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SCS 81: Intrinsic Topologies on Semilattices of Finite Breadth

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1. Introduction.

The purpose of this paper is to study some of the basic properties of certain intrinsic topologies one may define on a semilattice, particularly those in which every open set is an upper set. Some aspects of such "one-sided" topologies were considered previously in [3, VII-1]. Of particular interest is a topology which we call the "upper Z-topology". It can be defined by taking as a subbasis for the closed sets the solution sets to semilattice polynomial equations (hence it bears a resemblance to Zariski topologies); we give a different (but equivalent) formulation in Section 4. Section 2 contains preliminary material on the notion of locally finite breadth. In Section 3 we define the Z-topology and discuss some of its basic properties. The major topologies are introduced and some basic comparisons obtained in Section 4. Section 5 introduces the dual topology and in Section 6 we show that the Scott topology and the upper Z-topology agree for complete meet-continuous lattices of finite breadth.

2. Distributive Lattices of Locally Finite Breadth.

2.1. Definition. (i) Let L be a lattice and let $A \subseteq L$ be a finite subset. If for every $a \in A$ we have $\inf(A \setminus \{a\}) \not\leq a$, then A is called $\wedge$ - *irredundant*. The term $\vee$ - *irredundant* is defined dually.

(ii) If $x \in L$ is an element, then the $\wedge$-*breadth* of x is defined as

$$\mathrm{br}_{\wedge}(x) = \sup\{|A| \ : \ A \text{ is } \wedge \text{ -irredundant and } \inf A = x\}$$

(iii) if $\mathrm{br}_{\wedge}(x)$ is finite for every $x \in L$, then L is said to have *locally finite* $\wedge$ *-breadth.*

(iv) The $\vee$-*breadth* of an element $x \in L$ as well as the property of having *locally finite* $\vee$ *-breadth* are defined dually.

(v) If L has both locally finite $\wedge$-breadth and locally finite $\vee$-breadth, then L has *locally finite breadth.* []

Clearly, every lattice of finite breadth has locally finite breadth. Examples show that the converse is not true (take a linear sum of the Boolean algebras 2^n, $n \in N$, i.e. a sequence of increasing Boolean algebras piled on top of each other). Moreover, the lattice of finite subsets of N demonstrates that a lattice can have locally finite $\vee$-breadth but infinite $\wedge$-breadth.

For every $x \in L$ we let

$$\mathcal{P}(x) = \{P \subseteq L \ : \ P \text{ is a minimal prime ideal containing } x\}.$$

The following result connects the number of minimal prime ideals containing x with $\mathrm{br}_\wedge(x)$:

2.2. Proposition. Let L be a distributive lattice and let $x \in L$. Then $\mathrm{br}_\wedge(x)$ is finite if and only if $\mathcal{P}(x)$ is finite. In this case we have $\mathrm{br}_\wedge(x) = |\mathcal{P}(x)|$.

Proof. A standard result in lattice theory tells us that

$$\downarrow x = \bigcap \mathcal{P}(x).$$

Assume that $\mathcal{P}(x)$ is finite, say $\mathcal{P}(x) = \{P_1, \ldots, P_n\}$. If A is a finite set with $\inf A = x \in P_1 \cap \ldots \cap P_n$, then for every $1 \leq i \leq n$ there is an element $a_i \in A \cap P_i$. We obtain $x \leq a_1 \wedge \ldots \wedge a_n \in P_1 \cap \ldots \cap P_n = \downarrow x$, i.e. $x = a_1 \wedge \ldots \wedge a_n$. Hence, if A is irredundant, $A = \{a_1, \ldots, a_n\}$ showing that $\mathrm{br}_\wedge(x)$ is finite and that $\mathrm{br}_\wedge(x) \leq |\mathcal{P}(x)|$.

Conversely, let $P_1, \ldots, P_n \in \mathcal{P}(x)$ be n different prime ideals each of which is minimal with respect to containing x. We show that the $\wedge$-breadth at x is at least n. Hence, we have to construct an irredundant set A with $|A| = n$ and $\inf A = x$. Firstly, we show

(C) For every $i \in \{1, \ldots, n\}$ there is an element $x_i \in (P_1 \cap \ldots \cap P_{i-1} \cap P_{i+1} \cap \ldots \cap P_n) \setminus P_i$ such that $x_i \wedge x_j \leq x$ if $i \neq j$.

