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SCS 82: Compactifying Distributive Lattices

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In this SCS-memo, we will discuss some more properties of the Zariski-topology. Recall that the closed sets of the Zariski-topology on a distributive lattice is generated by sets of the form

$$[a \wedge x \leq b] = \{x \in L \mid a \wedge x \leq b\}$$
$$[a \vee x \geq b] = \{x \in L \mid a \vee x \geq b\}$$

(see [GS 83] for details). As a main result, we shall show that lattices which are Hausdorff in the Z-topology (= Zariski topology) allow a compactification which to a large extend has the same properties as the one-point compactification of a locally compact Hausdorff space, i.e. a compactification which is minimal in a certain sense.

Some notations: A distributive complete lattice L which is both meet-continuous and join-continuous will be called *infinitely distributive.* If L is a complete lattice and if $M \subseteq L$ is a sublattice such that L is the smallest complete sublattice of L containing M, then M is called *dense* in L. Note that for compact lattices this notions of denseness agrees with the topological notion. Whenever L is a distributive lattice and we refer to a topological property of L, then this property has to be interpreted in the Z-topology. Note that for infinitely distributive lattices the Z-topology and the interval topology agree.

1. The Main Result

1.1. Theorem. Let L be a distributive lattice and assume that L is Hausdorff in the Z-topology. Then there is a completely distributive lattice $\rho(L)$ and a (topological and algebraical) dense embedding $i : L \to \rho(L)$ such that whenever $f : L \to M$ is a lattice embedding into an infinitely distributive lattice M such that $f(L)$ is dense in M then there is a unique complete lattice homomorphism $g : M \to \rho(L)$ satisfying $g \circ f = i$. Moreover, the lattice $\rho(L)$ is uniquely determined by these properties.

Remark. It will turn out that we require the assumption that f be a lattice homomorphism at only one place, namely in the proof of Proposition (1.4) below.

However, this proposition is trivial in the case where L itself is completely distributive and hence $L = \rho(L)$. In this case it is enough to postulate that the mapping $f : L \to M$ is an order embedding (i.e. we have $x \le y$ iff $f(x) \le f(y)$).

We will split the proof of (1.1) into several small pieces. First of all, we have to define the lattice $\rho(L)$ and the embedding $i : L \to \rho(L)$. Let $B(L)$ be the maximal essential extension of L (see [BB 68] and [GS 82] for the details). We know that $B(L)$ is a complete Boolean algebra. In fact, $B(L)$ may be identified with the subset of all pseudocomplements in the congruence lattice of L (note that the congruence lattice is meet-continuous and distributive; hence may be thought of as a Heyting algebra).

1.2. Definition. Define

$$i : L \to B(L)$$

by

$$i(x) = \theta_x = \{(a,b) \in L^2 \mid a \vee x = b \vee x\}$$

and let $\rho(L)$ be the smallest complete sublattice of $B(L)$ containing $i(L)$. $\Box$

It was shown in [GS 82] and [GS 83] that i is indeed a lattice embedding, that i is a topological embedding for the Z-topologies, and that $\rho(L)$ is completely distributive. If we let

$$\theta^x = \{(a,b) \in L^2 \mid a \wedge x = b \wedge x\}$$

then θ^x is the complement of θ_x in $B(L)$.

1.3. Proposition. If N and M are subsets of L, then $\inf i(N) \le \sup i(M)$ if and only if the congruences θ_x, $x \in N$, and θ^y, $y \in M$, intersect in the diagonal Δ (i.e. in the least congruence).

Proof. The proof of Proposition 3 is an easy application of de Morgan's law: We have $\inf i(N) \le \sup i(M)$ if and only if $\inf i(N) \wedge (\sup i(M))^{\perp} = \Delta$, where $^{\perp}$ of course denotes complements in $B(L)$. Now the assertion follows from

$$\begin{aligned}(\sup i(M))^{\perp} &= \inf\{i(y)^{\perp} \mid y \in M\} \\ &= \inf\{\theta_y^{\perp} \mid y \in M\} \\ &= \inf\{\theta^y \mid y \in M\}. \Box\end{aligned}$$

From now on we fix a dense embedding $f : L \to M$, where M is an infinitely distributive lattice.

