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BERRY-ESSEEN BOUNDS FOR APPROXIMATE MAXIMUM LIKELIHOOD ESTIMATORS IN THE α -BROWNIAN BRIDGE

KHALIFA ES-SEBAIY, JABRANE MOUSTAAID*, AND IDIR OUASSOU

ABSTRACT. Let $T > 0$, $\alpha > \frac{1}{2}$. In this work we consider the problem of estimating the drift parameter of the α -Brownian bridge defined as $dX_t = -\alpha \frac{X_t}{T-t} dt + dW_t$, $0 \leq t < T$, where W is a standard Brownian motion. Assume that the process X is observed equidistantly in time with the step size $\Delta_n := \frac{T}{n+1}$, $t_i = i\Delta_n$, $i = 0, \dots, n$. We will propose two approximate maximum likelihood estimators $\hat{\alpha}_n$ and $\tilde{\alpha}_n$ for the drift parameter α based on the discrete observations X_{t_i} , $i = 0, \dots, n$. The consistency of those estimators is studied. Explicit bounds for the Kolmogorov distance in the central limit theorem for the estimators $\hat{\alpha}_n$ and $\tilde{\alpha}_n$ are obtained.

1. Introduction

Let $T \in (0, \infty)$ be fixed. We consider the α -Brownian bridge process $X := \{X_t, t \in [0, T)\}$, defined as the solution to the stochastic differential equation

$$X_0 = 0; \quad dX_t = -\alpha \frac{X_t}{T-t} dt + dW_t, \quad 0 \leq t < T, \quad (1.1)$$

where W is a standard Brownian motion, and $\alpha > 0$ is unknown parameter to be estimated.

Because (1.1) is linear, it is immediate to solve it explicitly; one then gets the following formula:

$$X_t = (T-t)^\alpha \int_0^t (T-s)^{-\alpha} dW_s, \quad 0 \leq t < T. \quad (1.2)$$

An important problem related to the α -Brownian bridge (1.1) is to estimate the parameter α when one observes the whole trajectory of X . For more information and further references concerning the subject, we refer the reader to [2], as well as [9] and [8].

The maximum likelihood estimator (MLE) of the unknown parameter α based on continuous observations, is given by

$$\tilde{\alpha}_t = - \left(\int_0^t \frac{X_u}{T-u} dX_u \right) / \left(\int_0^t \frac{X_u^2}{(T-u)^2} du \right), \quad t < T. \quad (1.3)$$

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In (1.3), the integral with respect to X must of course be understood in the Itô sense.

From a practical point of view, in parametric inference, it is more realistic and interesting to consider asymptotic estimation for the diffusion processes based on discrete observations. We assume that the process X is observed equidistantly in time with the step size $\Delta_n := \frac{T}{n+1}$, $t_i = i\Delta_n$, $i = 0, \dots, n$ and T_n denotes the length of the observation window. Then we will consider the following approximate maximum likelihood estimators

$$\hat{\alpha}_n := -\frac{\sum_{i=1}^n \frac{X_{t_{i-1}}}{(T-t_{i-1})} (X_{t_i} - X_{t_{i-1}})}{\Delta_n \sum_{i=1}^n \frac{X_{t_{i-1}}^2}{(T-t_{i-1})^2}}, \quad (1.4)$$

and

$$\bar{\alpha}_n := -\frac{\frac{1}{2} \left(\frac{X_{T_n}^2}{\Delta_n} - \Delta_n \sum_{i=1}^n \frac{X_{t_{i-1}}^2}{(T-t_{i-1})^2} - \log(n+1) \right)}{\Delta_n \sum_{i=1}^n \frac{X_{t_{i-1}}^2}{(T-t_{i-1})^2}}. \quad (1.5)$$

To our knowledge there is no study of the asymptotic behavior of the estimators $\hat{\alpha}_n$ and $\bar{\alpha}_n$. Our goal in the present paper is to investigate the consistency and the rate of convergence to normality of the estimators $\hat{\alpha}_n$ and $\bar{\alpha}_n$.