(Note that this is trivial for the case $\mathcal{P}(x) = \{P_1, \ldots, P_n\}$.)

Our first observation is that $P_1 \cap \ldots \cap P_{i-1} \cap P_{i+1} \cap \ldots \cap P_n \not\subseteq P_i$ for every $1 \leq i \leq n$, since otherwise $P_j \subseteq P_i$ for some $j \neq i$. Because both P_i and P_j are minimal with respect to containing x, this implies $P_i = P_j$, contradicting $i \neq j$.

Now pick $x_1 \in (P_2 \cap \ldots \cap P_n) \setminus P_1$ arbitrarily and assume that

$$x_i \in (P_1 \cap \ldots P_{i-1} \cap P_{i+1} \cap \ldots \cap P_n) \setminus P_i$$

is already chosen so that $x_i \wedge x_j \leq x$ for all $j < i$. We then will assure the existence of x_{i+1} in the following manner: Let

$$y = x_1 \vee \ldots \vee x_i$$

and assume that $z \wedge y \not\leq x$ whenever $z \in (P_1 \cap \ldots \cap P_i \cap P_{i+2} \cap \ldots \cap P_n) \setminus P_{i+1}$. Then, since

$$\begin{aligned}(P_1 \cap \ldots \cap P_i \cap P_{i+2} \cap \ldots \cap P_n) \setminus P_{i+1} &= \\ &= (P_1 \cap \ldots \cap P_i \cap P_{i+2} \cap \ldots \cap P_n) \cap (L \setminus P_{i+1})\end{aligned}$$

is an ideal of the filter $L \setminus P_{i+1}$, the filter G generated by $L \setminus P_{i+1}$ and $\uparrow y$ is equal to the filter generated by $(P_1 \cap \ldots \cap P_i \cap P_{i+2} \cap \ldots \cap P_n) \setminus P_{i+1}$ and $\uparrow y$ and hence does not contain x. Pick a prime ideal P such that $x \in P$ and $P \cap G = \emptyset$.

Since $L \setminus P_{i+1} \subseteq G$, this implies $P \cap (L \setminus P_{i+1}) = \emptyset$, i.e. $P \subseteq P_{i+1}$. Since P_{i+1} was a minimal prime ideal containing x, $P = P_{i+1}$. Now note that $x_j \in P_{i+1}$ for every $j \leq i$ and hence $y \in P_{i+1} = P$ contradicting $P \cap \uparrow y \subseteq P \cap G = \emptyset$.

Hence we conclude that there is an element

$$x_{i+1} \in (P_1 \cap \ldots \cap P_i \cap P_{i+2} \cap \ldots \cap P_n) \setminus P_{i+1}$$

such that

$$\begin{aligned} y \wedge x_{i+1} &= (x_1 \vee \ldots \vee x_i) \wedge x_{i+1} \\ &= (x_1 \wedge x_{i+1}) \vee \ldots \vee (x_i \wedge x_{i+1}) \\ &\leq x, \end{aligned}$$

i.e., $x_j \wedge x_{i+1} \leq x$ for all $j \leq i$.

Now let

$$y_j = x_j \vee x, \quad 1 \leq j \leq n.$$

Since $x \in P_j$ but $x_j \notin P_j$, we obtain $x_j \vee x \neq x$. Moreover, if $i \neq j$ then $y_i \wedge y_j = (x_i \vee x) \wedge (x_j \vee x) = (x_i \wedge x_j) \vee x = x$. Finally, let

$$a_i = y \vee \ldots \vee y_{i-1} \vee y_{i+1} \vee \ldots \vee y_n, \quad 1 \leq i \leq n.$$

Using the distributive law, we obtain

$$\begin{aligned} a_1 \wedge \ldots \wedge a_{i-1} \wedge a_{i+1} \wedge \ldots \wedge a_n &= y_i \\ a_1 \wedge \ldots \wedge a_n &= x \end{aligned}$$

Hence $\{a_1, \ldots, a_n\}$ is an irredundant set of cardinality n with infimum x. ▯

3

The next proposition tell us more about the nature of minimal prime ideals:

2.3. Proposition. Let L be a distributive lattice, let $x \in L$ an assume than $\mathrm{br}_{\wedge}(x)$ is finite. If $P \subseteq L$ is a prime ideal of L which is minimal with respect to containing x, then there exists an element $y \in L$ such that

$$P = \{z \in L \ : \ z \wedge y \leq x\}.$$

Proof. From (2.2) we know that $\mathcal{P}(x)$ is finite, say $\mathcal{P}(x) = \{P_1, \ldots, P_n\}$. Without loss of generality we assume that $P = P_1$. Since $P_2 \cap \ldots \cap P_n \not\subseteq P_1$, we may pick an element $y \in (P_2 \cap \ldots \cap P_n) \setminus P_1$. With this element y we obtain:

$$P = \{z \in L \ : \ z \wedge y \leq x\}.$$

Indeed, if $z \wedge y \leq x \in P$, then $z \in P$ since P is a prime ideal and $y \notin P$. Conversely, if $z \in P$, then $z \wedge y \in P \cap (P_2 \cap \ldots \cap P_n) = P_1 \cap \ldots \cap P_n = \downarrow x$, i.e. $z \wedge y \leq x$. []

2.4. Theorem. Let L be a meet-continuous up-complete distributive lattice and let $x \in L$. Then $\mathrm{br}_{\wedge}(x) = n < \infty$ iff there are pairwise incomparable prime elements $p_1, \ldots, p_n \in L$ such that $x = p_1 \wedge \ldots \wedge p_n$.

Proof. Standard arguments in lattice theory show: If $x = p_1 \wedge \ldots \wedge p_n$, where the p_i are pairwise incomparable prime elements, then $\mathrm{br}_{\wedge}(x) = n$.

Conversely, suppose that $\mathrm{br}_{\wedge}(x) = n < \infty$ and let $\mathcal{P}(x)$ be again the minimal prime ideals of L containing x. From (2.2) and (2.3) we know that there are elements $y_1, \ldots, y_n \in L$ such that $\mathcal{P}(x) = \{P_1, \ldots, P_n\}$ where $P_i = \{z \ : \ z \wedge y_i \leq x\}$. Let $p_i = \sup P_i$. Since L is meet-continuous, we obtain

$$\begin{aligned} p_i \wedge y_i &= (\sup P_i) \wedge y_i \\ &= \sup(P_i \wedge y_i) \\ &= \sup\{z \wedge y_i \ : \ z \wedge y_i \leq x\} \\ &\leq x, \end{aligned}$$

i.e. $p_i \in P_i$ and hence $P_i = \downarrow p_i$. Thus, p_i is a prime element and

$$x \leq p_1 \wedge \ldots \wedge p_n \in P_1 \cap \ldots \cap P_n = \downarrow x. []$$

2.5. Corollary. Let L be a meet-continuous an up-complete distributive lattice. Then L has locally finite $\wedge$-breadth if and only if every element of L is an infimum of finitely many prime elements. ▯

3. The Zariski Topology for Semilattices.

Before we continue out study of semilattices of finite breadth, we would like to introduce a topology on semilattices which up to now has not received much attention, namely the Zariski topology. As in the case of rings, the Zariski topology is the coarsest topology for which solution sets of semilattice polynomials in one variable are closed. For semilattices, we have three different types of polynomials in one variable, namely

$$p(x) = a$$
$$p(x) = a \wedge x$$
$$p(x) = x$$

Let us introduce some notations for solution set of equations: If p and q are two semilattice polynomials, then we let

$$[p(x) = q(x)] = \{x \in L \,:\, p(x) = q(x)\}$$
$$[p(x) \leq q(x)] = \{x \in L \,:\, p(x) \leq q(x)\}$$

3.1. Definition. The *Zariski topology* (or Z-topology for short) on a semilattice L is the coarsest topology for which all sets of the form $[p(x) = q(x)]$, p and q semilattice polynomials, are closed. ▯

Since we have only 3 different types of semilattice polynomials, taking into account all possible combinations of those we obtain the following 4 different "typical" closed sets:

$$[a \wedge x = b \wedge x] = \{x \in L : x \wedge x = b \wedge x\}$$
$$[a \wedge x = b] = \{x \in L \,:\, x \wedge a = b\}$$
$$[x = a] = \{a\}$$
$$[a \wedge x = x] = \{x \in L : a \wedge x = x\}$$

Obviously, $[a \wedge x = b \wedge x]$ is a lower set and we have $[a \wedge x = a] = \uparrow a$ and $[a \wedge x = x] = \downarrow a$. Moreover:

3.2. Proposition. On every semilattice L the sets $[a \wedge x \leq b]$ and $\uparrow a$, $a, b \in L$ form a subbase for the closed sets of the Z-topology.

