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SCS 76: The Trace of the Weak Topology and of the Γ-Topology of L^{op} Coincide on the Pseudo-Meet-Prime Elements of a Continuous Lattice L

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TOPIC: The trace of the weak toplogy and of the Γ-topology of L^{op} coincide on the pseudo-meet-prime elements of a continuous lattice L

REFERENCES: The Fell compactification revisited. Preprint. (Preliminary version in :"Continuous Lattices and Related Topics", pp. 68- 141. Mathematik Arbeitspapiere Nr. 27 , Universität Bremen, 1982)

and literature quoted there

This is a partial response to a private communication in which Karl H. Hofmann attempts to delineate a somewhat different approach to some of the results of my paper mentioned above (employing - implicitly - to some extent the apparatus of $[HL_2]$ and $[HM]$).

Recall that, in a 1,$\wedge$-semilattice L,

$$a \vdash b \quad (\text{"a is relatively meet-prime below b"})$$

for a, $b \in L$ iff whenever $\inf\{x_1,\dots,x_n\} \leq a$ for $x_1,\dots, x_n \in L$ ($n \in \mathbb{N}$, the set of natural numbers including o), then $x_i \leq b$ for some $i \in \mathbb{N}$, $o \leq i \leq n$. The sets

$$\Gamma^*(x) := \{y \in L \mid x \vdash y\} ,$$

with x ranging through L, form a (sub-)basis (cf. $[H_8]$ 1.3(ii)) of the closed sets of the Γ^*-topology (the Γ-topology of L^{op}, cf. $[H_3]$ §3, $[H_8]$). The sets

$$\uparrow x := \{y \in L \mid x \leq y\} \quad (x \in L)$$

form a subbasis of the closed sets of the lower topology ω_L of L (= the weak topology of L^{op}).

An element p of a complete lattice L is called

1) meet-prime iff $p \vdash p$,
2) pseudo-meet-prime iff p = sup P for a prime ideal P of L,
3) a γ^*-element (i.e. a γ-element of L^{op}) iff $\uparrow p$ is closed in (L, Γ^*) iff $p = \sup\{x \in L \mid x \vdash p\}$ ($[H_8]$ 1.5, 2.7).

Every meet-prime element is pseudo-meet-prime. Every pseudo--meet-prime element is a γ^*-element ($[H_8]$ 3.4). In a distributive complete lattice, every γ^*-element is a supremum of pseudo-meet-prime elements.($[H_8]$ 3.6).

(I hope that Karl H. Hofmann will once realize that, even in the present context, the notion of a γ-element is not a "red herring [das Objekt einer Fixierung auf eine Nebensächlichkeit, die einem den Weg zu einer direkten Einsicht versperrt]", but one of the most intriguing and central themes of today's continuous lattice theory research .)

Endowing the set

$$\psi^* L$$

of pseudo-meet-prime elements of a distributive continuous lattice L with the trace τ of ω_L , Karl H. Hofmann sketches a proof for

$$\underline{DO}(\psi^* L, \tau) \cong \mathrm{Filt}_\sigma L$$

which closely parallels my result

$$\underline{DO}(\psi^* L) \cong \mathrm{Filt}_\sigma L$$

where $\psi^* L$ carries the trace of the Γ^*-topology of L (see Corollary 4.1o of my paper). From a comparison one may be inclined to infer (note that one cannot !) that these topologies coincide on $\psi^* L$.

Indeed, arguments very similar to those used in the proof of theorem 4.9 of my paper provide a "direct proof" of this (which "is bound to exist").

PROPOSITION:
For a continuous lattice L (not necessarily distributive), the weak topology ω_L of L^{op} and the Γ^*-topology of L have the same trace on the set $\psi^* L$ of pseudo-meet-prime elements of L. For every $x \in L$, we have

$$\psi^* L \cap \uparrow x = \psi^* L \cap \bigcap\{\Gamma^*(y) \mid y \in L,\ y \ll x\} .$$

Proof:
Since the Γ^*-topology of L is always weaker than ω_L ($[H_3]$3.3), the above formula suffices to establish the assertion.
(a) Let $z \in \bigcap\{\Gamma^*(y) \mid y \in L,\ y \ll x\}$. Then $z \in \Gamma^*(y)$, i.e. $y \vdash z$ for every $y \in L$ with $y \ll x$. Thus $y \leq z$ for every such y. Since L is a continuous lattice, it results that

$$x = \sup\{y \in L \mid y \ll x\} \leq z,$$

hence

$$\bigcap\{\Gamma^*(y) \mid y \in L,\ y \ll x\} \subseteq \uparrow x .$$

(b) Now let $z \in \psi^* L \cap \uparrow x$ and let $y \in L$ with $y \ll x$. Then

$$z = \sup P$$

for some prime ideal P of L. However, $x \leq \sup P$ implies $y \in P$, since $y \ll x$. On the other hand, since P is a prime ideal, $y \in P$

and $z = \sup P$ imply $y \vdash z$, i.e.

$$z \in \Gamma^{*}(y)$$

for every $y \in L$ with $y \ll x$. Thus

$$\psi^{*}L \cap \uparrow x \subseteq \bigcap \{\Gamma^{*}(y) \mid y \in L,\ y \ll x\} .$$

This completes the proof.

REMARK 1 :

For every complete lattice L, the Γ^{*}-topology and ω_L have the same trace on $\mathrm{Spec}^{*}L$, the set of meet-prime elements of L (cf. $[H_3]$ 3.7).

REMARK 2 :

It is an open question whether the above proposition holds for all complete lattices L. It is also unknown whether it extends to $\gamma^{*}L$, the set of γ^{*}-elements of L (i.e. γ-elements of L^{op}). The latter extension is known to be true for completely distributive complete lattices L - cf. $[H_9]$ 5.5. (A revised draft of $[H_9]$, entitled "The injective hull and the CL-compactification of a continuous poset" will be distributed in a Seminarbericht, Fern-Universität Hagen.)

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