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SCS 74: Distributive Semilattices

Hans Dobbertin
Ruhr University of Bochum, Bochum, Germany

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NAME: Hans Dobbertin   Date: M 11 D 12 Y 82

TOPIC: Distributive semilattices

REFERENCES: [C] Compendium.
[D1] H. Dobbertin, On Vaught's criterion for isomorphisms of countable Boolean algebras, Algebra Univ. 15 (1982), in press.
[D2] H. Dobbertin, Verfeinerungsmonoide, Vaught Monoide und Boolesche Algebren, Dissertation, Universität Hannover, 1982.
[G] G. Grätzer, Lattice Theory, Birkhäuser, 1978.
[K] J. Ketonen, The structure of countable Boolean algebras, Ann. of Math. 108 (1978), 41 - 89.
[M] D. Myers, Structures and measures on Boolean algebras, unpublished manuscript.

The study of the monoid, under direct sums, of all isomorphism types of countable Boolean algebras has led to the notion of a refinement monoid [M, D1], cf. [K]:

A commutative monoid $M = (M;+,0)$ is called a refinement monoid provided that

(RM1) $x+y = 0$ only for $x = y = 0$ $(x, y \in M)$,

(RM2) M has the refinement property, that is, whenever $\Sigma x_i = \Sigma y_j$ for $x_i, y_j \in M$ $(i < n, j < m)$ then there are $z_{ij} \in M$ with $x_i = \Sigma_j z_{ij}$ and $y_j = \Sigma_i z_{ij}$.

A homomorphism $h : M \longrightarrow N$ between commutative monoids is said to be a V-homomorphism if $h(x) = y_1 + y_2$ $(x \in M,\ y_1, y_2 \in N)$ implies $x = x_1 + x_2$ and $h(x_i) = y_i$ for some $x_1, x_2 \in M$, and $h(x) = 0$ only for $x = 0$ $(x \in M)$. Observe that a V-homomorphic image of a refinement monoid is again a refinement monoid.

PROPOSITION 1. A semilattice $L = (L;+,0)$ with zero is distributive (in the sense of [G; p. 99]) iff L is a refinement monoid.

It is well-known that the category of distributive semilattices with zero and homomorphisms having the property that pre-images of prime filters are always prime filters is dually equivalent to the category of Stone spaces (sober T_0-spaces having a base of compact sets) and strongly continuous mappings (pre-images of compact-open sets are compact-open); see [G; II.5].

Let DSL be the category of distributive semilattices with zero and V-homomorphisms. We want to supply the category STS of Stone spaces with suitable morphisms so that DSL and STS become equivalent categories: First, let us call a subset U of a space X almost open if there is a smallest open set, say $\tilde{U}$, containing U, and U is a strict subset of $\tilde{U}$ (i. e., the inclusion map from U into $\tilde{U}$ is strict in the sense of [C; V.5.8]). Note that, for instance, every space is almost open in its sobrification. Of course, open sets are almost open. Now suppose that X and Y are Stone spaces, then $\mathrm{mor}(X,Y)$ consists of all continuous functions from X into Y mapping open sets onto almost open sets. Thus, all continuous-open mappings are morphisms in *STS*. Probably, the converse is false. However, I have no counterexample.

PROPOSITION 2. *DSL* and *STS* are equivalent categories.

Proof. Let $L_1, L_2 \in \mathrm{obj}(DSL)$ and $h \in \mathrm{mor}(L_1,L_2)$. Then the associated mapping $f_h : X(L_1) \longrightarrow X(L_2)$ between the prime filter spaces is given by setting $f_h(P) = \uparrow h(P)$. Conversely, if $X_1, X_2 \in \mathrm{obj}(STS)$ and $f \in \mathrm{mor}(X_1,X_2)$ then $h_f : L(X_1) \longrightarrow L(X_2)$ is defined by $h_f(C) = f(C)^{\sim}$, where $L(X)$ denotes the semilattice of compact-open subsets of a Stone space X. It is not difficult to show that in fact $f_h \in \mathrm{mor}(X(L_1),X(L_2))$ and $h_f \in \mathrm{mor}(L(X_1),L(X_2))$. The remainder of the proof is similar as in the case of the previously mentioned duality.

Question A. Is every distributive semilattice L with zero a $\vee$-homomorphic image of some generalized Boolean lattice? (Note that the converse is obvious.)

In [D2] it has been shown that the answer is positive when L is a lattice or has not more than $\aleph_1$ many elements. Moreover, if it will turn out that the morphisms of *STS* are not necessarily continuous-open then the following question arises:

Question B. Is every Stone space X the image of a locally compact, zero-dimensional Hausdorff space under a continuous-open mapping?

At present, I only have an affirmative result when X is first countable and in addtion $L(X)$ is a lattice or $|L(X)| \leq \aleph_1$.