Proof. We have $[a \wedge x \leq b] = [a \wedge x = b \wedge a \wedge x]$ and $[a \wedge x = a] = \uparrow a$, hence these sets will generated a topology which is coarser than or equal to the Z-topology.

Conversely, we have to show that the 4 "typical" Z-closed sets mentioned above are also closed in the topology generated by sets of the form $[a \wedge x \leq b]$ and $\uparrow a$. But this follows from the following 4 equalities:

$$[a \wedge x = b \wedge x] = [a \wedge x \leq b] \cap [b \wedge x \leq a]$$

$$[a \wedge x = b] = \begin{cases} [a \wedge x \leq b] \cap \uparrow b & \text{if } a \geq b \\ \emptyset & \text{if } a \ngeq b \end{cases}$$

$$\downarrow a = [a \wedge x = x] = \bigcap_{c \notin \downarrow a} [x \wedge c \leq a]$$

$$[x = a] = \downarrow a \cap \uparrow a. \quad \square$$

The following result is immediate from (3.2):

3.3. Proposition. If L is a Heyting algebra (i.e., a relatively pseudocomplemented distributive lattice), then the Z-topology is equal to the interval topology $\square$

It is clear that for meet-continuous, up-complete semilattices the Zariski-topology is equal to or coarser than the CL-topology (for the definition of the CL-topology see [3]). However, in general the CL-topology on a semilattice will be strictly finer than the Z-topology. In order to demonstrate this fact, take any distributive continuous lattice which is not completely distributive. In this case, the CL-topology will be Hausdorff and hence strictly finer than the interval topology. (Recall that the interval topology on a distributive complete lattice is Hausdorff if and only if the lattice is completely distributive.) However, on a meet-continuous distributive lattice the Z-topology is equal to the interval topology by (3.3).

We will show later on that on lattices of finite breadth the CL-topology and the Z-topology actually agree on meet-continuous semilattices (see the results in section 6). At this point we would also like to mention that every semilattice is a semi-topological semilattice in the Z-topology, i.e., the mapping $\wedge : L \times L \to L$ is separately continuous. As a matter of fact, the Z-topology is the coarsest topology with this property containing the interval topology (see also Proposition (5.1)).

There is also a very canonical way to define the Z-topology on a lattice by using lattice polynomials instead of semilattice polynomials (and of course, to invent similar definitions for universal algebras). The Z-topology for distributive lattices was discussed in [GS 83].

4. Comparing Topologies on Lattices of Locally Finite Breadth.

Normally, one is interested in topologies on lattices which are Hausdorff or compact. Thus, these topologies should have sufficiently many open upper sets and open lower sets. However, it is often more convenient and very fruitful to consider just one 'half' of these topologies, namely either the one given by all open upper sets or the one given by all open lower sets. For instance, the Scott topology is just one half of the CL-topology, but, of course, this Scott-topology is also important in its own right.

We introduce four topologies on an $\wedge$-semilattice L, which are the 'upper halfs' of other better known topologies:

The upper topology: A subbase for the closed sets in the upper topology is given by sets of the form $\downarrow x$, $x \in L$. (This is the upper half of the interval topology.)

The upper Z-topology: A subbase for the closed sets is given by sets of the form $[b \wedge x \leq a]$ and $\downarrow a$, $a, b \in L$.

The upper Frink-topology: This time, we define a topology on L be specifying a subbase for the open sets: Let $x \in L$ be given and let F be a filter maximal with respect to not containing x. Then F is a subbasic open set.

The Scott-topology: A set $U \subseteq L$ is said to be Scott-open, if for every directed set $D \subseteq L$ we have $\sup D \in U$ if and only if $D \cap U \neq \emptyset$.

The upper topology and the Scott-topology are discussed in detail in [3]. The upper Frink-topology is a one sided version of Frink's ideal topology (see [2]); as a matter of fact, Frink's ideal topology is the supremum of the upper Frink-topology and the lower Frink-topology (defined dually).

4.1. Examples. (i) If $L = [0,1]^n$, $n \in N$, then all four topologies agree. A subset $A \subseteq L$ is closed in any of those topologies if and only if A is a lower set which is closed in the usual Euclidean topology.

(ii) If $L = 2^N$, then the upper topology, the upper Z-topology and the Scott topology agree (closed sets are again lower sets which are closed in the product topology) whereas the upper Frink-topology is much finer, since every ultrafilter of the Boolean algebra 2^N is open in the latter topology.