1.4. Proposition. If N and M are subsets of L and if $\inf f(N) \leq \sup f(M)$, then $\inf i(N) \leq \sup i(M)$.

Proof. Assume the inequality $\inf i(N) \leq \sup i(M)$ is not true. Then applying Proposition (1.3) we can find a pair of distinct elements $(a, b) \in L^2$ which belong to θ_x for all $x \in N$ and to θ^y for all $y \in M$. Hence we have

$$\begin{aligned} a \vee x &= b \vee x && \text{if } x \in N, \\ a \wedge y &= b \wedge y && \text{if } y \in M. \end{aligned}$$

Now let $l = \inf f(M)$ and let $r = \sup f(N)$. Since f is a lattice embedding, $f(a)$ and $f(b)$ are still distinct. Moreover, since M is infinitely distributive, we conclude that

$$\begin{aligned} f(a) \vee l &= f(b) \vee l, \\ f(a) \wedge r &= f(b) \wedge r. \end{aligned}$$

Since by assumption $l \leq r$, the latter of these two equations implies

$$\begin{aligned} f(a) \wedge l &= f(a) \wedge r \wedge l \\ &= f(b) \wedge r \wedge l \\ &= f(b) \wedge l. \end{aligned}$$

Now these equations tell us that both $f(a)$ and $f(b)$ are relative complements of l. In a distributive lattice relative complements are unique, so we may conclude that $f(a) = f(b)$ contradicting the injectivity of f. ▯

Let us agree on the following notations:

Spec $\rho(L)$ denotes the set of all prime elements of $\rho(L)$. Prime elements (and only those) are denoted by the letters p and q.

Cospec $\rho(L)$ denotes the set of all coprimes of $\rho(L)$. Coprimes (and only those) are denoted by the letters c and d.

We write $x \gg y$ if x is way above y, i.e. if a down directed set has infimum less than or equal to y, then one of the elements of the down directed set in question is actually below x. Note that this is not the same as $y \ll x$!

We recall the following facts from [La 79]:

(*) For every element in $p \in \mathit{Spec}\, \rho(L)$ the set $\{q \mid q \gg p, q \text{ a prime element}\}$ is down directed and has infimum p.

(**) Whenever p and q are primes such that $q \gg p$ then there is a prime element $r \in \mathit{Spec}\, \rho(L)$ such that $q \gg r \gg p$

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Furthermore, since $\rho(L)$ is a completely distributive lattice, we have

(***) Every element of $\rho(L)$ is an infimum of prime elements and a supremum of coprime elements.

Instead of defining the mapping $g : M \to \rho(L)$ directly, we will construct the upper and lower adjoint of g. We begin this construction by defining a map on the primes of $\rho(L)$ with values in M by

$$\begin{aligned} \tilde{\phi} : Spec\, \rho(L) &\to M \\ p &\mapsto \inf\{\sup\{f(x) \mid i(x) \leq q, x \in L\} \mid q \gg p, q \in Spec\, \rho(L)\}. \end{aligned}$$

Then $\tilde{\phi}$ preserves directed infima: Indeed, let $A \subseteq \rho(L)$ be a filtered set of primes and assume that $p = \inf A$. We have to show that

$$\tilde{\phi}(p) = \inf \tilde{\phi}(A).$$

Clearly, since $\tilde{\phi}$ is monotone, we have $\tilde{\phi}(p) \leq \inf \tilde{\phi}(A)$. In order to verify the other inequality, we prove the following claim:

(C) For every $q \gg p$ there is an element $r \in A$ such that

$$\sup\{f(x) \mid i(x) \leq q, x \in L\} \geq \tilde{\phi}(r).$$

Indeed, pick $r \in A$ such that $q \gg r \geq p$, which is possible by the interpolation property for $\gg$. Then we have

$$\begin{aligned} \sup\{f(x) \mid i(x) \leq q, x \in L\} &\leq \inf\{\sup\{f(x) \mid i(x) \leq q', x \in L\} \mid q' \gg r\} \\ &= \tilde{\phi}(r). \end{aligned}$$

Now from (C) we conclude that

$$\begin{aligned} \tilde{\phi}(p) &= \inf\{\sup\{f(x) \mid i(x) \leq q, x \in L\} \mid q \gg p\} \\ &\geq \inf\{\tilde{\phi}(r) \mid r \in A\} \\ &= \inf \tilde{\phi}(A). \end{aligned}$$

In the next step, we define a map

$$\begin{aligned} \phi : \rho(L) &\to M \\ u &\mapsto \inf\{\tilde{\phi}(p) \mid u \leq p, p \text{ prime }\}. \end{aligned}$$

Obviously, ϕ is monotone. Moreover:

ϕ preserves finite infima.

