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CONVERGENCE TO WEIGHTED FRACTIONAL BROWNIAN SHEETS*

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ABSTRACT. We define weighted fractional Brownian sheets, which are a class of Gaussian random fields with four parameters that include fractional Brownian sheets as special cases, and we give some of their properties. We show that for certain values of the parameters the weighted fractional Brownian sheets are obtained as limits in law of occupation time fluctuations of a stochastic particle model. In contrast with some known approximations of fractional Brownian sheets which use a kernel in a Volterra type integral representation of fractional Brownian motion with respect to ordinary Brownian motion, our approximation does not make use of a kernel.

1. Introduction

Fractional Brownian sheets have been studied by several authors for their mathematical interest and their applications. One of the first papers on the subject is [12]. Some types of approximations of fractional Brownian sheets have been obtained recently (e.g., [2], [3], [9], [13], [14], [15]). In this paper we give a new type of approximation for certain values of the parameters by means of occupation time fluctuations of a stochastic particle model. The limits that are obtained in this way are a more general class of Gaussian random fields.

We consider centered Gaussian random fields $W = (W_{s,t})_{s,t\geq 0}$ with parameters (a_i, b_i), $i = 1, 2$, whose covariance is given by

$$K_W((s,t),(s',t')) = E\left(W_{s,t}W_{s',t'}\right) = C^{(1)}(s,s')C^{(2)}(t,t'), \tag{1.1}$$

where each $C^{(i)}$ is of the form

$$C^{(i)}(u,v) = \int_0^{u\wedge v} r^{a_i}[(u-r)^{b_i} + (v-r)^{b_i}]dr, \quad i = 1, 2, \tag{1.2}$$

with the following ranges for the parameters:

$$a_i > -1, \ -1 < b_i \leq 1, \ |b_i| \leq 1 + a_i. \tag{1.3}$$

$C^{(i)}$ is the covariance of weighted fractional Brownian motion with parameters (a_i, b_i). Weighted fractional Brownian motions were introduced in [7]. We call W a weighted fractional Brownian sheet with parameters (a_i, b_i), $i = 1, 2$. In the case $a_1 = a_2 = 0$ (the weight functions are 1) W is a fractional Brownian sheet with

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parameters $(\frac{1}{2}(1+b_1), \frac{1}{2}(1+b_2))$. The case $a_1 = a_2 = b_1 = b_2 = 0$ corresponds to the ordinary Brownian sheet. If $b_1 = b_2 = 0$, and at least one a_i is not 0, then W is a time-inhomogeneous Brownian sheet.

Due to the covariance structure (1.1), (1.2), many properties of W are consequences of those of weighted fractional Brownian motion. We will prove an approximation in law of W for a_i and b_i of the form $a_i = -\gamma_i/\alpha_i, b_i = 1 - 1/\alpha_i$, with $0 \leq \gamma_i < 1$ and $1 < \alpha_i \leq 2$; hence the approximation is restricted to values of a_i and b_i such that $-1 < a_i \leq 0$ and $0 < b_i < 1 + a_i$, $i = 1, 2$.

The approximations of fractional Brownian sheets in [2], [15] are based on a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_+$ and a kernel which appears in a Volterra type integral representation of fractional Brownian motion with respect to ordinary Brownian motion. The approximation in [3], analogous to the functional invariance theorem, also uses the kernel. Our approach does not use a kernel. We also use a Poisson random measure, but on $\mathbb{R} \times \mathbb{R}$ instead of $\mathbb{R}_+ \times \mathbb{R}_+$ and in a different way from [2], [15]. Some of the other approximations cited above are motived by simulation of fractional Brownian sheets. Our approximation is not intended for simulation, but rather to show that weighted fractional Brownian sheets emerge in a natural way from a simple particle model.

In section 2 we give the properties of W, in particular long-range dependence. In section 3 we describe the particle system and we prove convergence to W of rescaled ocupation time fluctuations of the system for the above mentioned values of the parameters.

2. Properties

We consider $\mathbb{R}_+^2$ with the following partial order: for $z = (s,t)$ and $z' = (s',t')$, $z \preceq z'$ iff $s \leq s'$ and $t \leq t'$, $z \prec z'$ iff $s < s'$ and $t < t'$, and if $z \prec z'$ we denote by $(z, z']$ the rectangle $(s, s'] \times (t, t']$. We refer to elements of $\mathbb{R}_+^2$ as "times" for simplicity of exposition.

If $X = (X_z)_{z \in \mathbb{R}_+^2}$ is a two-time stochastic process, the increment of X over the rectangle $(z, z']$ with $z = (s,t), z' = (s',t')$ is defined by

$$X((z,z']) \equiv \Delta_{s,t} X(s',t') := X_{(s',t')} - X_{(s,t')} - X_{(s',t)} + X_{(s,t)}.$$

We denote the covariance of the increments of the process X over the rectangles $((s,t),(s',t')]$, $((p,r),(p',r')]$ by

$$K_X\left((s,t),(s',t');(p,r),(p',r')\right) = Cov\left(\Delta_{s,t}X(s',t'), \Delta_{p,r}X(p',r')\right).$$

The covariance of W over rectangles is given by

$$\begin{aligned}
&K_W((s,t),(s',t');(p,r),(p',r')) \\
&= (C^{(1)}(s',p') - C^{(1)}(s,p') - C^{(1)}(s',p) + C^{(1)}(s,p)) \\
&\quad \times (C^{(2)}(t',r') - C^{(2)}(t,r') - C^{(2)}(t',r) + C^{(2)}(t,r)) \\
&= Cov(Y_{s'}^{(1)} - Y_s^{(1)}, Y_{p'}^{(1)} - Y_p^{(1)}) Cov(Y_{t'}^{(2)} - Y_t^{(2)}, Y_{r'}^{(2)} - Y_r^{(2)}), \qquad (2.1)
\end{aligned}$$

where $Y^{(i)}$ is weighted fractional Brownian motion with parameters (a_i, b_i), $i = 1, 2$.

The next theorem contains some properties of weighted fractional Brownian sheets.

Theorem 2.1. *The weighted fractional Brownian sheet W with parameters (a_i, b_i), $i = 1, 2$, has the following properties:*

(1) *Self-similarity:*

$$(W_{hs,kt})_{s,t\geq 0} \stackrel{d}{=} h^{(1+a_1+b_1)/2} k^{(1+a_2+b_2)/2} (W_{s,t})_{s,t\geq 0} \quad \text{for each} \quad h, k > 0, \tag{2.2}$$

where $\stackrel{d}{=}$ denotes equality in distribution.