(iii) If $L = \Re \times \Re$, then the upper Z-topology, the upper Fink-topology and the Scott-topology agree, whereas the upper topology is coarser: $\{(x, y) : x > 0\}$ is open in the first three topologies but is not open in the upper topology. []

Further examples show that all four topologies are distinct in general.

4.2. Proposition. Let L be a $\wedge$-semilattice. Then the upper topology is equal to or coarser than all the other three topologies and the upper Z-topology is contained in the upper Frink-topology.

Proof. Only the last claim is worth verifying: Let $a, b \in L$ and let $x \notin [b \wedge x \leq a]$, i.e., $b \wedge x \not\leq a$. Pick a maximal filter F containing $b \wedge x$ but missing a. Then $x \in F$ and $F \cap [b \wedge x \leq a] = \emptyset$. []

If L is a distributive lattice, then the complements of maximal filter missing $x \in L$ are exactly the minimal prime ideals containing x. Hence (2.3) yields:

4.3. Proposition. If L is a distributive lattice having locally finite $\wedge$-breadth, then the upper Z-topology and the upper Fink-topology agree. []

It is obvious that for meet-continuous distributive lattices the upper topology and the upper Z-topology agree. Moreover, using (2.4) we obtain the "if" part of

4.4. Proposition. Let L be a meet-continuous up-complete distributive lattice. Then the upper topology and the upper Frink-topology on L agree if and only if L has locally finite $\wedge$-breadth.

Proof. Assume that the upper topology and the upper Frink-topology on L agree and let $x \in L$. We would like to show that x is the infimum of finitely many prime elements. Firstly, note that every minimal prime ideal containing x is closed in the upper topology and hence in the Scott-topology. Thus $p = \sup P \in P$ and p is a prime element. Now let F be the filter generated by $\{\sup P : P \in \mathcal{P}(x)\}$. If $x \in F$, then x is the infimum of finitely many prime elements. Hence, assume that $x \notin F$. Then we can pick a minimal prime ideal P containing x such that $P \cap F = \emptyset$. Again, $\sup P \in P$ and, by the construction of F, $\sup P \in F$, a contradiction. []

4.5. Theorem. Let L be a distributive complete lattice. Then the following statements are equivalent:

(I) The Frink topology is compact.
(II) L is a compact topological lattice in the Frink-topology.
(III) L has locally finite breadth and is infinitely distributive.
(IV) L has locally finite breadth and is completely distributive.
(V) Every element $x \in L$ is an infimum of finitely many primes and a supremum of finitely many coprimes.

Proof. (I) $\Rightarrow$ (II): Every distributive lattice is a topological lattice in the Frink-topology.

(I) $\Rightarrow$ (II): In a compact topological lattice L an ideal P is closed if and only if it is of the form $\downarrow x$ for some $x \in L$. Hence, if L is a compact topological lattice in the Frink-topology, then the lower Frink-topology is contained in (and hence equal to) the lower topology. Since every compact distributive lattice is infinitely distributive, (II) implies (III) by (4.4).

(III) $\Rightarrow$ (V) follows from (2.4).

(V) $\Rightarrow$ (IV): Firstly, note that L has locally finite breadth by (2.4). Moreover, every lattice which is generated by primes in the sense that every element is an infimum of prime element is meet-continuous (see [3, I.3.13]). Hence L is infinitely distributive. It follows from [4, 1.7] and [5, 4.7] that L is Hausdorff in the topology generated by the sets of the form $[a \wedge x \leq b]$ and $[c \vee x \geq d]$. Since we already know that L is infinitely distributive, this topology is the interval topology. Hence, L is a Hausdorff space in the interval topology and therefore completely distributive.

(IV) $\Rightarrow$ (I): Since every completely distributive lattice L is compact in the interval topology, it is enough to verify that the interval topology and the Frink-topology agree. But this follows immediately from (4.4). []

Using results from M. Mislove and J. Luikkonen (see [7, Corollary 2.2], [6, Proposition 2.1]), we may add another equivalent condition to (4.5), namely the following:

(VI) L is a compact topological lattice and for every closed subset $A \subseteq L$ there is a finite subset $F \subseteq A$ such that $\inf A = \inf F$ and $\sup A = \sup F$.

