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SCS 85: The Space of Compact Convex Subsets of a Locally Convex Topological Vector Space

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SCS 85

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DATE M 1 D 19 Y 1984

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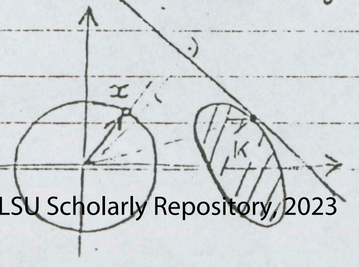
TOPIC
The space of compact convex subsets of a locally convex topological vector space

Introduction

One of the nicest examples of a (dually) continuous lattice is the lattice $\mathcal{C}\mathrm{Conv}(K)$ of all compact convex subsets of a compact convex set K which is embeddable into a locally convex topological vector space. J. Tiller has noticed that $\mathcal{C}\mathrm{Conv}(K)$ is also a compact convex set, a "konvexe Menge höherer Ordnung" as Blaschke may have said, and he has studied this object from an abstract point of view under the name of "convex continuous lattice". I would like to ask the question:

What is the structure of the set $\mathcal{C}\mathrm{Conv}(E)$ of all non-empty compact subsets of a locally convex topological vector space?

For this I would like to go back to a representation of compact convex subsets of $\mathbb{R}^n$ by continuous real valued functions on the sphere S^{n-1} as one may find it in the book of Fenchel:



For a compact convex set $K \subseteq \mathbb{R}^n$, the function $p_K : S^{n-1} \to \mathbb{R}$ is defined by $p_K(x) = \max_{y \in K} y \cdot x$, $x \in S^{n-1}$, where $y \cdot x$ is the inner product.

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As K is determined by its supporting hyperplanes, K is determined by p_K: K is the set of all $y \in \mathbb{R}^n$ such that $y \cdot x \le p_K(x)$ for all x, i.e. $K \mapsto p_K : \mathcal{C}\mathrm{Conv}(\mathbb{R}^n) \to C(S^{n-1}, \mathbb{R})$ is injective; moreover, $p_{K_1+K_2} = p_{K_1} + p_{K_2}$, $p_{\lambda K} = \lambda p_K$ $(\lambda \ge 0)$, $K_1 \subseteq K_2 \Leftrightarrow p_{K_1} \le p_{K_2}$. Finally, the Hausdorff metric on $\mathcal{C}\mathrm{Conv}(E)$ turns out to be the sup-norm topology in the functional representation. Thus, the functional representation clarifies the algebraic, topological and order structure of $\mathcal{C}\mathrm{Conv}(\mathbb{R}^n)$ provided that we characterise those functions $p : S^{n-1} \to \mathbb{R}$ which represent compact convex sets. This is more easily done by defining p_K on all of $\mathbb{R}^n$ (instead of S^{n-1}):

$$p_K(x) = \max_{y \in K} y \cdot x \qquad \text{for all } x \in \mathbb{R}^n.$$

Then all the above properties remain valid, and in addition:

$$p_K(x_1 + x_2) \le p_K(x_1) + p_K(x_2),$$
$$p_K(\lambda x) = \lambda p_K(x) \qquad \text{for } \lambda \ge 0;$$

i.e. p_K is sublinear on $\mathbb{R}^n$, and conversely, every sublinear functional on $\mathbb{R}^n$ represents a compact convex set. Thus we have seen:

The set $\mathcal{C}\mathrm{Conv}(\mathbb{R}^n)$ (of all compact subsets of $\mathbb{R}^n$) is algebraically, topologically and order theoretically faithfully represented by the set $S(\mathbb{R}^n)$ of all sublinear functionals on $\mathbb{R}^n$.

We shall see that this can be generalized to the infinite-dimensional case: Let E be a topological vector space and E^* its dual with the weak*topology. Then there is a one-to-one correspondance between the set $S^*(E)$ of all sublinear functionals on E and the set $\mathcal{C}\mathrm{Conv}(E^*)$ of all weak-*-compact convex

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subsets of E^*; this correspondance is given as follows:

To $p \in S^*(E)$ associate $C_p = \{ f \in E^* \mid f \le p \}$,

and conversely,

to $C \in \mathcal{C}\mathrm{Conv}(E^*)$ associate $p_C(x) = \sup_{f \in C} f(x)$,

as suggested by the finite dimensional case.