(2) *W has stationary increments only in the case $a_1 = a_2 = 0$.*

(3) *Covariance of increments: For $(0,0) \preceq (s,t) \prec (s',t') \preceq (p,r) \prec (p',r')$,*

$$K_W((s,t),(s',t');(p,r),(p',r')) = \int_s^{s'} u^{a_1}[(p'-u)^{b_1} + (p-u)^{b_1}]du \int_t^{t'} v^{a_2}[(r'-v)^{b_2} + (r-v)^{b_2}]dv, \tag{2.3}$$

hence

$$K_W((s,t),(s',t');(p,r),(p',r')) \begin{cases} > 0 & \text{if } b_1 b_2 > 0, \\ = 0 & \text{if } b_1 b_2 = 0, \\ < 0 & \text{if } b_1 b_2 < 0. \end{cases}$$

(4) *The one-time processes $(W_{s,t})_{s\geq 0}$ (t fixed) and $(W_{s,t})_{t\geq 0}$ (s fixed) are weighted fractional Brownian motions (multiplied by constants) with parameters (a_1, b_1) and (a_2, b_2), respectively.*

(5)

$$E((\Delta_{s,t} W_{s',t'})^2) = 4 \int_s^{s'} u^{a_1}(s'-u)^{b_1} du \int_t^{t'} v^{a_2}(t'-v)^{b_2} dv. \tag{2.4}$$

(6)

$$\lim_{\varepsilon,\delta\to 0} \varepsilon^{-b_1-1}\delta^{-b_2-1} E((\Delta_{s,t} W_{s+\varepsilon,t+\delta})^2) = \frac{4}{(1+b_1)(1+b_2)} s^{a_1} t^{a_2}, \tag{2.5}$$

$$\lim_{T,S\to\infty} S^{-(1+a_1+b_1)} T^{-(1+a_2+b_2)} E((\Delta_{s,t} W_{s+S,t+T})^2) = 4 \int_0^1 u^{a_1}(1-u)^{b_1} du \int_0^1 v^{a_2}(1-v)^{b_2} dv, \tag{2.6}$$

hence W has asymptotically stationary increments for long increments in $\mathbb{R}_+^2$, but not for short ones (if $a_1, a_2 \neq 0$).

(7) *The finite-dimensional distributions of the process*

$$(S^{-a_1/2} T^{-a_2/2} \Delta_{S,T} W_{s+S,t+T})_{s,t\geq 0}$$

converge as $T, S \to \infty$ to those of fractional Brownian sheet with parameters $(\frac{1}{2}(1+b_1), \frac{1}{2}(1+b_2))$ multiplied by $2/[(1+b_1)(1+b_2)]^{1/2}$.

(8) *Long-range dependence: for* $(s,t) \prec (s',t')$, $(p,u) \prec (p',u')$

$$\lim_{\tau,\kappa\to\infty} \tau^{1-b_1}\kappa^{1-b_2} K_W\left((s,t),(s',t');(p+\tau,u+\kappa),(p'+\tau,u'+\kappa)\right)$$
$$= \frac{b_1 b_2}{(1+a_1)(1+a_2)}(p'-p)((s')^{1+a_1}-s^{1+a_1})(u'-u)((t')^{1+a_2}-t^{1+a_2}). \quad (2.7)$$

(9) *For* $\theta > 0$, *we define the one-time process* $(Z_t)_{t\geq 0} = (W_{t,\theta t})_{t\geq 0}$, *i.e., the sheet restricted to a ray through the origen. (Note that* Z *is not a weighted fractional Brownian motion.) Then for* $0 \leq b_i < 1$, $i = 1,2$, *and not both* b_1, b_2 *equal to* 0, *this process has the long-range dependence property*

$$\lim_{\tau\to\infty} \tau^{1-(b_1+b_2)} Cov(Z_v - Z_u, Z_{t+\tau} - Z_{s+\tau})$$
$$= \frac{\theta^{1+a_2+b_2}(b_1+b_2)}{(1+a_1)(1+a_2)}(v^{2+a_1+a_2}-u^{2+a_1+a_2})(t-s), \quad u<v, s<t. \quad (2.8)$$

Proof. Except for part (9), the proofs follow directly from the form of K_W given by (1.1), (1.2) and properties of weighted fractional Brownian motion [7]. We give an outline of the proof of part (9).

We have, for $u < v$, $s < t$,

$$Cov(Z_v - Z_u, Z_{t+\tau} - Z_{s+\tau})$$
$$= \theta^{1+a_2+b_2}[C^{(1)}(v,t+\tau)C^{(2)}(v,t+\tau) - C^{(1)}(v,s+\tau)C^{(2)}(v,s+\tau)$$
$$- C^{(1)}(u,t+\tau)C^{(2)}(u,t+\tau) + C^{(2)}(u,s+\tau)C^{(2)}(u,s+\tau)], \quad (2.9)$$

$$C^{(1)}(v,t+\tau)C^{(2)}(v,t+\tau) - C^{(1)}(v,s+\tau)C^{(2)}(v,s+\tau)$$
$$= [C^{(1)}(v,t+\tau) - C^{(1)}(v,s+\tau)]C^{(2)}(v,t+\tau)$$
$$+ [C^{(2)}(v,t+\tau) - C^{(2)}(v,s+\tau)]C^{(1)}(v,s+\tau)$$
$$= \int_0^v r^{a_1}[(t-r+\tau)^{b_1} - (s-r+\tau)^{b_1}]dr \int_0^v r^{a_2}[(t-r+\tau)^{b_2} + (v-r)^{b_2}]dr$$
$$+ \int_0^v r^{a_2}[(t-r+\tau)^{b_2} - (s-r+\tau)^{b_2}]dr \int_0^v r^{a_1}[(s-r+\tau)^{b_1} + (v-r)^{b_1}]dr, \quad (2.10)$$

and similarly for the last two terms. The result follows from (2.9), (2.10) and the limits

$$\lim_{\tau\to\infty} \tau^{1-b}[(t_2+\tau)^b - (t_1+\tau)^b] = b(t_2-t_1)$$

and

$$\lim_{\tau\to\infty} \tau^{-b} \int_0^v r^a[(t+\tau)^b + (v-r)^b]dr = \frac{v^{1+a}}{1+a}.$$

□

Remark 2.2. There are three different long-range dependence regimes in property (9), and they are independent of a_1, a_2. The covariance of increments of Z has a power decay for $b_1 + b_2 < 1$, a power growth for $b_1 + b_2 > 1$, and a non-trivial limit for $b_1 + b_2 = 1$. We do not know if this property has been noted before for fractional Brownian sheets. It is worthwhile to observe that the non-Gaussian process $(Y_t^{(1)} Y_{\theta t}^{(2)})_{t\geq 0}$, where $Y^{(i)}$ are independent weighted fractional Brownian motions with parameters (a_i, b_i), $i = 1, 2$, has the same long-range dependence behavior.