In the case of continuous observations, the asymptotic behavior of the MLE $\tilde{\alpha}_t$, given by (1.3), of α based on the observation $\{X_s, 0 \leq s \leq t\}$ as $t \uparrow T$ has been studied in [2]. They proved that the MLE $\tilde{\alpha}_t$ of α is strongly consistent and asymptotically normal in the case $\alpha > \frac{1}{2}$. Recently, Es-Sebaiy and Moustaid [6] obtained, when $\alpha > \frac{1}{2}$, an optimal rate of Kolmogorov distance for central limit theorem of the MLE $\tilde{\alpha}_t$ in the following sense: there exist constants $0 < c < C < \infty$, depending only on α and T , such that for all t sufficiently near T ,

$$\begin{aligned} \frac{c}{\sqrt{|\log(T-t)|}} &\leq \sup_{z \in \mathbb{R}} \left| \mathbb{P} \left(\sqrt{\frac{|\log(T-t)|}{2\alpha-1}} (\alpha - \tilde{\alpha}_t) \leq z \right) - \mathbb{P}(Z \leq z) \right| \\ &\leq \frac{C}{\sqrt{|\log(T-t)|}}, \end{aligned}$$

where Z denotes a standard normal random variable.

On the other hand, there exists a rich literature on the parametric estimation problems based on discrete observations. In the case of the Ornstein-Uhlenbeck process defined as solution to the equation $dX_t = -\theta X_t dt + dW_t$, $t \geq 0$, $X_0 = 0$, with $\theta > 0$, Bishwal and Bose [3] obtained an upper bound in Kolmogorov distance for normal approximation of the approximate maximum likelihood estimators for the drift parameter θ on the basis of discrete observations of the process X . We mention that Es-Sebaiy [5] studied the least squares estimator of θ based on the sampling data $X_i, i = 1, \dots, n$ when the standard Brownian motion W is replaced by a fractional Brownian motion, see also [14, 15].

The rest of the paper is structured as follows. In Section 2 we give the basic tools of Malliavin calculus needed throughout the paper. In Section 3 we prove

the consistency and obtain the rate of convergence to normality of the estimators $\hat{\alpha}_n$ and $\bar{\alpha}_n$.

2. Preliminaries

In this section, we recall some elements from stochastic analysis that we will need in the paper. See [12], and [13] for details. Any real, separable Hilbert space \mathfrak{H} gives rise to an isonormal Gaussian process: a centered Gaussian family $(G(\varphi), \varphi \in \mathfrak{H})$ of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbf{E}(G(\varphi)G(\psi)) = \langle \varphi, \psi \rangle_{\mathfrak{H}}$. In this paper, it is enough to use the classical Wiener space, where $\mathfrak{H} = L^2([0, T])$, though any \mathfrak{H} will also work. In the case $\mathfrak{H} = L^2([0, T])$, G can be identified with the stochastic differential of a Wiener process $\{W_t, t \in [0, T]\}$ and one interprets $G(\varphi) := \int_0^T \varphi(s) dW(s)$.

The Wiener chaos of order p , denoted by \mathfrak{H}_p , is defined as the closure in $L^2(\Omega)$ of the linear span of the random variables $H_p(G(\varphi))$, where $\varphi \in \mathfrak{H}$, $\|\varphi\|_{\mathfrak{H}} = 1$ and H_p is the Hermite polynomial of degree p . The multiple Wiener stochastic integral I_p with respect to $G \equiv W$, of order p is an isometry between the Hilbert space $\mathfrak{H}^{\odot p} = L_{sym}^2([0, T]^p)$ (symmetric tensor product) equipped with the scaled norm $\sqrt{p!} \|\cdot\|_{\mathfrak{H}^{\otimes p}}$ and the Wiener chaos of order p under $L^2(\Omega)$'s norm, that is, the multiple Wiener stochastic integral of order p :

$$I_p : \left(\mathfrak{H}^{\odot p}, \sqrt{p!} \|\cdot\|_{\mathfrak{H}^{\otimes p}} \right) \longrightarrow \left(\mathfrak{H}_p, L^2(\Omega) \right)$$

is a linear isometry defined by $I_p(f^{\otimes p}) = H_p(G(f))$.