This result does not help to clarify the structure of the set $\mathcal{C}\mathrm{Conv}(E)$ of compact convex subsets for arbitrary locally convex topological vector spaces E but only for dual spaces with the weak topology. The situation in the topological situation is more complicated (!)

Our result seems to be tacitly known to functional analysts. J. Tiller has obtained a special version of it. The representation of $\mathcal{C}\mathrm{Conv}(E^*)$ as the function space $S^*(E)$ allows a (psychologically) easier investigation of the abstract order and topological properties of $\mathcal{C}\mathrm{Conv}(E^*)$. At the other hand, it shows up that properties that a "continuous lattice theorist" asks for from his point of view reflect basic theorems of functional analysis, like Hahn-Banach-theorems as König has propagated them. But at the other hand, our result allows a geometric visualisation of sublinear functionals; this allows easy proofs of some results of König on sublinear functionals as Tiller has remarked.

As in Note 1, there will be no references in this text; I would like to mention that I have profited of W. Roth's knowledge about sublinear functionals and Hahn-Banach-theorems. The exposition will be quite elementary, hopefully.

Notations

Throughout we shall use the following notations:

E will be a locally convex topological vectorspace over the reals

E^* the algebraic dual of E,

E' the topological dual,

$S^*(E)$ the set of all sublinear functionals $p : E \to \mathbb{R}$,

$S(E)$ the set of all continuous sublinear functionals on E.

2. Sublinear functionals

Recall that a functional $p : E \to \mathbb{R}$ is called sublinear, if

2.1 $\quad p(x+y) \le p(x) + p(y) \quad$ for all x, y in E

2.2 $\quad p(\lambda x) = \lambda p(x) \quad$ for all x in E and $\lambda \ge 0$

A sublinear functional is convex, i.e. for $0 \le \lambda \le 1$ and $x, y \in E$

2.3 $\quad p(\lambda x + (1-\lambda) y) \le \lambda p(x) + (1-\lambda) p(y)$

2.4 We note that a function $p : E \to \mathbb{R}$ is sublinear if and only if its hypergraph

$$H_p = \{ (x, \lambda) \in E \times \mathbb{R} \mid p(x) \le \lambda \}$$

is a convex cone.

2.5 If a sublinear functional p satisfies $p(-x) = -p(x)$ for all x, then p is linear.

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Indeed, $p(x) = p(x+y-y) \underset{2.1}{\leq} p(x+y) + p(-y) = p(x+y) - p(y)$, whence $p(x) + p(y) \leq p(x+y)$. Thus, p is also superlinear.

We shall use the following inequalities for sublinear functionals p and p_0:

2.6 $$-p(x) \leq p(-x) \quad \text{for all } x \text{ in } E$$

Indeed, $0 = p(0) = p(x-x) \leq p(x) + p(-x)$ by 2.1.

2.7 $$p(x) - p(y) \leq p(x-y) \quad \text{for all } x, y \text{ in } E$$

Indeed, $p(x) = p(x-y+y) \underset{2.1}{\leq} p(x-y) + p(y)$, whence 2.7.

2.8 $$|p(y) - p(x)| \leq \max(p(x-y), p(y-x))$$

This follows from 2.7, and by interchanging x and y in 2.7.

2.9 If $p \leq p_0$, then $-p_0(-x) \leq p(x) \leq p_0(x)$ for all x in E.

Indeed $-p(x) \underset{2.6}{\leq} p(-x) \leq p_0(-x)$, whence $-p_0(-x) \leq p(x)$

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3. Algebraic and order structure of $S^*(E)$.

The set $S^*(E)$ of all sublinear functionals $p: E \to \mathbb{R}$ should be considered as a subset of the vector space $\mathbb{R}^E$ with the pointwise defined addition, scalar multiplication and order. In particular $S^*(E)$ is ordered with

$$p \leq p_0 \quad \text{iff} \quad p(x) \leq p_0(x) \text{ for all } x \in E$$

3.1 One easily sees that

$$p_1, p_2 \in S^*(E) \text{ imply } p_1 + p_2 \in S^*(E)$$
$$p \in S^*(E) \text{ and } \lambda \geq 0 \text{ imply } \lambda p \in S^*(E)$$

i.e. $S^*(E)$ is a convex cone in $\mathbb{R}^E$.