In [7] it is shown that $A_{s,t} = \int_s^t u^a (t-u)^b du$, $0 \leq s < t$, has the following bounds: If $a \geq 0$, $s, t \leq T$ for any $T > 0$ and constant $M = M(T)$, and also if $a < 0$, $s, t \geq \varepsilon$ for any $\varepsilon > 0$ and constant $M = M(\varepsilon)$,

$$A_{s,t} \leq M \left| t - s \right|^{1+b}.$$

If $a < 0$, $1 + a + b > 0$, $s, t \geq 0$,

$$A_{s,t} \leq M \left| t - s \right|^{1+a+b}.$$

Then it follows from (2.4) that for $0 < \varepsilon \leq s < s' < T$, $0 < \varepsilon \leq t < t' < T$ and $i = 1, 2$,

$$E((\Delta_{s,t} W_{s',t'})^2) \leq M\, (s'-s)^{\delta_1}\, (t'-t)^{\delta_2}, \tag{2.11}$$

where

$$\delta_i = \begin{cases} 1 + a_i + b_i & \text{if} \quad a_i < 0 \quad \text{and} \quad 1 + a_i + b_i > 0, \\ 1 + b_i & \text{otherwise.} \end{cases} \tag{2.12}$$

The next lemma allows us to prove the continuity of W.

Lemma 2.3. [1], [10] *Let $X = (X_{s,t})_{s,t\geq 0}$ be a two-time stochastic process on a probability space $(\Omega, \mathfrak{F}, P)$ which is null almost surely on the axes and such that there exist $p > 0$, $a, b \in (1/p, \infty)$, such that*

$$(E(|\Delta_{s,t} X_{s+h,t+k}|^p))^{1/p} \leq M |h|^a |k|^b.$$

Then X has a modification $\tilde{X}$ with continuous trajectories. Also, the trajectories of $\tilde{X}$ are Hölder with exponents (a', b'), for $a' \in (0, a - 1/p)$, $b' \in (0, b - 1/p)$, that is, for any $\omega \in \Omega$ exists $M_\omega > 0$ such that for any s, s', t, t',

$$|\Delta_{s,t} \tilde{X}_{s',t'}(\omega)| \leq M_\omega (s'-s)^{a'} (t'-t)^{b'}, \qquad s < s',\ t < t'.$$

Proposition 2.4. *The weighted fractional Brownian sheet $(W_{s,t})_{s,t\geq 0}$ has a modification $(\tilde{W}_{s,t})_{s,t\geq 0}$ with continuous trajectories. Also, the trajectories of $\tilde{W}$ are Hölder with exponents (x, y) for any $x \in (0, \frac{1}{2}\delta_1), y \in (0, \frac{1}{2}\delta_2)$, where δ_i are as in (2.12).*

Proof. From the moments of the normal distribution and equations (2.4) and (2.11) we have

$$\begin{aligned} &(E(|\Delta_{s,t} W_{s+h,t+k}|^r))^{1/r} \\ &= C\Big(\int_s^{s+h} u^{a_1} (s+h-u)^{b_1} du \int_t^{t+k} v^{a_2} (t+k-v)^{b_2} dv \Big)^{1/2} \\ &\leq M h^{\delta_1/2} k^{\delta_2/2}, \end{aligned}$$

with some constants C and M. Taking $r > \max\{2/\delta_1, 2/\delta_2\}$ we have the conditions of Lemma 2.3, and the result follows. □

3. Approximation

The random field W, for some values of the parameters a_i, b_i, arises as a limit in distribution of occupation time fluctuations of a system of particles of two types that move as pairs in $\mathbb{R}\times\mathbb{R}$ according to independent stable Lévy processes. The system is described as follows. Given a Poisson random measure on $\mathbb{R}\times\mathbb{R}$ with intensity measure μ, $N_{0,0} = \mathrm{Pois}(\mu)$, from each point (x_1, x_2) of $N_{0,0}$ come out two independent Lévy processes, from x_1 comes out ξ^{x_1}, symmetric α_1-stable, and from x_2 comes out ζ^{x_2}, symmetric α_2-stable ($0 < \alpha_i \le 2$, $i = 1, 2$). Let $N = (N_{u,v})_{u,v\ge 0}$ denote random measure process on $\mathbb{R}\times\mathbb{R}$ such that $N_{u,v}$ represents the configuration of particles at time (u, v),

$$N_{u,v} = \sum_{(x_1,x_2)\in N_{0,0}} \delta_{(\xi_u^{x_1},\zeta_v^{x_2})} = \sum_{(x_1,x_2)\in N_{0,0}} \delta_{\xi_u^{x_1}} \otimes \delta_{\zeta_v^{x_2}}. \tag{3.1}$$

For $\varphi, \psi \in L^1(\mathbb{R})$ ($\varphi, \psi \neq 0$) fixed, we write

$$\langle N_{u,v}, \varphi\otimes\psi\rangle = \sum_{(x_1,x_2)\in N_{0,0}} \langle \delta_{\xi_u^{x_1}} \otimes \delta_{\zeta_v^{x_2}}, \varphi\otimes\psi\rangle = \sum_{(x_1,x_2)\in N_{0,0}} \varphi(\xi_u^{x_1})\psi(\zeta_v^{x_2}). \tag{3.2}$$

We define the occupation time process of N by

$$\langle L_{s,t}, \varphi\otimes\psi\rangle = \int_0^s\int_0^t \langle N_{u,v}, \varphi\otimes\psi\rangle\, dvdu, \qquad s, t \ge 0, \tag{3.3}$$

and the rescaled occupation time fluctuation process by

$$X_T(s,t) = \frac{1}{F_T}\left(\langle L_{Ts,Tt}, \varphi\otimes\psi\rangle - E(\langle L_{Ts,Tt}, \varphi\otimes\psi\rangle)\right), \qquad s, t \ge 0, \tag{3.4}$$

where T is the time scaling and F_T is a norming. We choose the intensity measure μ for the Poisson initial particle configuration as

$$\mu(dx_1, dx_2) = \mu_1 \otimes \mu_2(dx_1, dx_2) = \mu_1(dx_1)\mu_2(dx_2),$$

with

$$\mu_i(dx_i) = dx_i/\left|x_i\right|^{\gamma_i},\ 0 \le \gamma_i < 1,\ i = 1, 2. \tag{3.5}$$

The homogeneous case corresponds to $\gamma_1 = \gamma_2 = 0$ and it gives rise to the usual fractional Brownian sheet. We will show that for

$$F_T = F_T^{(1)}F_T^{(2)} \text{ with } F_T^{(i)} = T^{1-(1+\gamma_i)/2\alpha_i},\ 0 \le \gamma_i < 1 < \alpha_i,\ i = 1, 2, \tag{3.6}$$

the finite-dimensional distributions of the process X_T converge in law as $T \to \infty$ to those of weighted fractional Brownian sheet with parameters $a_i = -\gamma_i/\alpha_i$, $b_i = 1 - 1/\alpha_i$, $i = 1, 2$. In the case $a_1 = a_2 = 0$ we will also prove tightness.