• **Multiple Wiener-Itô integral.** If $f \in L^2([0, T]^p)$ is symmetric, we can also rewrite $I_p(f)$ as the following iterated adapted Itô stochastic integral:

$$\begin{aligned} I_p(f) &= \int_{[0, T]^p} f(t_1, \dots, t_p) dW_{t_1} \dots dW_{t_p} \\ &= p! \int_0^T dW_{t_1} \int_0^{t_1} dW_{t_2} \dots \int_0^{t_{p-1}} dW_{t_p} f(t_1, \dots, t_p). \end{aligned} \quad (2.1)$$

• **The Wiener chaos expansion.** For any $F \in L^2(\Omega)$, there exists a unique sequence of functions $f_p \in \mathfrak{H}^{\odot p}$ such that

$$F = \mathbf{E}[F] + \sum_{p=1}^{\infty} I_p(f_p),$$

where the terms are all mutually orthogonal in $L^2(\Omega)$ and

$$\mathbf{E} [I_p(f_p)^2] = p! \|f_p\|_{\mathfrak{H}^{\otimes p}}^2.$$

• **Product formula and contractions.** For any integers $p, q \geq 1$ and symmetric integrands $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$,

$$I_p(f)I_q(g) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(f \tilde{\otimes}_r g); \quad (2.2)$$

where $f \otimes_r g$ is the contraction of order r of f and g which is an element of $\mathfrak{H}^{\otimes(p+q-2r)}$ defined by

$$(f \otimes_r g)(s_1, \dots, s_{p-r}, t_1, \dots, t_{q-r}) := \int_{[0, T]^{p+q-2r}} f(s_1, \dots, s_{p-r}, u_1, \dots, u_r) \\ g(t_1, \dots, t_{q-r}, u_1, \dots, u_r) du_1 \dots du_r,$$

while $f \widetilde{\otimes}_r g$ denotes its symmetrization. More generally the symmetrization \tilde{f} of a function f is defined by $\tilde{f}(x_1, \dots, x_p) = \frac{1}{p!} \sum_{\sigma} f(x_{\sigma(1)}, \dots, x_{\sigma(p)})$ where the sum runs over all permutations σ of $\{1, \dots, p\}$.

• **Kolmogorov distance between random variables.** If X, Y are two real-valued random variables, then the Kolmogorov distance between the law of X and the law of Y is given by

$$d_{Kol}(X, Y) := \sup_{z \in \mathbb{R}} |\mathbb{P}(X \leq z) - \mathbb{P}(Y \leq z)|.$$

• **The derivative operator.** We denote by \mathcal{S} the class of smooth cylindrical functionals of the form

$$F = f(W(h_1), W(h_2), \dots, W(h_n)), \quad (2.3)$$

where $n \geq 1$ $f \in C_b^\infty(\mathbb{R}^n, \mathbb{R})$ and $h_1, h_2, \dots, h_n \in \mathfrak{H}$.

The derivative operator D of a smooth cylindrical random variable F of the form (2.3) is defined as the \mathfrak{H} -valued random variable

$$DF = \sum_{i=1}^n \partial_i f(W(h_1), W(h_2), \dots, W(h_n)) h_i.$$

In this way the derivative DF is an element of $L^2(\Omega; \mathfrak{H})$. We denote by $D^{1,2}$ the closure of \mathcal{S} with respect to the norm defined by

$$\|F\|_{1,2} = \mathbf{E}(F^2) + \mathbf{E}(\|DF\|_{\mathfrak{H}}^2).$$

Theorem 2.1 ([11]). *Let $F = I_q(g)$ with $q \geq 2$ and $g \in L^2([0, T]^q)$. Then,*

$$d_{kol}(F, N) \leq \sqrt{\mathbf{E} \left[\left(1 - \frac{1}{q} \|DF\|_{\mathfrak{H}}^2 \right)^2 \right]}, \quad (2.4)$$

where $N \sim \mathcal{N}(0, 1)$.