3.2 For every family $p_i \in S^*(E)$ such that $\bigvee_i p_i(x) < +\infty$ for all $x \in E$, (in particular for every family p_i dominated by some $p_0 \in S^*(E)$,) the pointwise sup $p(x) := \bigvee_i p_i(x)$ also is sublinear. (Such families p_i will be called simply bounded.)

Even if two sublinear p_1, p_2 have a sublinear minorant p, the pointwise inf need not be sublinear; it suffices to consider $p_1(x_1, x_2) = |x_1|$ and $p_2(x_1, x_2) = |x_2|$ on $E = \mathbb{R}^2$. Of course, if a family $p_i \in S^*(E)$ has a minorant p in $S^*(E)$, then by 3.2 it has a greatest minorant p in $S^*(E)$; we write $p = \bigwedge_i p_i$. But, two sublinear functionals need not have a sublinear minorant; for this it suffices to consider two independant linear functionals. But one has:

3.3 For any filtered (= down directed) family $p_i \in S^*(E)$, the pointwise inf $$p(x) = \bigwedge_i p_i(x)$$ also belongs to $S^*(E)$.

Proof. First we show that for every x, the set $p_i(x)$ is bounded below in $\mathbb{R}$; indeed, fix i_0; for all i with $p_i \leq p_{i_0}$ we have $-p_{i_0}(-x) \leq p_i(x)$ by 2.9. Now $p(\lambda x) = \lambda p(x)$ is clear and $p(x+y) \leq p(x) + p(y)$ follows from the down directedness.

3.4 For every $p \in S^*(E)$ and every (pointwise) bounded family p_i,

$$p + \bigvee_i p_i = \bigvee_i (p + p_i), \quad \lambda \cdot \bigvee p_i = \bigvee \lambda p_i$$

and for every filtered family p_i,

$$p + \bigwedge\!\!\downarrow p_i = \bigwedge\!\!\downarrow (p + p_i), \quad \lambda \cdot \bigwedge\!\!\downarrow p_i = \bigwedge\!\!\downarrow \lambda p_i$$

This is true, as it holds in $\mathbb{R}$ and as the operations are pointwise.

3.5 For every $p_0 \in S^*(E)$, the set $\downarrow p_0$ of all sublinear functionals p dominated by p_0 is a (dually) continuous lattice. More precisely, for $p \leq p_0$ we have $-p_0(-x) \leq p(x) \leq p_0(x)$; consequently, $\downarrow p_0$ is a subset of $\prod_{x \in E} [-p_0(-x), p_0(x)] \subseteq \mathbb{R}^E$ closed under arbitrary sups and filtered meets. Thus, the Lawson topology on $\downarrow p_0$ coincides with the product topology induced from $\mathbb{R}^E$. If we do not say the contrary, we will always consider this pointwise topology on $S^*(E)$, which we will call the weak topology.
Clearly, $\downarrow p_0$ also is a convex set, and consequently a compact convex set for the weak (= Lawson) topology embeddable in a locally convex topological vector space, namely $\mathbb{R}^E$ with the product topology.
By 3.4, addition and scalar multiplication on $S^*(E)$ are jointly Scott continuous. Clearly, these operations as well as $\vee$ are jointly continuous for the weak topology. Clearly, $S^*(E)$ is weakly closed in $\mathbb{R}^E$, whence weakly complete.

4. Continuous sublinear functionals

Until now our considerations were purely algebraic. From now on we take into account, that E is endowed with a locally convex vector space topology and we denote by $S(E)$ the set of all continuous sublinear functionals $p : E \to \mathbb{R}$. Clearly, $S(E)$ is a convex cone in $S^*(E)$. From 2.8 we obtain:

4.1 For a sublinear functional p on E, the following are equivalent

(i) p is uniformly continuous on E;

(ii) p is continuous on E;

(iii) p is continuous in 0.

From 2.9 we infer:

4.2 If a sublinear functional p is dominated by a continuous one, say p_0, then p is also continuous; i.e. if $p_0 \in S(E)$, then

$$\downarrow p_0 = \{p \in S^*(E);\ p \leq p_0\} \subseteq S(E).$$

Moreover this set is equicontinuous.