5. The Dual Topology.

Let P be a poset. The lower topology on P has a subbase for the open sets all sets of the form $P \setminus \uparrow x$, $x \in P$. By the Alexander subbasis theorem a lower set A is quasicompact in the lower topology if and only if every subbasis open cover has a finite subcover. This translates to the following: A lower set A is quasicompact in the lower topology if and only if the following condition holds:

(Q) If $\emptyset \neq B \subseteq A$ has the property that every finite subset F of B has an upper bound in A, then B has an upper bound in A.

We take the lower sets with this property, i.e., the lower sets quasicompact in the lower topology, as a subbasis for the closed sets of a new topology, which we call the *dual topology*[*].

5.1. Proposition. Let P be a poset. If U is open in the dual topology, then it is Scott-open. If P is an up-complete poset with the property that $a, b \leq x$ implies that $\sup\{a, b\}$ exists, then the dual topology agrees with the Scott-topology.

Proof. Let A be a subbasic closed set in the dual topology, and let $D \subseteq A$ be a directed set with $x = \sup D$. Since each finite subset of D is bounded above in D (and hence in A), there exists an upper bound y of D in A. Thus $x = \sup D \leq y$. Since A is a lower set, $x \in A$. Thus A is Scott-closed. It follows that every set open in the dual topology is Scott-open.

Conversely, suppose A is Scott-closed, and that $\emptyset \neq B \subseteq A$ has the property that every finite subset F of B has an upper bound in A. Then by hypothesis $\sup F$ exists for all finite subsets $F \subseteq B$. Since $A = \downarrow A$, $\sup F \in A$. Thus $D = \{\sup F \ : \ F \text{ finite }, F \subseteq B\}$ is a directed subset of A. Since A is Scott-closed and P is up-complete, $\sup D$ exists and is in A. But $B \subseteq \downarrow \sup D$, so A is closed in the dual topology. []

6. The Upper Z-Topology.

In this section we develop some basic properties of the upper Z-topology and compare it with other semilattice topologies (see Section 4 for the definitions).

6.1. Proposition. Let S be a semilattice. The lower Z-topology on S is the coarsest topology containing the lower topology for which the meet operation is separately continuous.

Proof. Let $A = [b \wedge x \leq a]$. Fix $s \in S$ and let $B = \{y \ : \ s \wedge y \in A\}$. If $y \in B$, then $(s \wedge y) \wedge b \leq a$. Thus $y \in [(b \wedge s) \wedge x \leq a]$. Conversely, if $y \in [(b \wedge s) \wedge x \leq a]$, then $s \wedge b \wedge y \leq a$, i.e. $b \wedge (s \wedge y) \leq a$, and thus $x \wedge y \in A$. Hence $y \in B$. We conclude $B = [(b \wedge s) \wedge x \leq a]$. This argument shows that the inverse image under translation by s of a subbasic closed set in the upper Z-topology is closed. Thus translation by s is continuous. Since $s \in S$ was arbitrary, the meet operation is separately continuous.

Let $\mathcal{M}$ be the collection of closed sets for some topology on S containing the lower topology and making the meet operation separately continuous. Then $\downarrow a \in \mathcal{M}$ for each $a \in S$. The inverse image of $\downarrow a$ under translation by $b \in S$ is equal to $[b \wedge x \leq a]$, hence closed. Thus $\mathcal{M}$ contains all sets which are closed in the upper

[*]This is the cocompact topology for the lower topology; see [3, p.312].

Z-topology since it contains the subbasis. ▯

6.2. Corollary. Let S be an up-complete semilattice for which the meet operation is meet-continuous. Then the upper Z-topology is contained in the Scott-topology.

Proof. It is immediate to verify that translations are continuous in the Scott-topology under the given hypothesis. The corollary then follows from Proposition (6.1). ▯

The main result of this section depends on the following lemma:

6.3. Lemma. Let S be a semilattice of finite breadth and assume that S is quasicompact in the lower topology. Then S can be written as a finite union of sets of the form $\downarrow a$ and $[a \wedge x \leq b]$, $a, b \in S$, $a \not\leq b$.

Proof. The proof proceeds by induction on n, the breadth of S.

If $n = 1$, then S is a chain. If S fails to have a largest element 1, then $\{S \setminus \uparrow x : x \in S\}$ is an open cover of S in the lower topology without a finite subcover. Thus S must have a 1; i.e. $S = \downarrow 1$.