Proposition 2.2 ([11]). *Let $F = I_q(g)$ with $q \geq 2$ and $g \in L^2([0, T]^q)$. Then,*

$$\mathbf{E} \left[\left(1 - \frac{1}{q} \|DF\|_{\mathfrak{H}}^2 \right)^2 \right] \quad (2.5)$$

$$\leq (1 - q! \|g\|_{\mathfrak{H}^{\otimes q}}^2)^2 \quad (2.6)$$

$$+ q^2 \sum_{r=1}^{q-1} (2q - 2r)! (r - 1)!^2 \binom{q-1}{r-1} \times \|g \otimes_r g\|_{\mathfrak{H}^{\otimes 2(q-r)}}^2$$

Lemma 2.3 (Wick's lemma). *Let (Z_1, Z_2, Z_3, Z_4) be a Gaussian random vector with zero mean. Then*

$$\mathbf{E}[Z_1 Z_2 Z_3 Z_4] = \mathbf{E}[Z_1 Z_2] \mathbf{E}[Z_3 Z_4] + \mathbf{E}[Z_1 Z_3] \mathbf{E}[Z_2 Z_4] + \mathbf{E}[Z_1 Z_4] \mathbf{E}[Z_2 Z_3].$$

Throughout the paper Z denotes a standard normal random variable. Also, C denotes a generic positive constant (perhaps depending on α and T , but not on anything else), which may change from line to line.

3. Asymptotic Behavior of the Approximate Maximum Likelihood Estimators

This section is devoted to study the consistency and the asymptotic distribution of the approximate maximum likelihood estimators $\hat{\alpha}_n$ and $\bar{\alpha}_n$ given by (1.4) and (1.5) respectively.

Let us fix the notations needed in what follows. Define for every $t \in [0, T)$,

$$\begin{aligned} Y_t &:= \frac{X_t}{T-t}, \quad B_{T_n} := \int_0^{T_n} Y_t dW_t, \\ I_n &:= \Delta_n \sum_{i=1}^n Y_{t_{i-1}}^2, \quad A_n := \sum_{i=1}^n \int_{t_{i-1}}^{t_i} Y_{t_{i-1}} [Y_{t_{i-1}} - Y_t] dt, \\ I_{T_n} &:= \int_0^{T_n} Y_t^2 dt, \quad B_n := \sum_{i=1}^n \int_{t_{i-1}}^{t_i} Y_{t_{i-1}} dW_t. \end{aligned}$$

Notice that for every $t, s \in [0, T)$, the covariance of X_t and X_s is

$$\text{Cov}(X_t, X_s) = \begin{cases} \frac{(T-s)^\alpha (T-t)^\alpha}{1-2\alpha} \left(T^{1-2\alpha} - (T - (s \wedge t))^{1-2\alpha} \right) & \text{if } \alpha \neq \frac{1}{2}, \\ (T-s)^\alpha (T-t)^\alpha \log\left(\frac{T}{T-s \wedge t}\right) & \alpha = \frac{1}{2}. \end{cases} \quad (3.1)$$

In particular X_t is a normally distributed random variable with mean $\mathbf{E}X_t = 0$ and with variance

$$\mathbf{E}[X_t^2] = \begin{cases} \frac{T}{1-2\alpha} \left(\frac{T-t}{T}\right)^{2\alpha} - \frac{T-t}{1-2\alpha} & \text{if } \alpha \neq \frac{1}{2}, \\ (T-t) \log\left(\frac{T}{T-t}\right) & \alpha = \frac{1}{2}. \end{cases} \quad (3.2)$$

In order to prove our main results we will make use of the following lemmas.