Theorem 3.1. *If X_T is the process defined in (3.4), $0 \le \gamma_i < 1 < \alpha_i$, $i = 1, 2$, with F_T defined by (3.6), then the finite-dimensional distributions of X_T converge as $T \to \infty$ to the finite-dimensional distributions of DW, where W is weighted fractional Brownian sheet with parameters $a_1 = -\gamma_1/\alpha_1, b_1 = 1 - 1/\alpha_1$, $a_2 = -\gamma_2/\alpha_2$, $b_2 = 1 - 1/\alpha_2$, and D is the constant*

$$D = \int_{\mathbb{R}}\varphi(x)dx\int_{\mathbb{R}}\psi(x)dx\left(\prod_{i=1}^{2}\frac{1}{1-1/\alpha_i}p_1^{\alpha_i}(0)\left(\int_{\mathbb{R}}\frac{p_1^{\alpha_i}(x)}{|x|^{\gamma_i}}dx\right)\right)^{1/2}, \tag{3.7}$$

where $p_t^\alpha(x)$ is the density of the symmetric α-stable Lévy process, which is given by

$$p_t^\alpha(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \exp\{-(ixy + t|y|^\alpha)\}\, dy.$$

Proof. For each $k \in \mathbb{N}$, $d_1, \cdots, d_k \in \mathbb{R}$ and $(s_1, t_1), \cdots, (s_k, t_k) \in \mathbb{R}_+^2$, we must show that

$$\sum_{j=1}^k d_j X_{s_j,t_j}^T \quad \text{converges in law to} \quad \sum_{j=1}^k d_j W_{s_j,t_j} \quad \text{as } T \to \infty,$$

which we do by proving convergence of the corresponding characteristic functions. From the fact that $N_{0,0} = \mathrm{Pois}(\mu_1 \otimes \mu_2)$, we have for each $\theta \in \mathbb{R}$,

$$\begin{aligned}
&C_T(\theta) := E\exp\Big\{i\theta\sum_{j=1}^k d_j X_{s_j,t_j}^T\Big\} \\
&= \exp\Big\{-\frac{i\theta}{F_T}\sum_{j=1}^k d_j E(\langle L_{s_j,t_j}^T, \varphi\otimes\psi\rangle)\Big\} \\
&\quad\times \exp\Big\{-\int_{\mathbb{R}\times\mathbb{R}}\Big[1 - E_{(x_1,x_2)}\Big(\exp\Big\{\frac{i\theta}{F_T}\sum_{j=1}^k d_j\langle L_{s_j,t_j}^T, \varphi\otimes\psi\rangle\Big\}\Big)\Big]\mu_1(dx_1)\mu_2(dx_2)\Big\},
\end{aligned} \tag{3.8}$$

where $E_{(x_1,x_2)}$ denotes expectation starting with one pair of initial particles in (x_1, x_2), (see e.g., [11], mixed Poisson process).

We also need the mean and the covariance of N. From the Poisson initial condition, the first and second moments are given by

$$\begin{aligned}
E(\langle N_{u,v}, \varphi\otimes\psi\rangle) &= \int_{\mathbb{R}\times\mathbb{R}} E_{(x_1,x_2)}(\langle N_{u,v}, \varphi\otimes\psi\rangle)\,\mu_1(dx_1)\mu_2(dx_2) \\
&= \int_{\mathbb{R}\times\mathbb{R}} E(\varphi(\xi_u^{x_1})\psi(\zeta_v^{x_2}))\,\mu_1(dx_1)\mu_2(dx_2)
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
&E(\langle N_{u_1,v_1}, \varphi\otimes\psi\rangle\langle N_{u_2,v_2}, \varphi\otimes\psi\rangle) \\
&= \int_{\mathbb{R}\times\mathbb{R}} E_{(x_1,x_2)}(\langle N_{u_1,v_1}, \varphi\otimes\psi\rangle\langle N_{u_2,v_2}, \varphi\otimes\psi\rangle)\,\mu_1(dx_1)\mu_2(dx_2) \\
&\quad+ \int_{\mathbb{R}\times\mathbb{R}} E_{(x_1,x_2)}(\langle N_{u_1,v_1}, \varphi\otimes\psi\rangle)\,\mu_1(dx_1)\mu_2(dx_2) \\
&\quad\times \int_{\mathbb{R}\times\mathbb{R}} E_{(x_1,x_2)}(\langle N_{u_2,v_2}, \varphi\otimes\psi\rangle)\,\mu_1(dx_1)\mu_2(dx_2) \\
&= \int_{\mathbb{R}\times\mathbb{R}} E\big(\varphi(\xi_{u_1}^{x_1})\psi(\zeta_{v_1}^{x_2})\varphi(\xi_{u_2}^{x_1})\psi(\zeta_{v_2}^{x_2})\big)\,\mu_1(dx_1)\mu_2(dx_2) \\
&\quad+ \int_{\mathbb{R}\times\mathbb{R}} E\big(\varphi(\xi_{u_1}^{x_1})\psi(\zeta_{v_1}^{x_2})\big)\,\mu_1(dx_1)\mu_2(dx_2)\int_{\mathbb{R}\times\mathbb{R}} E\big(\varphi(\xi_{u_2}^{x_1})\psi(\zeta_{v_2}^{x_2})\big)\,\mu_1(dx_1)\mu_2(dx_2),
\end{aligned}$$

hence, by the independence of ξ and ζ, and the Markov property,

$$\begin{aligned}
&Cov\left(\langle N_{u_1,v_1},\varphi\otimes\psi\rangle,\langle N_{u_2,v_2},\varphi\otimes\psi\rangle\right)\\
&=\int_{\mathbb{R}\times\mathbb{R}}E_{(x_1,x_2)}\left(\langle N_{u_1,v_1},\varphi\otimes\psi\rangle\langle N_{u_2,v_2},\varphi\otimes\psi\rangle\right)\mu_1(dx_1)\mu_2(dx_2)\\
&=\int_{\mathbb{R}}\mathcal{T}^{\alpha_1}_{u_1\wedge u_2}(\varphi\mathcal{T}^{\alpha_1}_{|u_1-u_2|}\varphi)(x_1)\mu_1(dx_1)\int_{\mathbb{R}}\mathcal{T}^{\alpha_2}_{v_1\wedge v_2}(\psi\mathcal{T}^{\alpha_2}_{|v_1-v_2|}\psi)(x_2)\mu_2(dx_2),\quad(3.10)
\end{aligned}$$

where $\mathcal{T}^{\alpha_i}_t$ denotes the semigroup of the symmetric α_i-stable process.