Indeed, let u and v be two arbitrary elements of $\rho(L)$. Then the monotonicity of ϕ implies $\phi(u \wedge v) \leq \phi(u) \wedge \phi(v)$. Conversely, let p be a prime element above the infimum of u and v. Then p is actually above u or above v, yielding $\{\tilde{\phi}(p) \mid u \wedge v \leq p\} = \{\tilde{\phi}(p) \mid u \leq p\} \cup \{\tilde{\phi}(p) \mid v \leq q\}$. Thus, $\phi(u \wedge v) \geq \phi(u) \wedge \phi(v)$.

Our next claim is:

ϕ preserves down directed infima.

Let A be a down directed subset of $\rho(L)$. The inequality $\phi(\inf A) \leq \inf \phi(A)$ is again obvious. Conversely, let $\inf A \leq p$ and assume that $q \gg p$. We then may pick a prime element p' such that $q \gg p' \gg p$. Since A is down directed and has an infimum less than or equal to p, there is an element $a \in A$ such that $a \leq p'$. We now obtain

$$\begin{aligned} \inf \phi(A) &\leq \phi(a) \\ &\leq \tilde{\phi}(p') \\ &\leq \tilde{\phi}(q). \end{aligned}$$

Since $q \gg p$ was arbitrary, since $\tilde{\phi}$ preserves infima of down directed sets of primes and since $\{q \mid q \gg p\}$ is down directed with infimum p, we conclude that

$$\inf \phi(A) \leq \tilde{\phi}(p).$$

Finally, since p was an arbitrary prime element such that $\inf A \leq p$, we obtain

$$\inf \phi(A) \leq \phi(\inf A)$$

from the definition of ϕ.

We conclude:

*1.5. **Proposition.*** The mapping

$$\phi : \rho(L) \to M$$

preserves arbitrary infima. $\square$

Dually, we may define maps

$$\begin{aligned} &\tilde{\psi} : \mathit{Cospec}\, \rho(L) \to M \\ &\qquad c \mapsto \sup\{\inf\{f(x) \mid d \leq i(x), x \in L\} \mid c \ll d\} \end{aligned}$$

and

$$\begin{aligned}\psi : \rho(L) &\to M\\ u &\mapsto \sup\{\tilde{\psi}(c) \mid c \leq u\}.\end{aligned}$$

1.6. Proposition. The mapping $\psi : \rho(L) \to M$ preserves arbitrary suprema. []

We will now show that the mapping ϕ and ψ are the upper and lower adjoint of a certain mapping $g : M \to \rho(L)$. This mapping g will be the mapping we are looking for.

1.7. Proposition. For given elements $a, b \in \rho(L)$ we have $\psi(a) \leq \phi(b)$ if and only if $a \leq b$.

Proof. Firstly, let us assume that $a \leq b$. We have to show: If c is a coprime and if p is a prime such that $c \leq a \leq b \leq p$, then $\tilde{\psi}(c) \leq \tilde{\phi}(p)$. This is the same as: If $d \ll c \leq a$ and $q \gg p \geq b$ then

$$\inf\{f(x) \mid d \leq i(x), x \in L\} \leq \sup\{f(y) \mid i(y) \leq q, y \in L\}.$$

Now note that the set $\{z \in \rho(L) \mid d \ll z \text{ and } q \gg z\}$ is a neighborhood of the elements $a, b \in \rho(L)$. Since $i(L)$ is dense in $\rho(L)$, this set contains an element of the form $i(x')$, $x' \in L$. Hence there is an element $x' \in L$ such that $d \leq i(x') \leq q$. This yields

$$\begin{aligned}\inf\{f(x) \mid d \leq i(x), x \in L\} &\leq f(x')\\ &\leq \sup\{f(y) \mid i(y) \leq q, y \in L\},\end{aligned}$$

as desired.

