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# SCS 75: Distributive Semilattices, Heyting Algebras, and V-Homomorphisms

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TOPIC: Distributive semilattices, Heyting algebras and V-homomorphisms

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The study of the monoid, under direct sums, of all isomorphism types of countable Boolean algebras has led to the notion of a refinement monoid [M, D1], cf. [K]:

A commutative monoid $M = (M;+,0)$ is called a refinement monoid provided that

(RM1) $x+y = 0$ only for $x = y = 0$ $(x, y \in M)$,

(RM2) $M$ has the refinement property, that is, whenever $\Sigma x_i = \Sigma y_j$ for $x_i, y_j \in M$ $(i < n, j < m)$ then there are $z_{ij} \in M$ with $x_i = \Sigma_j z_{ij}$ and $y_j = \Sigma_i z_{ij}$.

A homomorphism $h : M \longrightarrow N$ between commutative monoids is said to be a V-homomorphism if $h(x) = y_1 + y_2$ $(x \in M,\ y_1, y_2 \in N)$ implies $x = x_1 + x_2$ and $h(x_i) = y_i$ for some $x_1, x_2 \in M$, and $h(x) = 0$ only for $x = 0$ $(x \in M)$. Observe that a V-homomorphic image of a refinement monoid is again a refinement monoid.

PROPOSITION 1. A semilattice $L = (L;+,0)$ with zero is distributive (in the sense of [G; p. 117]) iff $L$ is a refinement monoid.

It is well-known that the category of distributive semilattices with zero and homomorphisms having the property that pre-images of prime filters are always prime filters is dually equivalent to the category of Stone spaces (sober $T_0$-spaces having a base of compact sets) and strongly continuous mappings (pre-images of compact-open sets are compact-open); see [G; 2.11].

Let $DSL$ be the category of distributive semilattices with zero



...ply the category $STS$ of Stone spaces with suitable morphisms so that $STS$ and $DSL$ become equivalent categories. First let us call a subset $U$ of a space $X$ almost-open if there is a smallest open set, say $\tilde{U}$, containing $U$, and $U$ is a strict subset of $\tilde{U}$ (i. e., the inclusion map from $U$ into $\tilde{U}$ is strict [C; V.5.8]). Note that, for instance, every space is almost-open in its sobrification. Of course, open sets are almost-open. Now suppose that $X$ and $Y$ are Stone spaces, then $mor(X,Y)$ consists of all continuous functions from $X$ into $Y$ mapping open sets (or equivalently, almost-open sets) onto almost-open sets. Thus all continuous-open mappings are morphisms of $STS$. Probably, the converse is false. However, I have no counter-example.

We call a mapping $h : L \longrightarrow K$ between complete lattices a strong V-homomorphism if $h$ is Sup-preserving, $h(x) = \mathrm{Sup}_{i \in I}\, y_i$ always implies $x = \mathrm{Sup}_{i \in I}\, x_i$ and $h(x_i) = y_i$ for some elements $x_i$, and $h(x) = 0$ only for $x = 0$.

LEMMA 2. Let $H_1$ and $H_2$ be complete Heyting algebras, and suppose that $g : H_1 \longrightarrow H_2$ and $h : H_2 \longrightarrow H_1$ form an adjunction [C; p. 18]. Then the following are equivalent:

(i) $g$ preserves Sups, Infs and $\Rightarrow$,

(ii) $h$ is a strong V-homomorphism.

Let $AHA_0$ (resp. $AHA_1$) be the category of algebraic complete Heyting algebras and strong V-homomorphisms (resp. $\Rightarrow$-preserving, complete homomorphisms).

PROPOSITION 3. The categories $STS$, $DSL$, $AHA_0$, $AHA_1^{op}$ are equivalent.

Of course, the emphasis in Proposition 3 lies on the morphisms; on the object level this is well-known.

THEOREM 4. [D2] Let $L$ be a distributive semilattice with zero. If $L$ is a lattice or $|L| \leq \aleph_1$ then $L$ is a V-homomorphic image of some generalized Boolean lattice.

Question A. Does Theorem 4 hold for all $L$?

A "dual version" of Thm. 4 is the following: Let $H$ be an algebraic complete Heyting algebra such that the set $K(H)$ of compact elements of H is a lattice or $|K(H)| \leq \aleph_1$, then H is embeddable into the ideal lattice Id(B) of some generalized Boolean algebra B under a mapping preserving Sups, Infs and =>. As a consequence, every Heyting algebra can be embedded into Id(B) for some Boolean algebra B (under a mapping preserving sups, infs and =>). A similar result for distributive pseudo-complemented lattices has been shown by Lakser (see [G; p. 180]).

Question B. Is every Stone space X the image of a locally compact zero-dimensional Hausdorff space under a continuous-open mapping?

It is not difficult to see that if X is first-countable then, for all Stone spaces Y, mor(Y,X) consists only of continuous-open mappings. Thus, in this case, it follows from Thm. 4 that Question B has an affirmative answer provided that the set L(X) of compact-open subsets of X is closed under finite intersections or $|L(X)| \leq \aleph_1$.

etwas erweiterte Fassung des "Memos" vom 12.11.82

Gruß
HD