Using an expansion of the characteristic function (see e.g., [5], p. 297) in the integrand with respect to (x_1,x_2) in (3.8), it is equal to

$$1+\frac{i\theta}{F_T}E_{(x_1,x_2)}\left(\sum_{j=1}^k d_j\langle L^T_{s_j,t_j},\varphi\otimes\psi\rangle\right)-\frac{\theta^2}{2F_T^2}E_{(x_1,x_2)}\left(\sum_{j=1}^k d_j\langle L^T_{s_j,t_j},\varphi\otimes\psi\rangle\right)^2+\delta^T_{(x_1,x_2)},$$

where

$$|\delta^T_{(x_1,x_2)}|\leq\frac{\theta^3}{F_T^3}E_{(x_1,x_2)}\left(\sum_{j=1}^k d_j\langle L^T_{s_j,t_j},\varphi\otimes\psi\rangle\right)^3.\quad(3.11)$$

Since

$$\sum_{j=1}^k d_jE(\langle L^T_{s_j,t_j},\varphi\otimes\psi\rangle)=\int_{\mathbb{R}\times\mathbb{R}}\sum_{j=1}^k d_jE_{(x_1,x_2)}(\langle L^T_{s_j,t_j},\varphi\otimes\psi\rangle)\mu_1(dx_1)\mu_2(dx_2),$$

then (3.8) becomes

$$C_T(\theta)=\exp\Bigg\{-\int_{\mathbb{R}\times\mathbb{R}}\Bigg[\frac{\theta^2}{2F_T^2}E_{(x_1,x_2)}\left(\sum_{j=1}^k d_j\langle L^T_{s_j,t_j},\varphi\otimes\psi\rangle\right)^2+\delta^T_{(x_1,x_2)}\Bigg]\mu_1(dx_1)\mu_2(dx_2)\Bigg\}\quad(3.12)$$

and by (3.3) and a previous calculation,

$$\begin{aligned}
&\int_{\mathbb{R}\times\mathbb{R}}\frac{1}{F_T^2}E_{(x_1,x_2)}\left(\sum_{j=1}^k d_j\langle L^T_{s,t},\varphi\otimes\psi\rangle\right)^2\mu_1(dx_1)\mu_2(dx_2)\\
=&\frac{1}{F_T^2}\sum_{j=1}^k d_j\sum_{j'=1}^k d_{j'}\int_{\mathbb{R}\times\mathbb{R}}\int_0^{Ts_j}\int_0^{Tt_j}\int_0^{Ts_{j'}}\int_0^{Tt_{j'}}\\
&E_{(x_1,x_2)}\left(\langle N_{u_1,v_1},\varphi\otimes\psi\rangle\langle N_{u_2,v_2},\varphi\otimes\psi\rangle\right)dv_2du_2dv_1du_1\mu_1(dx_1)\mu_2(dx_2)\\
=&\sum_{j=1}^k d_j\sum_{j'=1}^k d_{j'}\frac{1}{F_T^{(1)2}}\int_{\mathbb{R}}\int_0^{Ts_j}\int_0^{Ts_{j'}}\mathcal{T}^{\alpha_1}_{u_1\wedge u_2}(\varphi\mathcal{T}^{\alpha_1}_{|u_1-u_2|}\varphi)(x_1)du_2du_1\frac{dx_1}{|x_1|^{\gamma_1}}\\
&\times\frac{1}{F_T^{(2)2}}\int_{\mathbb{R}}\int_0^{Tt_j}\int_0^{Tt_{j'}}\mathcal{T}^{\alpha_2}_{v_1\wedge v_2}(\psi\mathcal{T}^{\alpha_2}_{|v_1-v_2|}\psi)(x_2)dv_2dv_1\frac{dx_2}{|x_2|^{\gamma_2}}.\quad(3.13)
\end{aligned}$$

Now, recalling (3.6) we have

$$\frac{1}{\left(T^{1-(1+\gamma)/2\alpha}\right)^2}\int_{\mathbb{R}}\int_0^{Ts_1}\int_0^{Ts_2}\mathcal{T}^{\alpha}_{u_1\wedge u_2}(\varphi\mathcal{T}^{\alpha}_{|u_1-u_2|}\varphi)(x)du_2du_1\frac{dx}{|x|^{\gamma}}$$
$$=\frac{1}{\left(T^{1-(1+\gamma)/2\alpha}\right)^2}\int_{\mathbb{R}}\int_0^{Ts_1}\int_0^{Ts_2}\int_{\mathbb{R}}p^{\alpha}_{u_1\wedge u_2}(x-y)\varphi(y)$$
$$\times\int_{\mathbb{R}}p^{\alpha}_{|u_1-u_2|}(y-z)\varphi(z)dzdydu_2du_1\frac{dx}{|x|^{\gamma}},$$

substituting $u_1=Tu_1', u_2=Tu_2'$, using the self-similarity of the α-stable process in $\mathbb{R}$, i.e., $p^{\alpha}_t(x)=t^{-1/\alpha}p^{\alpha}_1(t^{-1/\alpha}x)$, and then substituting $x=(T(u_1'\wedge u_2'))^{1/\alpha}x'$,

$$=T^{(1+\gamma)/\alpha}\int_{\mathbb{R}}\int_0^{s_1}\int_0^{s_2}\int_{\mathbb{R}}p^{\alpha}_{T(u_1'\wedge u_2')}(x-y)\varphi(y)$$
$$\times\int_{\mathbb{R}}p^{\alpha}_{T|u_1'-u_2'|}(y-z)\varphi(z)dzdydu_2'du_1'\frac{dx}{|x|^{\gamma}}$$
$$=T^{(\gamma-1)/\alpha}\int_{\mathbb{R}}\int_0^{s_1}\int_0^{s_2}(u_1'\wedge u_2')^{-1/\alpha}|u_1'-u_2'|^{-1/\alpha}$$
$$\times\int_{\mathbb{R}}p^{\alpha}_1\left((T(u_1'\wedge u_2'))^{-1/\alpha}(x-y)\right)\varphi(y)$$
$$\times\int_{\mathbb{R}}p^{\alpha}_1\left((T|u_1'-u_2'|)^{-1/\alpha}(y-z)\right)\varphi(z)dzdydu_2'du_1'\frac{dx}{|x|^{\gamma}}$$
$$=\int_{\mathbb{R}}\int_0^{s_1}\int_0^{s_2}(u_1'\wedge u_2')^{-\gamma/\alpha}|u_1'-u_2'|^{-1/\alpha}\int_{\mathbb{R}}p^{\alpha}_1\left((x'-(T(u_1'\wedge u_2'))^{-1/\alpha}y)\right)\varphi(y)$$
$$\times\int_{\mathbb{R}}p^{\alpha}_1\left((T|u_1'-u_2'|)^{-1/\alpha}(y-z)\right)\varphi(z)dzdydu_2'du_1'\frac{dx'}{|x'|^{\gamma}}. \tag{3.14}$$