Lemma 3.1. *Suppose that $\alpha > \frac{1}{2}$. Let $0 < \varepsilon < 1$. Then*

$$\mathbb{P}(|\lambda_n I_{T_n} - 1| > \varepsilon) \leq \frac{C}{\varepsilon^2 \log(n+1)}, \quad (3.3)$$

and

$$\mathbb{P}(|\lambda_n (I_n - I_{T_n})| > \varepsilon) \leq \frac{C}{\varepsilon^2 \log^2(n+1)}, \quad (3.4)$$

where $\lambda_n = \frac{2\alpha+1}{\log(n+1)}$.

Proof. Applying the Itô formula, see [7], we have

$$\frac{2\alpha - 1}{\log(n+1)} I_{T_n} - 1 = \frac{2}{\log(n+1)} \int_0^{T_n} \frac{X_t}{(T-t)} dW_t - \frac{X_{T_n}^2}{\Delta_n \log(n+1)}. \quad (3.5)$$

On the other hand, by combining the Lemma 2.3 and (3.2), we get that

$$\mathbf{E} [X_{T_n}^4] = 3 (\mathbf{E} [X_{T_n}^2])^2 = \frac{3 (\Delta_n - T^{1-2\alpha} \Delta_n^{2\alpha})^2}{(2\alpha - 1)^2},$$

and

$$\begin{aligned} \mathbf{E} [I_{T_n}] &= \int_0^{T_n} \mathbf{E} \left[\frac{X_t^2}{(T-t)^2} \right] dt \\ &= \int_0^{T_n} \frac{(T-t) - T^{1-2\alpha} (T-t)^{2\alpha}}{(2\alpha - 1) (T-t)^2} dt \\ &= \frac{(2\alpha - 1) \log(n+1) + T^{1-2\alpha} \Delta_n^{2\alpha-1} - 1}{(2\alpha - 1)^2}. \end{aligned} \quad (3.6)$$

Hence, by Chebyshev inequality we have

$$\begin{aligned} &\mathbb{P} (|\lambda_n I_{T_n} - 1| > \varepsilon) \\ &\leq \frac{1}{\varepsilon^2} \mathbf{E} [|\lambda_n I_{T_n} - 1|^2] \\ &= \frac{1}{\varepsilon^2} \mathbf{E} \left[\left| \frac{2}{\log(n+1)} \int_0^{T_n} \frac{X_t}{(T-t)} dW_t - \frac{X_{T_n}^2}{\Delta_n \log(n+1)} \right|^2 \right] \\ &\leq \frac{1}{\varepsilon^2} \left\{ \frac{8}{\log^2(n+1)} \mathbf{E} \left[\left(\int_0^{T_n} \frac{X_t}{(T-t)} dW_t \right)^2 \right] + 2 \mathbf{E} \left[\frac{X_{T_n}^4}{\Delta_n^2 \log^2(n+1)} \right] \right\} \\ &= \frac{2}{\varepsilon^2} \left\{ \frac{4}{\log^2(n+1)} \mathbf{E} [I_{T_n}] + \frac{\mathbf{E} [X_{T_n}^4]}{\Delta_n^2 \log^2(n+1)} \right\} \\ &= \frac{2}{\varepsilon^2} \left\{ \frac{4 ((2\alpha - 1) \log(n+1) + T^{1-2\alpha} \Delta_n^{2\alpha-1} - 1)}{(2\alpha - 1)^2 \log^2(n+1)} \right. \\ &\quad \left. + \frac{3 (\Delta_n - T^{1-2\alpha} \Delta_n^{2\alpha})^2}{(2\alpha - 1)^2 \Delta_n^2 \log^2(n+1)} \right\} \\ &\leq \frac{C}{\varepsilon^2 \log(n+1)}. \end{aligned}$$