Assume the lemma is true for all semilattices which have breadth less than n. We then want to verify the result for all semilattices S which have breadth equal to n. For every non-empty finite set $F \subseteq S$, we define

$$A_F = \downarrow F \cup \bigcup \{[\inf(G \setminus \{g\} \wedge x \leq \inf G] : g \in G \subseteq F, \\ |G| = n, G \text{ is irredundant } \}$$

If for any F, $A_F = S$, then the proof is complete. Hence we assume for every finite set F, there exists $y_F \in S \setminus A_F$.

Let $G \subseteq F$ be an irredundant subset, $|G| = n$. Since $y_F \notin \downarrow F$, we conclude that $y_F \notin G$, and therefore the set $H = G \cup \{y_F\}$ has $n + 1$ elements. Thus H is redundant since S has breadth n. Pick $z \in H$ such that

$$\inf H = \inf(H \setminus \{z\}).$$

If $z \in G$, then

$$\begin{aligned} y_F \wedge \inf G &= \inf H \\ &= \inf(H \setminus \{z\}) \\ &= y_F \wedge \inf(G \setminus \{z\}) \end{aligned}$$

It follows that

$$\inf G \geq y_F \wedge \inf(G \setminus \{z\}),$$

and hence

$$y_F \in [(\inf G \setminus \{z\}) \wedge x \leq \inf G] \subseteq A_F,$$

an impossibility. We conclude that

$$z = y_F.$$

Thus $y_F \wedge \inf G = \inf G$, i.e., $\inf G \leq y_F$.

Consider the set

$$B = \{\inf G \ : \ |G| = n, G \text{ is irredundant }\}.$$

We claim that B has an upper bound. Indeed, if this would be not the case, then $\{S \setminus \uparrow b \ : \ b \in B\}$ would be an open cover of S in the lower topology. Since S is quasicompact, there exist $b_1, \ldots, b_n \in B$ such that $S = \bigcup_{i=1}^n S \setminus \uparrow b_i$. Let $F = \bigcup_{i=1}^n G_i$ where $|G_i| = n$, G_i is irredundant, $\inf G_i = b_i$. By the preceding paragraph $\inf G_i \leq y_F$ for all i. Hence $y_F \notin \bigcup_{i=1}^n S \setminus \uparrow b_i$, a contradiction. Thus B has some upper bound, call it u. Then u has the following property:

($*$) For every irredundant set G with $|G| = n$ we have $\inf G \leq u$.

We now repeat the preceding argument with a slight variation. For F finite, let

$$\begin{aligned} D_F = {\downarrow} F \cup \bigcup \{[\inf G \wedge x \leq u] \ : \ G \subseteq F, |G| = n-1, G \text{ irredundant }, \inf G \not\leq u\} \\ \cup \bigcup \{[\inf G \wedge x \leq \inf(G \setminus \{g\})] \ : \ g \in G \subseteq F, |G| = n-1, \\ G \text{ irredundant }, \inf G \not\leq u\} \end{aligned}$$

Again if $D_F = S$ for some finite set F, then the proof is complete. Thus we assume that for every finite set F there exists $z_F \in S \setminus D_F$. Let $G \subseteq F$ be irredundant, $|G| = n-1$, and $\inf G \not\leq u$. As before $|G \cup \{z_F\}| = n$ since $z_F \notin {\downarrow} F$. If $G \cup \{z_F\}$ were irredundant, then

$$\inf(G \cup \{z_F\}) = z_F \wedge \inf G \leq u$$

by ($*$), and hence

$$z_F \in [\inf G \wedge x \leq u] \subseteq D_F,$$

an impossibility. Hence $G \cup \{z_F\}$ is redundant. Thus

$$z_F \wedge \inf G = \inf(G \cup \{z_F\} \setminus \{x_F\})$$

12

for some $x_F \in G \cup \{z_F\}$. Reasoning as before, we conclude $x_F = z_F$ and hence $\inf G \leq z_F$.

By an argument involving the quasicompactness analogous to the one given earlier, we obtain $w \in S$ such that $\inf G \leq w$ for all irredundant G with $|G| = n-1$. Thus we have

(**) If $G \subseteq S$ is irredundant, $|G| = n-1$, and $\inf G \not\leq u$, then $\inf G \leq w$.

Let $I = \downarrow u \cup \downarrow w$ and let T be the Rees quotient S/T obtained by shrinking I to a point. Let $\phi : S \to T$ be the canonical semilattice homomorphism. Since by (**) for any irredundant set G in T with $|G| = n-1$ we have $\inf G = 0$, it follows that T has breadth less than n.

