U. For example, if U is an open disk in $\mathbb{R}^2$ then *every* element of $\mathfrak{U}_U$ is irreducible. On the other hand, if U is an open square in $\mathbb{R}^2$ then only those open squares $V \subset U$ which have a boundary in common with U are irreducible.

U

$V \in \twoheaddownarrow U$ V not irreducible

$V \notin \twoheaddownarrow U$ V irreducible

In the following constellation, the two open squares have neither a join nor a meet:



EXAMPLE 11 . The system of all open rectangles

$$Q_{a,b,c,d} = \{ (x,y) \in \mathbb{R}^2 : a < x < b, c < y < d \}$$
$$(0 \leq a \leq b \leq 1, 0 \leq c \leq d \leq 1)$$

contained in the unit square $[0,1]^2$ is a complete continuous lattice without atoms. On the other hand, the system of all closed rectangles contained in $[0,1]^2$ is also a complete lattice but not (weakly) upper continuous, by similar reasons as in Example 8. Dropping the atoms of this lattice, we obtain a complete continuous lattice isomorphic to the lattice of open rectangles.

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