Now assume that we are given two elements $a, b \in \rho(L)$ such that $a \not\leq b$. Since primes and coprimes both order generate the lattice $\rho(L)$, we can find primes p, q and coprimes c, d such that

$$\begin{aligned}d \ll c &\leq a,\\ q \gg p &\geq b,\\ d &\not\leq q.\end{aligned}$$

We would like to show that already the inequality

$$\inf\{f(x) \mid d \leq i(x), x \in L\} \leq \sup\{f(y) \mid i(y) \leq q, y \in L\}$$

fails to be true, since this would certainly imply that $\tilde{\psi}(c) \not\leq \tilde{\phi}(p)$, hence $\psi(a) \not\leq \phi(b)$. Thus, let us assume that the above inequality is true. Then Proposition (1.4) would yield

$$\begin{aligned} d &\leq \inf\{i(x) \mid d \leq i(x), x \in L\} \\ &\leq \sup\{i(y) \mid i(y) \leq q, y \in L\} \\ &\leq q, \end{aligned}$$

contradicting the choice of d and q. ▯

1.8. Proposition. The intervals $[\psi(u), \phi(u)]$, $u \in \rho(L)$, are pairwise disjoint; their union is a complete sublattice of M.

Proof. If the intersection of $[\psi(u), \phi(u)]$ and $[\psi(v), \psi(v)]$ would be non-empty, we could conclude that $\psi(u), \psi(v) \leq \phi(u) \wedge \phi(v) = \phi(u \wedge v)$. Now Proposition (1.7) would imply that both u and v are less than or equal to $u \wedge v$, i.e. $u = v$.

Now let A be a subset of $\bigcup_{u \in \rho(L)} [\psi(u), \phi(u)]$. Then for every $a \in A$ there is a unique $\tilde{a} \in L$ such that

$$\psi(\tilde{a}) \leq a \leq \phi(\tilde{a}).$$

Let b be the supremum of A and let $\tilde{b}$ be the supremum of the $\tilde{a}$. Then the fact that ψ preserves suprema and that h is monotone implies

$$\begin{aligned} \psi(\tilde{b}) &= \psi(\sup\{\tilde{a} \mid a \in A\}) \\ &= \sup\{\psi(\tilde{a}) \mid a \in A\} \\ &\leq \sup\{a \mid a \in A\} \\ &= b \\ &\leq \sup\{\phi(\tilde{a}) \mid a \in A\} \\ &\leq \phi(\sup\{\tilde{a} \mid a \in A\}) \\ &= \phi(\tilde{b}). \end{aligned}$$

Hence $\sup A$ belongs to the interval $[\psi(\tilde{b}), \phi(\tilde{b})]$. ▯

1.9. Proposition. For every $x \in L$ we have $\psi(i(x)) \leq f(x) \leq \phi(i(x))$

Proof. For every $x \in L$ we have

$$\phi(i(x)) = \inf\{\tilde{\phi}(p) \mid i(x) \leq p\}.$$

Hence, in order to verify $f(x) \leq \phi(i(x))$ we have to show that $f(x) \leq \tilde{\phi}(p)$ whenever $i(x) \leq p$. By the definition of $\tilde{\phi}$ this is equivalent to

$$f(x) \leq \sup\{f(y) \mid i(y) \leq q, y \in L\}. \tag{1}$$

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whenever $q \gg p$. But if $q \gg p$, then $i(x) \leq p \leq q$, hence $f(x) \in \{f(y) \mid i(y) \leq , y \in L\}$ and therefore the inequality (1) holds trivially.

Similarly, we show that $\psi(i(x)) \leq f(x)$ for every $x \in L$. []

1.10. Proposition. The intervals $[\psi(u), \phi(u)]$, $u \in \rho(L)$, cover the whole lattice M.

Proof. By Proposition (1.9), the union of these intervals contains the image if f, and by Proposition (1.8) this union is a complete sublattice of M. Since the image of L under f is dense in M, i.e., since M is the smallest complete sublattice of M containing $f(L)$, the union of the intervals in question is equal to M. []

We are now ready to define the complete lattice homomorphism $g : M \to \rho(L)$:

$$\begin{aligned} g : M &\to \rho(L) \\ m &\mapsto u \text{ iff } m \in [\psi(u), \phi(u)]. \end{aligned}$$

The same arguments as the one given in the proof of Proposition (1.8) show that g is a complete lattice homomorphism. Finally, Proposition (1.10) shows that $g \circ f = i$. This completes the hard part of the proof of theorem 1.