Taking $T\to\infty$ in (3.14) we obtain the limit

$$p^{\alpha}_1(0)\left(\int_{\mathbb{R}}\varphi(y)dy\right)^2\int_{\mathbb{R}}\frac{p^{\alpha}_1(x)}{|x|^{\gamma}}dx\int_0^{s_1}\int_0^{s_2}(u_1'\wedge u_2')^{-\gamma/\alpha}|u_1'-u_2'|^{-1/\alpha}du_2'du_1'$$
$$=p^{\alpha}_1(0)\left(\int_{\mathbb{R}}\varphi(y)dy\right)^2\int_{\mathbb{R}}\frac{p^{\alpha}_1(x)}{|x|^{\gamma}}dx$$
$$\times\frac{1}{1-1/\alpha}\int_0^{s_1\wedge s_2}u^{-\gamma/\alpha}\left[(s_1-u)^{1-1/\alpha}+(s_2-u)^{1-1/\alpha}\right]du. \tag{3.15}$$

By (3.13), (3.14) and (3.15),

$$\lim_{T\to\infty}\frac{1}{F_T^2}\int_{\mathbb{R}\times\mathbb{R}}E_{(x_1,x_2)}\left(\sum_{j=1}^k d_j\langle L^T_{s_j,t_j};\varphi\otimes\psi\rangle\right)^2\mu_1(dx_1)\mu_2(dx_2)$$
$$=\frac{p^{\alpha_1}_1(0)}{1-1/\alpha_1}\frac{p^{\alpha_2}_1(0)}{1-1/\alpha_2}\left(\int_{\mathbb{R}}\varphi(y)dy\right)^2\left(\int_{\mathbb{R}}\psi(y)dy\right)^2\int_{\mathbb{R}}\frac{p^{\alpha_1}_1(x)}{|x|^{\gamma_1}}dx\int_{\mathbb{R}}\frac{p^{\alpha_2}_1(x)}{|x|^{\gamma_2}}dx$$

$$\times \sum_{j,j'=1}^{k} d_j d'_j \int_0^{s_j \wedge s_{j'}} u^{-\gamma_1/\alpha_1} [(s_j-u)^{1-1/\alpha_1} + (s_{j'}-u)^{1-1/\alpha_1}] du$$
$$\times \int_0^{t_j \wedge t_{j'}} v^{-\gamma_2/\alpha_2} [(t_j-v)^{1-1/\alpha_2} + (t_{j'}-v)^{1-1/\alpha_2}] dv$$
$$= D^2 \sum_{j,j'=1}^{k} d_j d'_j C^{(1)}(s_j, s_{j'}) C^{(2)}(t_j, t_{j'}), \tag{3.16}$$

where D is defined by (3.7) and $C^{(i)}$ is as in (1.2) with $a_i = -\gamma_i/\alpha_i, b_i = 1-1/\alpha_i$.

Proceeding similary with the third order term we find

$$\int_{\mathbb{R}\times\mathbb{R}} \frac{1}{F_T^3} E_{(x_1,x_2)} \left(\sum_{j=1}^{k} d_j \langle L_{s_j,t_j}, \varphi \otimes \psi \rangle \right)^3 \mu_1(dx_1)\mu_2(dx_2)$$
$$= \frac{1}{F_T^3} \sum_{i=1}^{k} d_i \sum_{j=1}^{k} d_j \sum_{l=1}^{k} d_l \int_{\mathbb{R}\times\mathbb{R}} E_{(x_1,x_2)} (\langle L^T_{s_i,t_i}, \varphi\otimes\psi\rangle \langle L^T_{s_j,t_j}, \varphi\otimes\psi\rangle$$
$$\times \langle L^T_{s_l,t_l}, \varphi\otimes\psi\rangle) \mu_1(dx_1)\mu_2(dx_2)$$
$$= \frac{1}{F_T^3} \sum_{i=1}^{k} d_i \sum_{j=1}^{k} d_j \sum_{l=1}^{k} d_l \int_{\mathbb{R}\times\mathbb{R}} \int_0^{Ts_i} \int_0^{Tt_i} \int_0^{Ts_j} \int_0^{Tt_j} \int_0^{Ts_l} \int_0^{Tt_l}$$
$$E_{(x_1,x_2)} (\langle N_{u_i,v_i}, \varphi\otimes\psi\rangle \langle N_{u_2,v_2}, \varphi\otimes\psi\rangle \langle N_{u_3,v_3}, \varphi\otimes\psi\rangle)$$
$$dv_3 du_3 dv_2 du_2 dv_1 du_1 \mu_1(dx_1)\mu_2(dx_2)$$
$$= \sum_{i=1}^{k} d_i \sum_{j=1}^{k} d_j \sum_{l=1}^{k} d_l \frac{1}{F_T^{(1)3}} \int_{\mathbb{R}} \int_0^{Ts_i} \int_0^{Ts_j} \int_0^{Ts_l} \mathcal{T}^{\alpha_1}_{\tilde u_1} \varphi(\mathcal{T}^{\alpha_1}_{\tilde u_2 - \tilde u_1} \varphi(\mathcal{T}^{\alpha_1}_{\tilde u_3 - \tilde u_2} \varphi))(x_1)$$
$$d\tilde u_3 d\tilde u_2 d\tilde u_1 \frac{dx_1}{|x_1|^{\gamma_1}}$$
$$\times \frac{1}{F_T^{(2)3}} \int_{\mathbb{R}} \int_0^{Tt_i} \int_0^{Tt_j} \int_0^{Tt_l} \mathcal{T}^{\alpha_2}_{\tilde v_1} \psi(\mathcal{T}^{\alpha_2}_{\tilde v_2 - \tilde v_1} \psi(\mathcal{T}^{\alpha_2}_{\tilde v_3 - \tilde v_2} \psi))(x_2) d\tilde v_3 d\tilde v_2 d\tilde v_1 \frac{dx_2}{|x_2|^{\gamma_2}}, \tag{3.17}$$

$\tilde u_1, \tilde u_2, \tilde u_3$ denoting u_1, u_2, u_3 in increasing order, and similarly for $\tilde v_1, \tilde v_2, \tilde v_3$.