The equalities

$$\begin{aligned} \phi^{-1}(\uparrow 0) &= S \\ \phi^{-1}(\uparrow \phi(s)) &= \uparrow s, \quad \text{if } \phi(s) \neq 0 \\ \phi^{-1}(\downarrow \phi(s)) &= \downarrow s \cup \downarrow u \cup \downarrow w \\ \phi^{-1}([\phi(p) \wedge x \leq \phi(q)]) &= [p \wedge x \leq q] \cup [p \wedge x \leq u] \cup [p \wedge x \leq w] \end{aligned}$$

imply that ϕ is continuous for the upper Z-topologies and for the lower topologies. Thus T is quasicompact in the lower topology. By the inductive hypothesis T can be written as a finite union of sets of the form $\downarrow t$ and $[t_1 \wedge x \leq t_2]$ where $t_1 \not\leq t_2$. Again by the preceding equalities these sets can be pulled back to sets of the same type in S. This completes the proof. ▯

6.4. Theorem. Let S be a semilattice of finite breadth. Then the dual topology is contained in the Z-topology. If in addition S is conditionally complete, up-complete and meet-continuous, then the Scott-topology, the dual topology and the upper Z-topology all agree.

Proof. Let $x_0 \in S$ and let U be a subbasic open set around x_0 in the dual topology. Then $G = S \setminus U$ is quasicompact in the lower topology on S. Let $T = \downarrow x_0 \cap G$. Then T is a lower set, hence a subsemilattice. We will show that T is quasicompact in its own lower topology: Let $A \subseteq T$ be such that $\{T \setminus \uparrow a : a \in A\}$ is a covering of T by subbasic open sets in the lower topology. We claim that $\{G \setminus \uparrow a : a \in A\}$ is a covering of G. If it is not, then there exists $y \in G$ such that $y \notin G \setminus \uparrow a$ for all $a \in A$, i.e. $a \leq y$ for all $a \in A$. Since $A \subseteq T$, $a \leq y \wedge x_0$ for all $a \in A$. But $y \wedge x_0 \in T$ since G is a lower set. This contradicts the assumption that $\{T \setminus \uparrow a : a \in A\}$ covers T.

Since G is quasicompact, there exists a finite subset $F \subseteq A$ such that $G \subseteq \{G \setminus \uparrow a : a \in F\}$. Then $T \subseteq \{T \setminus \uparrow a : a \in F\}$. This argument shows that T is quasicompact in its own lower topology.

Applying Lemma (6.3), we conclude that T can be written as a finite union of sets of the form $\downarrow y$ and $\{x \in T : z \wedge x \leq y\}$ where $y, z \in T$, $z \not\leq y$. For each such set $\{x \in T : z \wedge x \leq y\}$, form the corresponding $\{x \in S : z \wedge x \leq y\}$. These sets contain $\{x \in T : z \wedge x \leq y\}$ and miss x_0 since $x_0 \wedge z = z$. Thus by uniting the given sets of the form $\downarrow y$ and $[z \wedge x \leq y]$, we obtain a set B which is closed in the upper Z-topology, misses x_0, and contains T. Let $C = \{w \in S : x_0 \wedge w \in B\}$. Then C misses x_0 and is closed in the upper Z-topology by Proposition 6.1. If $s \in G$, then $s \wedge x_0 \in T \subseteq B$, so $s \in C$. Thus $G \subseteq C$. Hence $V = S \setminus C$ is open in the upper Z-topology, and we have $x \in V \subseteq U$. This shows that the identity function from the upper Z-topology to the dual topology is continuous, i.e. the dual topology is contained in the upper Z-topology.

The last part of the theorem now follows from Proposition 5.1 and Corollary 6.2. []

6.5. Example. Let $L = [0,1] \times [0,1]$ and let

$$S = L \setminus \{(x,1) : 0 \leq x < \frac{1}{2}\}.$$

Then S is a sublattice of L. If the Scott topology on L is restricted to S, then this topology agrees with the upper Z-topology on S, but not with the Scott-topology on S. The upper Z-topology thus seems a more appropriate topology for many purposes on semilattices of finite breadth which are not meet-continuous.

We conclude these notes with a corollary:

6.6. Corollary. If S is a continuous lattice of finite breadth, then the Z-topology and the CL-topology agree. Especially, the Z-topology on S is compact and Hausdorff.[]

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