Note that by 2.8 the equicontinuity in 0 implies equicontinuity everywhere.

4.3 COROLLARY. For every family p_i dominated in $S(E)$ the pointwise sup belongs to $S(E)$; likewise the pointwise inf of a filtered family in $S(E)$.

Again we endow $S(E)$ with the weak topology induced from $\mathbb{R}^E$.

If $C \subseteq S(E)$ is equicontinuous, the weak closure $\overline{C}$ in $\mathbb{R}^E$ is also equicontinuous and, a fortiori, contained in $S(E)$.

For a subset $C \subseteq S(E)$, the following are equivalent:

(i) C is equicontinuous;

(ii) C is order bounded in $S(E)$;

(iii) the pointwise sup $p_0(x) := \bigvee_{p \in C} p(x)$ belongs to $S(E)$.

Proof. (ii) and (iii) are equivalent by 4.3. (ii) implies (i) by 4.2.
(i) ⇒ (iii): C is equicontinuous. Thus there is a neighborhood $\mathcal{U}$ of 0 in E such that $|p(x)| \leq 1$ for all $x \in \mathcal{U}$ and all $p \in C$. Then $p_0(x) := \bigvee_{p \in C} p(x)$ satisfies $|p_0(x)| \leq \varepsilon$ for all $x \in \varepsilon\mathcal{U}$. Thus $p_0 \in S(E)$.

Every equicontinuous subset $C \subseteq S(E)$ is simply bounded (i.e. relatively weakly compact). If E is barrelled, e.g. a Banach space or a Fréchet space, then every simply bounded set $C \subseteq S(E)$ has its pointwise sup in $S(E)$, and is consequently equicontinuous.

Proof. The first assertion is straightforward. For the converse, let $C \subseteq S(E)$ be simply bounded. Then $p_0(x) := \bigvee_{p \in C} p(x) \in S^*(E)$ by 3.2. Let $\mathcal{U} := \{x ; p_0(x) \leq 1 \text{ and } p_0(-x) \leq 1\}$. Clearly $\mathcal{U}$ is convex, circled and closed. In order to see that $\mathcal{U}$ absorbs every $y \in E$ it suffices to note that $\frac{1}{\lambda} y \in \mathcal{U}$ for $\lambda := \max\{p_0(y), p_0(-y)\} + 1$. Thus, $\mathcal{U}$ is a barrel and, thus, a neighborhood of 0 by hypothesis; i.e. p_0 is equicontinuous.

COROLLARY. Let E be barrelled. Then $S(E)$ is closed in $S^*(E)$ for sups (as they exist in $S^*(E)$); $S(E)$ is closed in $S^*(E)$ for lim sup $\mathcal{F}$ for simply bounded filters $\mathcal{F}$.

4.8 PRINCIPLE OF UNIFORM BOUNDEDNES. If E is a Banach space, then every simply bounded subset of $S(E)$ is uniformly bounded on the unit ball of E.

It seems to me that everything done for "spaces of linear mappings" in Schaefer "Topological vector spaces" pp. 79-87, carries over to $S(E)$ "spaces of sublinear functionals", including the Banach-Steinhaus theorem. I have just written down part of it.
One may also try to replace $\mathbb{R}$ by a cone K which has "the properties" that $S(E)$ had before, say $K = S(E)$, and consider $S(E,K)$ the set of sublinear functionals $f : E \to K$. One might also try to replace E by a cone. (A certain amount of results will not carry over to this situation, I guess.)

5. Compact convex sets and sublinear functionals

5.1 LEMMA! For a sublinear functional $p \in S^*(E)$ the following are equivalent :

(i) p is linear,

(ii) p is minimal in $S^*(E)$,

(iii) p is $\vee$-irreducible in $S^*(E)$.

holds for continuous sublinear functionals with respect

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is evident. (iii) $\Rightarrow$ (i) : Suppose that $p \in S(E)$ nonlinear. We shall exhibit sublinear functionals $p_1, p_2\ (\neq p)$ such $p(x) = p_1(x) \vee p_2(x)$ for all x.

