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SCS 69: Order Generation and Distributive Laws in Complete Lattices

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SEMINAR ON CONTINUITY IN SEMILATTICES

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Topic: Order generation and distributive laws in complete lattices

REFERENCES:

[Av] S.P.Avann : Dependence of finiteness conditions in distributive lattices. Math. Zeitschr. 85 (1964) 245 - 256

[BF] G.Birkhoff and O.Frink: Representations of lattices by sets. Trans. AMS 64 (1948) 299 - 316

[Br] G.Bruns: Verbandstheoretische Kennzeichnung vollständiger Mengenringe. Arch. Math.10 (1959) 109 - 112

[CD] P.Crawley and R.P.Dilworth: Algebraic lattices theory of . Prentice Hall Inc., Englewood Cliffs, N.J., 1973

[E1] M.Erné: Order and topology. Preprint Hannover 1978

[E2] ———: On the existence of decompositions in lattices. Algebra Universalis (to appear)

[Kah] Karl H. Hofmann: A remark on the complete distributivity of algebraic lattices. SCS - Memo 7 - 16 - 82

[Me] W.Menzel: Über den Untergruppenvergand einer abelschen Operatorgruppe, Teil I: m - Verbände. Math. Zeitschr. 74 (1960) 39 - 51

In a recent Memo [Kah] Karl H. Hofmann asked for references concerning the following equivalent statements on a complete lattice:

(a) L is "join Brouwerian" and algebraic.

(b) L is completely distributive and algebraic.

(c) Every element of L is a join of completely join-primes.

It is well known that these conditions are selfdual, being equivalent to

(d) L is isomorphic to a complete ring of sets.

For the sake of convenience, we call such lattices <u>A - lattices</u>.
A lattice is <u>weakly atomic</u> (cf. [CD]) if for all $a < b$ there are elements $u,v \in [a,b]$ such that v covers u (i.e. $u < v$ and $|[u,v]| = 2$).

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Already in the late Fifties, G.Bruns [Br] has shown that a complete lattice L is an A - lattice iff it is weakly atomic and infinitely distributive (i.e. L and L^{op} are complete Heyting algebras). It is also well known (cf. [BF] and [CD]) that every algebraic lattice is weakly atomic and (meet-)continuous. Hence the above equivalences are immediate consequences of Bruns' theorem. They occur implicitly in [Me, 3.6 and 3.8] and in [GG, Cor.1,Cor.2 and Prop.6] , and explicitly in [Av, Thm 5.3] and in [E1, 1.7.59 - 1.7.63] where a list of alternative descriptions of A-lattices can be found. The following more general theorem was established in 1979. It does not require any distributivity assumption (see [E2]).

THEOREM. Consider the following conditions on a complete lattice L:

(a) Every element of L is a join of completely join-irreducibles, and L is meet-continuous.
(b) L is algebraic.
(c) L is weakly atomic and continuous.
(d) L is weakly atomic and meet-continuous.
(e) L is weakly atomic, and whenever v covers u in L then there is a $q \in L$ maximal subject to $q \wedge v = u$.
(f) Every element is a meet of completely meet-irreducibles (i.e. Irr L is "order generating").

In general, (a) => (b) => (c) => (d) => (e) => (f), and none of these implications can be inverted. However,
(1) if L is modular then (e) <=> (f) ,
(2) if L is distributive then (d) <=> (e) <=> (f),
(3) if L is join-continuous then all six conditions are equivalent and imply their duals.

In particular, if L is "join Brouwerian" then each of these conditions is necessary and sufficient for L to be an A-lattice.

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