Again, recalling (3.6), substituting $\tilde u_i = T\tilde u'_i$ $i = 1,2,3$, using self-similarity of the α-stable process, and then substituting $x = (T\tilde u'_1)^{1/\alpha} x'$, we have

$$\frac{1}{\left(T^{1-(1+\gamma)/2\alpha}\right)^3} \int_{\mathbb{R}} \int_0^{Ts_1} \int_0^{Ts_2} \int_0^{Ts_3} \mathcal{T}^{\alpha}_{\tilde u_1} \varphi(\mathcal{T}^{\alpha}_{\tilde u_2 - \tilde u_1} \varphi(\mathcal{T}^{\alpha}_{\tilde u_3 - \tilde u_2} \varphi))(x) d\tilde u_3 d\tilde u_2 d\tilde u_1 \frac{dx}{|x|^{\gamma}}$$
$$= T^{(\gamma-1)/2\alpha} \int_{\mathbb{R}} \int_0^{s_1} \int_0^{s_2} \int_0^{s_3} \int_{\mathbb{R}} \tilde u_1^{-\gamma/\alpha} (\tilde u_2 - \tilde u_1)^{-1/\alpha} (\tilde u_3 - \tilde u_2)^{-1/\alpha}$$
$$\times \int_{\mathbb{R}} p_1^{\alpha}(x' - (T\tilde u_1)^{-1/\alpha} w) \varphi(w) \int_{\mathbb{R}} p_1^{\alpha}((T(\tilde u_2 - \tilde u_1))^{-1/\alpha}(w-y)) \varphi(y)$$
$$\times \int_{\mathbb{R}} p_1^{\alpha}((T(\tilde u_3 - \tilde u_2))^{-1/\alpha}(y-z)) \varphi(z) dz dy dw d\tilde u_3 d\tilde u_2 d\tilde u_1 \frac{dx'}{|x'|^{\gamma}}, \tag{3.18}$$

then from (3.17) and (3.18),

$$\frac{1}{F_T^3}\int_{\mathbb{R}\times\mathbb{R}} E_{(x_1,x_2)}\left(\sum_{j=1}^k d_j\langle L^T_{s_j,t_j},\varphi\otimes\psi\rangle\right)^3\mu_1(dx_1)\mu_2(dx_2)$$
$$=\prod_{i=1}^2 T^{(\gamma_i-1)/2\alpha_i}\int_{\mathbb{R}\times\mathbb{R}} A_T(x_1,x_2)\mu_1(dx_1)\mu_2(dx_2), \tag{3.19}$$

where

$$A_T(x_1,x_2)=\int_0^{s_1}\int_0^{s_2}\int_0^{s_3}\tilde{u}_1^{-\gamma_1/\alpha_1}(\tilde{u}_2-\tilde{u}_1)^{-1/\alpha_1}(\tilde{u}_3-\tilde{u}_2)^{-1/\alpha_1}$$
$$\times\int_{\mathbb{R}} p_1^{\alpha_1}(x_1-(T\tilde{u}_1)^{-1/\alpha_1}w)\varphi(w)\int_{\mathbb{R}} p_1^{\alpha_1}((T(\tilde{u}_2-\tilde{u}_1))^{-1/\alpha_1}(w-y))\varphi(y)$$
$$\times\int_{\mathbb{R}} p_1^{\alpha_1}((T(\tilde{u}_3-\tilde{u}_2))^{-1/\alpha_1}(y-z))\varphi(z)dzdydwdu_3du_2du_1$$
$$\times\int_0^{t_1}\int_0^{t_2}\int_0^{t_3}\int_{\mathbb{R}}\tilde{v}_1^{-\gamma_2/\alpha_2}(\tilde{v}_2-\tilde{v}_1)^{-1/\alpha_2}(\tilde{v}_3-\tilde{v}_2)^{-1/\alpha_2}$$
$$\times\int_{\mathbb{R}} p_1^{\alpha_2}(x_2-(T\tilde{v}_1)^{-1/\alpha_2}w)\psi(w)\int_{\mathbb{R}} p_1^{\alpha_2}((T(\tilde{v}_2-\tilde{v}_1))^{-1/\alpha_2}(w-y))\psi(y)$$
$$\times\int_{\mathbb{R}} p_1^{\alpha_2}((T(\tilde{v}_3-\tilde{v}_2))^{-1/\alpha_2}(y-z))\psi(z)dzdydwdv_3dv_2dv_1. \tag{3.20}$$

From (3.20) we obtain

$$\lim_{T\to\infty}\int_{\mathbb{R}\times\mathbb{R}} A_T(x_1,x_2)\mu_1(dx_1)\mu_2(dx_2)$$
$$=\int_0^{s_1}\int_0^{s_2}\int_0^{s_3}\tilde{u}_1^{-\gamma_1/\alpha_1}(\tilde{u}_2-\tilde{u}_1)^{-1/\alpha_1}(\tilde{u}_3-\tilde{u}_2)^{-1/\alpha_1}du_3du_2du_1$$
$$\times\int_0^{t_1}\int_0^{t_2}\int_0^{t_3}\tilde{v}_1^{-\gamma_2/\alpha_2}(\tilde{v}_2-\tilde{v}_1)^{-1/\alpha_2}(\tilde{v}_3-\tilde{v}_2)^{-1/\alpha_2}dv_3dv_2dv_1$$
$$\times\left(\int_{\mathbb{R}}\varphi(x)dx\right)^3\left(\int_{\mathbb{R}}\psi(x)dx\right)^3\prod_{i=1}^2(p_1^{\alpha_i}(0))^2\int_{\mathbb{R}}\frac{p_1^{\alpha_i}(x)}{|x|^{\gamma_i}}dx. \tag{3.21}$$

Then, from (3.11), (3.19) and (3.21) we get

$$\lim_{T\to\infty}\int_{\mathbb{R}\times\mathbb{R}}\delta^T_{(x_1,x_2)}\mu_1(dx_1)\mu_2(dx_2)=0. \tag{3.22}$$

Finally, putting (3.12), (3.16) and (3.22) together we obtain

$$\lim_{T\to\infty} C_T(\theta)=\exp\left\{-\frac{\theta^2}{2}D^2\sum_{j=1}^k\sum_{j'=1}^k d_jd_{j'}C^{(1)}(s_j,s_{j'})C^{(2)}(t_j,t_{j'})\right\},$$

and convergence of finite-dimensional distributions of X_T to finite-dimensional distributions of weighted fractional Brownian sheet DW has been proved. □

Theorem 3.2. *Under the hypotheses of Theorem 3.1, if $\gamma_1=\gamma_2=0$, then X_T converges in law to DW in the space of continuous functions $C([0,\tau]\times[0,\tau],\mathbb{R})$*

for any $\tau > 0$ as $T \to \infty$, where W is fractional Brownian sheet with parameters $(1 - \frac{1}{2\alpha_1}, 1 - \frac{1}{2\alpha_2})$, and

$$D = \int_{\mathbb{R}} \varphi(x)dx \int_{\mathbb{R}} \psi(x)dx \left[\prod_{i=1}^{2} \frac{1}{1 - 1/\alpha_i} p_i^{\alpha_i}(0)\right]^{1/2}.$$