If p is not linear, then there is an $x_0 \in E$ such that $-p(-x_0) \neq p(x_0)$ by 2.5. Define

$$p_1(x) = \inf_{\lambda \geq 0} \{ p(x + \lambda x_0) - \lambda m \}, \text{ where } m = \frac{p(x_0) - p(-x_0)}{2}$$

$$p_2(x) = \inf_{\lambda \geq 0} \{ p(x - \lambda x_0) + \lambda m \}$$

p_1, p_2 are sublinear :

$$p_1(x+y) = \inf_{\lambda \geq 0} \{ p(x+y+\lambda x_0) - \lambda m \} = \inf_{\lambda,\mu \geq 0} \{ p(x+y+(\lambda+\mu)x_0) - \overbrace{\quad}^{(\lambda+\mu)m} \}$$

$$\leq \inf_{\lambda,\mu \geq 0} \{ p(x+\lambda x_0) - \lambda m + p(y+\mu x_0) - \mu m \}$$

$$= \inf_{\lambda \geq 0} \{ p(x+\lambda x_0) - \lambda m \} + \inf_{\mu \geq 0} \{ p(x+\mu x_0) - \mu m \}$$

$$= p_1(x) + p_1(y)$$

$p_1(x) \leq p(x)$ and $p_2(x) \leq p(x)$ is seen by putting $\lambda = 0$

$p_1(x_0) = -m < p(-x_0)$, $p_2(x_0) = m < p(x_0)$ whence $p_1 \neq p$

$p_2 \neq p$

indeed: $p_1(-x_0) = \inf_{\lambda \geq 0} \{ p(-x_0 + \lambda x_0) - \lambda m \}$

$$\lambda \geq 1 : \quad p(-x_0 + \lambda x_0) - \lambda m = (\lambda - 1)\, p(+x_0) - \lambda \frac{p(x_0) - p(-x_0)}{2}$$

$$= -p(+x_0) + \lambda \underbrace{\frac{p(x_0) + p(-x_0)}{2}}_{\geq 0}$$

$$0 \leq \lambda \leq 1 : \quad p(-x_0 + \lambda x_0) - \lambda m = (1-\lambda)\, p(-x_0) - \lambda \frac{p(x_0) - p(-x_0)}{2}$$

$$= p(-x_0) - \lambda \underbrace{\frac{p(x_0) + p(-x_0)}{2}}_{\geq 0}$$

In beiden Fällen wird das Infimum für $\lambda = 1$ angenommen, und für diesen Wert erhält man

$$p_1(-x_0) = -p(x_0) + \frac{p(x_0) + p(-x_0)}{2} = -m$$

Similarly $p_2(x_0) = m$.

$$-\quad p_1(x) \vee p_2(x) = p(x)$$

Indeed, suppose that $p_1(x) < p(x)$ and $p_2(x) < p(x)$ for some x. Then we could find λ_1 and $\lambda_2 \geq 0$ such that

$$p(x + \lambda_1 x_0) - \lambda_1 m < p(x)$$

and

$$p(x - \lambda_2 x_0) + \lambda_2 m < p(x)$$

Multiplying by λ_2 and λ_1 respectively, and adding the resulting inequalities we would obtain :

$$\lambda_2\, p(x + \lambda_1 x_0) + \lambda_1\, p(x - \lambda_2 x_0) < (\lambda_2 + \lambda_1)\, p(x)$$

but $(\lambda_1 + \lambda_2)\, p(x) = p(\lambda_1 x + \lambda_2 x) = p(\lambda_1 x - \lambda_1 \lambda_2 x_0 + \lambda_2 x + \lambda_1 \lambda_2 x_0)$

$$\leq p(\lambda_1 x - \lambda_1 \lambda_2 x_0) + p(\lambda_2 x + \lambda_1 \lambda_2 x_0)$$

$$= \lambda_2\, p(x + \lambda_1 x_0) + \lambda_1\, p(x - \lambda_2 x_0) \quad ↯$$

5.2 $\boxed{\text{HAHN-BANACH-THEOREM}}$ Every [continuous] sublinear functional on E is the pointwise sup of [continuous] linear functionals.

Proof. $\downarrow p$ is a (dually) continuous lattice by 3.5. By the Compendium, p is a sup of $\vee$-irreducibles, and these are linear by 5.1.