Again, by Chebyshev inequality we have

$$\mathbb{P} (|\lambda_n (I_n - I_{T_n})| > \varepsilon) \leq \frac{\lambda_n^2}{\varepsilon^2} \mathbf{E} [|(I_n - I_{T_n})|^2]. \quad (3.7)$$

On the other hand

$$\mathbf{E} [|(I_n - I_{T_n})|^2] = \mathbf{E} \left[\left| \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (Y_{i-1}^2 - Y_t^2) dt \right|^2 \right]$$

$$\begin{aligned}
&= \mathbf{E} \left[\sum_{i=1}^n \left(\int_{t_{i-1}}^{t_i} (Y_{t_{i-1}} + Y_t) (Y_{t_{i-1}} - Y_t) dt \right)^2 \right] \\
&\quad + 2 \sum_{i < j} \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} \left\{ \mathbf{E} [(Y_{t_{i-1}} - Y_t) (Y_{t_{j-1}} + Y_s) \right. \\
&\quad \quad \left. (Y_{t_{i-1}} + Y_t) (Y_{t_{j-1}} - Y_s)] \right\} ds dt \\
&=: b_{1,n} + b_{2,n}.
\end{aligned}$$

We begin with studying the term $b_{1,n}$. Using Lemma 2.3 together with Cauchy Schwarz inequality, we have

$$\mathbf{E} \left[(Y_{t_{i-1}} + Y_t)^2 (Y_{t_{i-1}} - Y_t)^2 \right] \leq 3 \mathbf{E} \left[(Y_{t_{i-1}} + Y_t)^2 \right] \mathbf{E} \left[(Y_{t_{i-1}} - Y_t)^2 \right].$$

Moreover, by (3.1), we have

$$\begin{aligned}
&(2\alpha - 1) \mathbf{E} \left[(Y_{t_{i-1}} - Y_t)^2 \right] \\
&= (T - t_{i-1})^{-1} - T^{1-2\alpha} (T - t_{i-1})^{2\alpha-2} + (T - t)^{-1} - T^{1-2\alpha} (T - t)^{2\alpha-2} \\
&\quad - 2(T - t_{i-1})^{-\alpha} (T - t)^{\alpha-1} + 2T^{1-2\alpha} (T - t_{i-1})^{\alpha-1} (T - t)^{\alpha-1} \\
&= (T - t_{i-1})^{-1} + (T - t)^{-1} - 2(T - t_{i-1})^{-\alpha} (T - t)^{\alpha-1} \\
&\quad - T^{1-2\alpha} \left((T - t_{i-1})^{\alpha-1} - (T - t)^{\alpha-1} \right)^2 \\
&\leq (T - t_{i-1})^{-1} + (T - t)^{-1}.
\end{aligned}$$

Similarly, we can obtain

$$(2\alpha - 1) \mathbf{E} \left[(Y_{t_{i-1}} + Y_t)^2 \right] \leq 4(T - t)^{-1}.$$

This implies that,

$$\begin{aligned}
b_{1,n} &\leq \mathbf{E} \left[\sum_{i=1}^n \Delta_n \int_{t_{i-1}}^{t_i} (Y_{t_{i-1}} + Y_t)^2 (Y_{t_{i-1}} - Y_t)^2 dt \right] \\
&\leq C \Delta_n \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (T - t)^{-1} \left((T - t_{i-1})^{-1} + (T - t)^{-1} \right) ds dt \\
&= C \Delta_n \left\{ \sum_{i=1}^n \left((T - t_i)^{-1} - (T - t_{i-1})^{-1} \right) \right. \\
&\quad \left. + \sum_{i=1}^n (T - t_{i-1})^{-1} (\log(T - t_{i-1}) - \log(T - t_i)) \right\} \\
&= C \Delta_n \left\{ \frac{1}{\Delta_n} + \frac{1}{T} + \frac{1}{\Delta_n} \sum_{k=1}^n \frac{\log(k+1) - \log(k)}{k+1} \right\} \\
&\leq C \Delta_n \left\{ \frac{1}{\Delta_n} + \frac{1}{T} + \frac{1}{\Delta_n} \sum_{k=1}^n \frac{1}{k^2} \right\} \leq C'. \tag{3.8}
\end{aligned}$$

