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SCS 80: Compact Ordered Spaces and Prime Wallman Compactifications: Summary of Results

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TOPIC: Compact ordered spaces and prime Wallman compactifications; Summary of Results

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The Wallman compactification of a T_1 space X, defined as the set of maximal closed filters on X, provided with the hull-kernel topology, has abysmal functorial properties. This changes radically if prime closed filters are used.

We have a contravariant functor $\Gamma : \mathrm{TOP}^{op} \longrightarrow \mathrm{LAT}$ which assigns to a space X the lattice of closed sets of X. Adjoint on the right is $\Sigma : \mathrm{LAT}^{op} \longrightarrow \mathrm{TOP}$, with ΣL the set of prime filters in L, provided with the hull-kernel topology for which the sets $a^* = \{\varphi \in \Sigma L : a \in \varphi\}$, for $a \in L$, form a basis of open sets. Maps $f : X \longrightarrow \Sigma L$ in TOP and $g : L \longrightarrow \Gamma X$ in LAT are adjoint if always $a \in f(x) \Longleftrightarrow x \in g(a)$.

The adjunction of Σ and Γ produces a monad $\mathbb{W} = (W,\eta,\mu)$ on TOP, with $W = \Sigma\Gamma^{op}$ and $\eta_X : X \longrightarrow WX$ the <u>prime Wallman compactification</u> of X.

An algebra (X,α) for $\mathbb{W}$ turns out to be a compact ordered space Z (compact pospace in [1]), where X is Z with the upper topology, i.e. open sets of X are increasing open sets of Z, and the order of Z is the specialization order of X, with $x \leq y$ iff $x \in cl_X\{y\}$, with $\alpha(cl_X\varphi)$ the limit of φ in Z for an ultrafilter φ on Z.

In this situation, X is a quasicompact locally quasicompact sober space, and Z has the patch topology for X.

Put $C \gg A$ for closed sets A and C of X if C is in every ultrafilter φ on X with all limits of φ for X in A ; this is dual to "way below" for open sets. A topological space X has at most one $\mathfrak{w}$-algebra structure, and we have the following theorem.

THEOREM. For a quasicompact and locally quasicompact sober space X , the following statements are logically equivalent.

(i) X has a $\mathfrak{w}$-algebra structure.

(ii) X has the upper topology for a compact ordered space.

(iii) The patch topology of X is compact.

(iv) The intersection of two saturated quasicompact sets in X is always quasicompact.

(v) If $C \gg A$ and $C \gg B$ for closed sets in X , then always $C \gg A \cup B$.

(vi) The adherence of an ultrafilter on X is always an irreducible closed set.

The equivalence of (ii) through (vi) is already in [1].

For maps, we have:

THEOREM. If (X,α) and (Y,β) are $\mathfrak{w}$-algebras, then the following are logically equivalent for a mapping $f : X \longrightarrow Y$.

(i) f is a homomorphism of $\mathfrak{w}$-algebras.

(ii) f is a continuous and order preserving map of compact ordered spaces.

(iii) $f : X \longrightarrow Y$ is continuous, and $f^{-1}(Q)$ is quasicompact in X for every quasicompact saturated subset Q of Y .

(iv) f is continuous for the given topologies of X and Y, and also continuous for the patch topologies.

If the spaces and maps characterized by these theorems are called spectral spaces and spectral maps, then WX is always a spectral space, with the following

UNIVERSAL PROPERTY. If Y is a spectral space and $f : X \longrightarrow Y$ a continuous map, then there is a unique spectral map $f^* : WX \longrightarrow Y$ such that $f = f^* \eta_X$.