5.3 $\boxed{\text{NOTE}}$ In the proof of 5.1 we have not used the Hahn-Banach theorem, nor Zorn's lemma. Using the Hahn-Banach-Theorem, 5.1 is easily proved by means of THE LEMMA, as J. Tiller has noticed. We have preferred this access in order to illustrate the relation between the Hahn-Banach-Theorem and the theory of continuous lattices.

To every weakly compact convex set $C \subseteq E^*$ we may – as such set C is simply bounded – associate the sublinear functional

$$p_C(x) = \sup\{f(x);\ f \in C\} = \bigvee_{f\in C} f \in S^*(E)$$

By 3.4 we have:

For all compact convex subsets $C_1, C_2 \subseteq E^*$ and all $\lambda \geq 0$

$$C_1 \subseteq C_2 \Rightarrow p_{C_1} \leq p_{C_2}$$

$$p_{\lambda C_1} = \lambda p_{C_1}$$

$$p_{C_1+C_2} = p_{C_1} + p_{C_2}$$

i.e. the map $C \mapsto p_C : \text{Comp Conv}(E^*) \to S^*(E)$ is monotone and linear (more precisely positively linear). We also define a map the other way around:

For every $p \in S^*(E)$, let

$$C_p = \{f \in E^*;\ f \leq p\} = \downarrow p \cap E^*$$

As $\downarrow p$ is a compact convex set in $\mathbb{R}^E$ and E^* closed and convex, C_p is a compact convex subset of E^*. Clearly $p_1 \leq p_2$ implies $C_{p_1} \subseteq C_{p_2}$, whence the map

$$p \mapsto C_p : S^*(E) \to \text{Comp Conv}(E^*)$$

is monotone. By the Hahn-Banach-Theorem 5.2, $p = \bigvee C_p = p_{C_p}$. At the other hand, if C is a compact convex subset of E^* and $f_0 \in E^* \setminus C$, then by the separation theorem on compact convex sets (Question: is there a straightforward argument in the line of the preceding to replace the separation theorem?) there is an $x \in E$ such that $f(x) < f_0(x) - 1$ for all $f \in C$, whence $f_0 \nleq \sup\{f;\ f \in C\} = $ We conclude that $C_{p_C} = C$. We have established the following:

5.4 $\boxed{\text{THEOREM}}$ The maps $C \mapsto p_C = \sup C$, $p \mapsto C_p = \downarrow p \cap E$ establish an algebraic and order isomorphism between the set $\text{Comp Conv}(E^*)$ of compact convex subsets of E^* and the set $S^*(E)$ of sublinear functionals on E.

In particular, as $S^*(E)$ is a cone in a topological locally convex vector space $(\mathbb{R}^E)$, $\text{Comp Conv}(E)$ is also a cone which may be endowed with "locally convex" topology. For every fixed compact convex $C \subseteq E^*$, the topology on $\text{Comp Conv}(C)$ induced by this locally convex topology on $S^*(E)$ is just the Lawson (= hyperspace) topology by 3.5. An intrinsic characterisation of this topology on $\text{Comp Conv}(E^*)$ will be given later on.

If we restrict ourselves to the cone $S(E)$ of continuous sublinear functionals, then by 4.5 the correspondance of the THEOREM applies to the set $\mathrm{Equ\,Conv}(E')$ of weakly closed convex equicontinuous subsets of E'; in the case of barreled spaces this is exactly the collection of all weakly compact convex subsets of E' by 4.6. We have:

5.5 $\boxed{\text{COROLLARY}}$ $C \mapsto p_C := \sup C$ is an algebraic and an order isomorphism of the cone $\mathrm{Equ\,Conv}(E')$ of all weakly closed equicontinuous convex subsets of E' onto the cone $S(E)$ of all continuous sublinear functionals on E.

For reflexive spaces, E and E' may be interchanged. For reflexive Banach spaces one obtains:

5.6 $\boxed{\text{COROLLARY.}}$ For a reflexive Banach space E, the set $\mathrm{Comp\,Conv}(E)$ of all (weakly) compact convex subsets of E is algebraically and order isomorphic to the set $S(E')$ of all continuous sublinear functionals in E'.

It is not clear to me in more general cases how $\mathrm{Equ\,Conv}(E)$ (not E'!) can be characterised.