We now study $b_{2,n}$. Applying Lemma 2.3, we can write

$$\begin{aligned}
& \mathbf{E} [(Y_{t_{i-1}} + Y_t) (Y_{t_{j-1}} + Y_s) (Y_{t_{i-1}} - Y_t) (Y_{t_{j-1}} - Y_s)] \\
&= \mathbf{E} [(Y_{t_{i-1}} + Y_t) (Y_{t_{j-1}} + Y_s)] \mathbf{E} [(Y_{t_{i-1}} - Y_t) (Y_{t_{j-1}} - Y_s)] \\
&+ \mathbf{E} [(Y_{t_{i-1}} + Y_t) (Y_{t_{i-1}} - Y_t)] \mathbf{E} [(Y_{t_{j-1}} + Y_s) (Y_{t_{j-1}} - Y_s)] \\
&+ \mathbf{E} [(Y_{t_{i-1}} + Y_t) (Y_{t_{j-1}} - Y_s)] \mathbf{E} [(Y_{t_{j-1}} + Y_s) (Y_{t_{i-1}} - Y_t)] \\
&=: U_{1,n} + U_{2,n} + U_{3,n}.
\end{aligned}$$

Hence,

$$\begin{aligned}
b_{2,n} &= 2 \sum_{i < j} \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} \{ \mathbf{E} [(Y_{t_{i-1}} + Y_t) (Y_{t_{j-1}} + Y_s)] \\
&\quad \mathbf{E} [(Y_{t_{i-1}} - Y_t) (Y_{t_{j-1}} - Y_s)] \} ds dt \\
&+ 2 \sum_{i < j} \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} \{ \mathbf{E} [(Y_{t_{i-1}} + Y_t) (Y_{t_{i-1}} - Y_t)] \\
&\quad \mathbf{E} [(Y_{t_{j-1}} + Y_s) (Y_{t_{j-1}} - Y_s)] \} ds dt \\
&+ 2 \sum_{i < j} \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} \{ \mathbf{E} [(Y_{t_{i-1}} + Y_t) (Y_{t_{j-1}} - Y_s)] \\
&\quad \mathbf{E} [(Y_{t_{j-1}} + Y_s) (Y_{t_{i-1}} - Y_t)] \} ds dt \\
&=: V_1(n) + V_2(n) + V_3(n).
\end{aligned}$$

First we study the term $V_1(n)$. Let us consider two cases. Assume first that $\alpha \geq 1$, so

$$\begin{aligned}
& (2\alpha - 1) \mathbf{E} [(Y_{t_{i-1}} - Y_t) (Y_{t_{j-1}} - Y_s)] \\
&= \left((T - t_{j-1})^{\alpha-1} - (T - s)^{\alpha-1} \right) \left((T - t_{i-1})^{-\alpha} - (T - t)^{-\alpha} \right) \\
&+ T^{1-2\alpha} \left((T - t)^{\alpha-1} - (T - t_{i-1})^{\alpha-1} \right) \\
&\leq 0,
\end{aligned}$$

where we used $t_{i-1} \leq t < t_i$, $t_{j-1} \leq s < t_j$ and $\alpha \geq 1$ in the last inequality. Moreover

$$\begin{aligned}
& (2\alpha - 1) \mathbf{E} [(Y_{t_{i-1}} + Y_t) (Y_{t_{j-1}} + Y_s)] \\
&= \left((T - t_{j-1})^{\alpha-1} + (T - s)^{\alpha-1} \right) \\
&\left((T - t_{i-1})^{-\alpha} - T^{1-2\alpha} (T - t_{i-1})^{\alpha-1} + (T - t)^{-\alpha} - T^{1-2\alpha} (T - t)^{\alpha-1} \right) \\
&\geq 0.
\end{aligned}$$