Note the the unicity of g follows from the fact that $f(L)$ is dense in M. The fact that $\rho(L)$ is uniquely determined by all these properties follows from general categorical nonsense. []

It is now the time to list same consequences of Theorem 1 and the Remark following the statement of this theorem:

1.11. Corollary. Let L be a completely distributive lattice densely embedded (as a partially ordered set) into a infinitely distributive M. Then L is a retract of M under a complete lattice homomorphism. []

1.12. Corollary. Let M be a compact distributive lattice. If M contains a dense sublattice whose $Z - topology$ is Hausdorff, then M admits continuous homomorphisms into the unit interval. []

1.13. Corollary. Let M be a compact distributive lattice. If M contains an order isomorphic copy of a completely distributive lattice which is dense in M, then M admits continuous lattice homomorphism into the unit interval. []

It turns out that completely distributive lattice and distributive lattices whose

Z-topology is Hausdorff are also characterized by all those properties listed in the corollaries:

1.14. Theorem. If L is a lattice such that for every infinitely distributive lattice M and every dense embedding $i : L \to M$ there is a complete lattice homomorphism $f : M \to L$ such that $f \circ i = id_L$, then L is completely distributive.

The proof of this theorem follows from the fact that every distributive lattice admits a dense embedding into a completely distributive lattice and that every quotient of a completely distributive lattice under a complete lattice homomorphism is again completely distributive.[]

Now assume that we are given any lattice L together with a dense embedding $e : L \to L'$, where L' is an infinitely distributive lattice such that for every other dense embedding $f : L \to M$ into a infinitely distributive lattice there is a complete lattice homomorphism $g : M \to L'$ satisfying $g \circ f = e$. Since f is a dense embedding, the mapping g has to be uniquely determined. Let $\rho(L)$ be the "closure" of L in them maximal essential extension $B(L)$ of L (i.e. $\rho(L)$ is again the smallest complete sublattice of $B(L)$ containing the image of $i(L)$). Then L is densely embedded into $\rho(L)$, and, although $\rho(L)$ will not be necessarily completely distributive, it still will be infinitely distributive. (Recall that every complete Boolean algebra and hence every complete sublattice of a complete Boolean algebra is infinitely distributive). It follows that there is a complete lattice homomorphism $g : \rho(L) \to L'$. This mapping g has to be surjective; we would like to show that is also injective. One way to do this is the following: We first of all show that the injection $e : L \to L'$ is an essential embedding. Hence, let Θ be a non-trivial congruence relation on L' and let Λ be the preimage of Θ under the mapping $g : \rho(L) \to L'$. Since g is onto, Λ is a non-trivial congruence relation on $\rho(L)$. Therefore we can find a pair of distinct elements $x, y \in L$ such that $(i(x), i(y) \in \Lambda$. But his implies $(e(x), e(y)) = (g(i(x)), g(i(y))) \in g(\Lambda) = \Theta$. Hence the restriction of Θ to $e(L)$ is non-trivial.

Now, since $B(L)$ is the maximal essential extension of L, there is a mapping $j : L' \to B(L)$ such that

(i) $i = j \circ e$,

(ii) j preserves all infima and all suprema.

The mapping j is the embedding of L' into its maximal essential extension which happens to be identical with $B(L)$. Now j has to be continuous for the interval topologies on L' and $B(L)$ and that is the reason why j preserves all infima and all suprema.

We conclude that $j(L') \subseteq B(L)$ is a complete sublattice containing the image of $i(L)$. Thus, $\rho(L) \subseteq j(L')$. On the other hand, $i(L)$ is dense in both $\rho(L)$ and $j(L')$. This shows $\rho(L) = j(L')$. This shows that j is a bijection between $\rho(L)$ and

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L'.