6 Lower semicontinuous sublinear functionals

A sublinear functional $f: E \to \mathbb{R}$ is lower semicontinuous, if $\{x;\ f(x) \le \alpha\}$ is closed for every $\alpha \in \mathbb{R}$. (In fact, it suffices to have this property for $\alpha = \pm 1$.) We denote by $S_\ell(E)$ the set of all lower semicontinuous $p \in S^*(E)$.

Lower semicontinuity means that the hypograph of p is closed. From this and the Hahn-Banach-Theorem it follows that a $p \in S_\ell(E)$ is the pointwise sup of continuous linear functionals. At the other hand, the pointwise sup of any family of (lower semi-) continuous functions is always lower semicontinuous (as far as this sup is defined). Thus we get an algebraic and an order isomorphism from the set $\mathrm{B\,Conv}(E')$ of all simply bounded closed convex subsets of E' onto the set $S_\ell(E)$ of all lower semicontinuous sublinear functionals on E.

Note that not every $f \in E^*$ dominated by some $p \in S_\ell(E)$ needs to be continuous (for this would imply that a closed simply bounded convex subset of E' would be compact.)

In the case of a barrelled space, $S_\ell(E) = S(E)$, for if $p \in S_\ell(E)$ then $\{x;\ p(x) \le 1 \text{ und } p(-x) \le 1\}$ is a barrel, whence $p \in S(E)$.

Note that $S_\ell(E)$ is a convex cone (in particular additively stable). Thus $\mathrm{B\,Conv}(E')$ also is a convex cone (in particular the sum of two simply bounded closed convex sets is also closed and addition is cancellative) embeddable in a l.c.v.s.

Topologies on Equconv(E′) and S(E)

Let $\mathcal{S}$ be a family of bounded subsets of E which is updirected and covers E. On E' as well as on S(E) we may consider the topology $\tau_{\mathcal{S}}$ of uniform convergence on all $S \in \mathcal{S}$. This topology is known to be a locally convex vector space topology on E' as well as on $S(E)-S(E)$; in particular, S(E) is a cone in a lcvs $S(E)=S(E)$ with $\tau_{\mathcal{S}}$.

On Equ Conv(E′) we may consider the hyperspace uniformity induced from $\tau_{\mathcal{S}}$. Fortunately, the two agree:

$\boxed{\text{PROPOSITION}}$ $C \mapsto p_C = VC$: Equ.Conv$(E') \to S(E)$ is a uniform isomorphism.

Proof. We write down the proof for the simple convergence only. By adding a quantifier $\forall x \in S$ at the appropriate places one has a proof for the general case:

Given x, ε, p_1 and $p_2 \in S(E)$ are (ε, x)-close iff

(*) $p_1(x) - \varepsilon \le p_2(x) \le p_1(x) + \varepsilon$

At the other hand, C_1 and $C_2 \in$ Equ Conv(E′) are (ε, x)-close iff

(**) $C_1(x) \subseteq C_2(x) + [-\varepsilon, \varepsilon]$

and $C_2(x) \subseteq C_1(x) + [-\varepsilon, \varepsilon]$

Let $p_1 = VC_1$, $p_2 = VC_2$. From (**) we have

$p_1(x) = \sup C_1(x) \le \sup C_2(x) + \varepsilon = p_2(x) + \varepsilon$

$p_2(x) = \sup C_2(x) \le \sup C_1(x) + \varepsilon = p_1(x) + \varepsilon$

which implies (*). At the other hand (*) implies

$\sup C_1(x) \le \sup C_2(x) + \varepsilon$, $\sup C_2(x) \le \sup C_1(x) + \varepsilon$

Applying (*) with $-x$ for x, we obtain

$\sup -C_1(x) \le \sup -C_2(x) + \varepsilon$, $\sup -C_2(x) \le \sup -C_1(x) + \varepsilon$

i.e. $-\inf C_1(x) \le -\inf C_2(x) + \varepsilon$, $-\inf C_2(x) \le -\inf C_1(x) + \varepsilon$

$\inf C_1(x) \ge \inf C_2(x) - \varepsilon$, $\inf C_2(x) \ge \inf C_1(x) - \varepsilon$

As $C_1(x)$ and $C_2(x)$ are intervals this implies (**).