Proof. By Theorem 3.1 we have convergence of finite-dimensional distributions of X_T to those of DW. It remains to show that the family $\{X_T\}$ is tight. Since these processes are null on the axes, by the Bickel-Wichura theorem [4] we only need prove that there exist even $m \geq 2$ and positive constants C_m, δ_1, δ_2 such that $m\delta_1, m\delta_2 > 1$ and

$$\sup_T E\left((\Delta_{s_1,t_1} X_T(s_2,t_2))^m\right) \leq C_m(s_2 - s_1)^{m\delta_1}(t_2 - t_1)^{m\delta_2}, \quad \text{for all } s_1 < s_2, t_1 < t_2. \tag{3.23}$$

From (3.4),

$$\Delta_{s_1,t_1} X_T(s_2,t_2) = \frac{1}{F_T} \int_{Ts_1}^{Ts_2} \int_{Tt_1}^{Tt_2} \left(\langle N_{u,v}, \varphi \otimes \psi\rangle - E(\langle N_{u,v}, \varphi \otimes \psi\rangle)\right) dudv,$$

then, by (3.10),

$$\begin{aligned}
&E((\Delta_{s_1,t_1} X_T(s_2,t_2))^2)\\
&= \frac{1}{F_T^2} \int_{Ts_1}^{Ts_2} \int_{Tt_1}^{Tt_2} \int_{Ts_1}^{Ts_2} \int_{Tt_1}^{Tt_2} Cov\left(\langle N_{u_1,v_1}, \varphi \otimes \psi\rangle, \langle N_{u_2,v_2}, \varphi \otimes \psi\rangle\right) dv_2 du_2 dv_1 du_1\\
&= \frac{1}{F_T^2} \int_{Ts_1}^{Ts_2} \int_{Tt_1}^{Tt_2} \int_{Ts_1}^{Ts_2} \int_{Tt_1}^{Tt_2} \int_{\mathbb{R}} \mathcal{T}^{\alpha_1}_{u_1 \wedge u_2}(\varphi \mathcal{T}^{\alpha_1}_{|u_1 - u_2|}\varphi)(x_1)dx_1\\
&\qquad \times \int_{\mathbb{R}} \mathcal{T}^{\alpha_2}_{v_1 \wedge v_2}(\varphi \mathcal{T}^{\alpha_2}_{|v_1 - v_2|}\varphi)(x_2)dx_2 dv_2 du_2 dv_1 du_1\\
&= \frac{1}{T^{2-(1+\gamma_1)/\alpha_1}} \int_{Ts_1}^{Ts_2} \int_{Ts_1}^{Ts_2} \int_{\mathbb{R}} \mathcal{T}^{\alpha_1}_{u_1 \wedge u_2}(\varphi \mathcal{T}^{\alpha_1}_{|u_1 - u_2|}\varphi)(x_1)dx_1 du_2 du_1\\
&\quad \times \frac{1}{T^{2-(1+\gamma_2)/\alpha_2}} \int_{Tt_1}^{Tt_2} \int_{Tt_1}^{Tt_2} \int_{\mathbb{R}} \mathcal{T}^{\alpha_2}_{v_1 \wedge v_2}(\varphi \mathcal{T}^{\alpha_2}_{|v_1 - v_2|}\varphi)(x_2)dx_2 dv_2 dv_1\\
&= E(\langle X_T^{(1)}(s_2) - X_T^{(1)}(s_1), \varphi\rangle^2) E(\langle X_T^{(2)}(t_2) - X_T^{(2)}(t_1), \psi\rangle^2),
\end{aligned} \tag{3.24}$$

where $X_T^{(i)}, i = 1, 2$, are occupation time fluctuation processes of independent systems of particles moving in $\mathbb{R}$ according to symmetric α_i-stable processes with initial configurations given by Poisson random measures on $\mathbb{R}$ with intensities μ_i, i.e.,

$$X_T^{(1)}(t) = \frac{1}{T^{1-1/2\alpha_1}} \int_0^{Tt} (\langle N_u^{(1)}, \varphi\rangle - E(\langle N_u^{(1)}, \varphi\rangle))du, \qquad N_u^{(1)} = \sum_{x \in \mathrm{Pois}(\mu_1)} \delta_{\xi_u^x},$$

and

$$X_T^{(2)}(t) = \frac{1}{T^{1-1/2\alpha_2}} \int_0^{Tt} (\langle N_u^{(2)}, \varphi\rangle - E(\langle N_u^{(2)}, \varphi\rangle))du, \qquad N_u^{(2)} = \sum_{x \in \mathrm{Pois}(\mu_2)} \delta_{\zeta_u^x}.$$

In [6] such a one-time system is studied and it is shown that

$$E(\langle X_T^{(i)}(t) - X_T^{(i)}(s), \varphi\rangle^2) \leq C|t-s|^h, \tag{3.25}$$

where C is a positive constant (not depending on T) and $h = 2 - \frac{1}{\alpha_i} > 1$. From (3.24) and (3.25) we obtain (3.23). □

Remark 3.3. We make some remarks below.

(1) Theorem 3.2 gives a functional approximation of fractional Brownian sheet with parameters $(h_1, h_2) \in (1/2, 3/4]^2$, taking $h_i = 1 - \frac{1}{2\alpha_i}$, $i = 1, 2$.
(2) Proving tightness with $\gamma_1 \neq 0$ or $\gamma_2 \neq 0$ is considerably more difficult because it requires computing moments of arbitrarily high order (see [8] for the one time case), and this involves moments of arbitrarily high order of the Poisson random measure $\text{Pois}(\mu_1 \otimes \mu_2)$, which are cumbersome.
(3) In Theorem 3.1 we may also consider the measures μ_i of the form (3.5) with $\gamma_i < 0$, assuming that $|\gamma_i| < \alpha_i$ if $\alpha_i < 2$ (which implies that the mean is finite), and the result in the theorem holds.
(4) The role of the functions φ, ψ is only subsidiary since they are fixed, and in the occupation time fluctuation limit they appear only in the constant D given by (3.7). If φ, ψ are taken as variables in the space $\mathcal{S}(\mathbb{R})$ of smooth rapidly decreasing functions, then in principle it is possible to prove convergence of the occupation time fluctuations as $(\mathcal{S}'(\mathbb{R}))^2$-valued processes, where $\mathcal{S}'(\mathbb{R})$ is the space of tempered distributions (topological dual of $\mathcal{S}(\mathbb{R})$), the limit being the space-time random field

$$(Z_{s,t})_{s,t\geq 0} = K(\lambda \otimes \lambda)\,(W_{s,t})_{s,t\geq 0}\,,$$

where W is the weighted fractional Brownian sheet in Theorem 3.1,

$$K = \left(\prod_{i=1}^{2} \frac{1}{1-1/\alpha_i} p_1^{\alpha_i}(0) \int_{\mathbb{R}} \frac{p_1^{\alpha_i}(x)}{|x|^{\gamma_i}} dx\right)^{1/2},$$

and λ is the Lebesgue measure on $\mathbb{R}$. (See [8] for such a setup for a one-time particle system.)

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