This argument shows that $\rho(L)$ is the only choice we have in order to proof a result like Theorem (1.1). Now a similar argument as the one given in the proof of Theorem (1.16) yields

1.17. Theorem. Let L be a distributive lattice and suppose that L admits an dense embedding $e : L \to L'$ into an infinitely distributive lattice L' such that whenever $f : L \to M$ is a second such dense embedding into a infinitely distributive lattice M, then there is a (uniquely determined) complete lattice homomorphism $g : M \to L'$ such that $g \circ f = e$. Then, up to a canonical isomorphism, $L' = \rho(L)$ and $e = i$. Moreover, in this case $\rho(L)$ is completely distributive and therefore the Z-topology on L is Hausdorff. []

2. A Characterization of $\rho(L)$ by Means of Closed Filters and Ideals.

In this section we will try to give a description of $\rho(L)$ involving the Z-topology on L. Recall that the embedding of L into $\rho(L)$ is a topological embedding for the Z-topology. We will use this fact later.

2.1. Definition. Let L be a distributive lattice and let I be an ideal of L. If I is closed in the Z-topology of L, then L will be called a ***closed ideal.*** Closed filters are defined correspondingly.[]

Let us record the following result (see [GS 83]):

2.2. Proposition. Let L be a distributive lattice and let $I \subseteq L$ be an ideal of L. Then the following statements are equivalent:

(1) I is closed.

(2) If $D \subseteq I$ is a directed subset and if D converges to an element $x \in L$ in the Z-topology, then $x \in I$.

(3) If $D \subseteq I$ is a directed subset such that $\sup D$ exists in L and such that $a \wedge \sup D = \sup(D \wedge a)$ for every $a \in L$, then $\sup D \in I$.

(4) If $x \in L$ has the property that $x \wedge a = \sup(D \cap \downarrow(x \wedge a))$ for every $a \in L$, then $x \in I$. []

From now on, we will identify L with a subset of $\rho(L)$.

2.3. Proposition. If I and J are two different closed ideals of L, then I and J have different suprema in $\rho(L)$.

Proof. Let I and J be different closed ideals and assume that $\sup I = \sup J$.

Let $j \in J$ be an arbitrary element. Then $\sup_{\rho(L)} I$ is an upper bound of j in $\rho(L)$. Since $\rho(L)$ is infinitely distributive, we conclude that $j = \sup_{\rho(L)}(I \cap \downarrow j)$. Moreover, since the Z-topology on $\rho(L)$ is the interval topology, the directed set $I \cap \downarrow j$ converges to j in the Z-topology on $\rho(L)$. The embedding of L into $\rho(L)$ is topological, therefore $I \cap \downarrow j$ also converges to j in the Z-topology of L. Hence, by property (2) of Proposition (2.2), j belongs to I. This yields $J \subseteq I$. By symmetry, $I = J$. $\square$

Let M be any complete lattice and let A be a subset of M. Then we define

$$\begin{aligned} A^+ &= \{\sup D \mid D \text{ is a up-directed subset of } A\}, \\ A^- &= \{\sup F \mid F \text{ is a down-directed subset of } A\}. \end{aligned}$$

Furthermore, if L is again an arbitrary distributive lattice, we let

$$\begin{aligned} \mathcal{I}_c(L) &= \{I \subseteq L \mid I \text{ is a closed ideal of } L\}, \\ \mathcal{F}_c(L) &= \{F \subseteq L \mid F \text{ is a closed filter of } L\}. \end{aligned}$$

When ordered by inclusion, $\mathcal{I}_c(L)$ and $\mathcal{F}_c(L)$ are complete lattices. The infimum in those lattices agrees with the set theoretical intersection. However, it is not true that the supremum of two closed ideals taken in the lattice $\mathcal{I}_c(L)$ agrees with the supremum of those two ideals taken in the lattice of all ideals, i.e. the supremum in the ideal lattice of two closed ideals is not necessarily closed (see the following example). Therefore, it is not obvious that the lattice of all closed ideals (filters) is again distributive (see Proposition (2.5)).

2.4. Example. Let L be the open unit square enriched by the point (1.1). Then the Z-topology on L is the topology induced by the Euclidean topology of the plane. Let

$$\begin{aligned} I &= \{(x, y) \in L \mid y \leq \frac{1}{2}\}, \\ J &= \{(x, y) \in L \mid x \leq \frac{1}{2}\}. \end{aligned}$$

The the supremum of I and J in the ideal lattice of L contains all the elements of L except the point $(1, 1)$ and hence is not closed.

In the following result, L will again be identified with a subset of $\rho(L)$. Hence, L^+ has to be evaluated in the lattice $\rho(L)$ and not in L. Even in the case where L is a complete lattice, L^+ does not have to agree with L. Now note that the meet continuity of $\rho(L)$ implies that L^+ is a sublattice of $\rho(L)$. Therefore, L^+ will always be a distributive lattice.

2.5. Theorem. The lattice L^+ is isomorphic with $I_c(L)$; and isomorphism between $I_c(L)$ and L^+ is given by the mapping

$$I \mapsto \sup I;$$

its inverse is the mapping

$$x \mapsto \{y \in L \mid y \leq x\}.$$

Especially, $I_c(L)$ is a distributive lattice.

Proof. By Proposition (2.3), the mapping $I \mapsto \sup I$ is injective. If $x \in L^+$, then x is a directed supremum (and hence the supremum of an ideal) of elements of L. Thus, x has to be the supremum of the closed ideal $\{y \in L \mid y \leq x\} = {\downarrow}x \cap L$. Since there is no more than one closed ideal with supremum x, the mapping $x \mapsto \{y \in L \mid y \leq x\}$ is the inverse of $I \mapsto \sup I$. []

Let us remark that in the case where L is a complete meet-continuous lattice, an ideal of L is closed if and only if it is a lower set of a point. In this case L and $I_c(L)$ will be isomorphic and therefore $I_c(L)$ will be meet-continuous, too. This last property remains true in general:

2.4. Proposition. If L is any distributive lattice, then $I_c(L)$ is meet-continuous.

Proof. This follows easily from the fact that $I_c(L)$ is isomorphic with L^+ and that L^+ is closed under finite infima and arbitrary suprema in the complete Boolean algebra $B(L)$. []

Now let us examen the join-continuity of $I_c(L)$: Unfortunately, the embedding $I \mapsto \sup I$ does not preserve arbitrary infima. Especially, $I_c(L)$ will not be join-continuous in general (Consult the open unit square for examples. If L is the open unit square, then $I_c(L) = L^+ = \{(x, y) \in \Re^2 \mid 0 < x, y \leq 1\}$ and this lattice is not join-continuous.) However, what happens if L is a complete join-continuous lattice to begin with? Is it then true that $I_c(L)$ is meet-continuous and join-continuous? We will see in a moment that this is the case, provided that the Z-topology on L is Hausdorff. But this does not solve the problem in general. I could not find any counterexamples. For instance, it seems to me that if L is the lattice of all closed subsets of a sober spaces, then $I_c(L)$ is meet- and join-continuous. This last statement may be completely wrong, because I did not work out all the details.

If it should be true that $I_c(L)$ is join continuous whenever L is join-continuous, then $\rho(L)$ will always be of the form $\rho(L) = L^{+-} = L^{-+}$. There are two indications why this is not too unlikely:

1. If L is a free distributive lattice, then $\rho(L) = L^{+-} = L^{-+}$. Note that the Z-topology on a free distributive lattice is Hausdorff if and only if L is finite.

2. $\mathcal{I}_c(L)$ has something to do with closed sets and the closed sets of any topological space form a join-continuous lattice.

If the Z-topology on L is Hausdorff, then $\rho(L)$ is completely distributive. Since L is order dense in $\rho(L)$, we conclude that $L^{+-} = L^{-+} = \rho(L)$. We may reformulate this in the following way:

2.7. Theorem. If L is a distributive lattice which is Hausdorff in the Z-topology, then $\rho(L)$ is isomorphic with the dual of the lattice $\mathcal{F}_c\mathcal{I}_c(L)$. The canonical embedding of L into this lattice is given by

$$\begin{aligned} i : L &\to \mathcal{F}_c\mathcal{I}_c(L) \\ x &\mapsto \{I \in \mathcal{I}_c(L) \mid x \in I\} \square \end{aligned}$$

This last theorem yields easily that if L is the open unit square, then $\rho(L)$ is the closed unit square.

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