This gives

$$V_1(n) \leq 0. \tag{3.9}$$

In the second case, we suppose $\frac{1}{2} < \alpha \leq 1$, so

$$\begin{aligned}
0 \leq (2\alpha - 1)\mathbf{E} [(Y_{t_{i-1}} - Y_t) (Y_{t_{j-1}} - Y_s)] &\leq \left((T - t_{j-1})^{\alpha-1} - (T - s)^{\alpha-1} \right) \\
&\quad \left((T - t_{i-1})^{-\alpha} - (T - t)^{-\alpha} \right) \\
&\leq C (s - t_{j-1}) (t - t_{j-1}) \\
&\quad (T - s)^{\alpha-2} (T - t)^{-\alpha-1},
\end{aligned}$$

and

$$0 \leq (2\alpha - 1)\mathbf{E} [(Y_{t_{i-1}} + Y_t) (Y_{t_{j-1}} + Y_s)] \leq C (T - s)^{\alpha-1} (T - t)^{-\alpha}.$$

Thus

$$\begin{aligned}
|V_1(n)| &\leq C \sum_{i < j} \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} (s - t_{j-1}) (t - t_{j-1}) (T - t)^{-2\alpha-1} (T - s)^{2\alpha-3} dt ds \\
&\leq C \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (t - t_{j-1}) (T - t)^{-2\alpha-1} dt \times \\
&\quad \sum_{j=1}^n \int_{t_{j-1}}^{t_j} (s - t_{j-1}) (T - s)^{2\alpha-3} ds \\
&\leq C \left(\Delta_n \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (T - t)^{-2\alpha-1} dt \right) \left(\Delta_n \sum_{j=1}^n \int_{t_{j-1}}^{t_j} (T - s)^{2\alpha-3} ds \right) \\
&\leq C (1 - T^{2\alpha-2} \Delta_n^{2-2\alpha} - T^{-2\alpha} \Delta_n^{2\alpha} + T^{-2} \Delta_n^2) \\
&\leq C. \tag{3.10}
\end{aligned}$$

For the term $V_2(n)$, when $\alpha \geq 1$, we have

$$\begin{aligned}
&(2\alpha - 1)^2 |U_{2,n}| \\
&= \left| \left((T - t_{i-1})^{-1} - (T - t)^{-1} - T^{1-2\alpha} (T - t_{i-1})^{2\alpha-2} + T^{1-2\alpha} (T - t)^{2\alpha-2} \right) \right. \\
&\quad \left. \left((T - t_{j-1})^{-1} - (T - s)^{-1} - T^{1-2\alpha} (T - t_{j-1})^{2\alpha-2} + T^{1-2\alpha} (T - s)^{2\alpha-2} \right) \right| \\
&\leq \left((T - t)^{-1} - (T - t_{i-1})^{-1} + C \right) \left((T - s)^{-1} - (T - t_{j-1})^{-1} + C \right).
\end{aligned}$$

This implies

$$|V_2(n)| \leq C \sum_{i < j} \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} \left((T - t)^{-1} - (T - t_{i-1})^{-1} + C \right) \tag{3.11}$$

$$\begin{aligned}
&\quad \left((T - s)^{-1} - (T - t_{j-1})^{-1} + C \right) ds dt \\
&\leq C \sum_{0 \leq i, j \leq n} \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} \left((T - t)^{-1} - (T - t_{i-1})^{-1} + C \right) \tag{3.12} \\
&\quad \left((T - s)^{-1} - (T - t_{j-1})^{-1} + C \right) ds dt
\end{aligned}$$

$$\begin{aligned}
&= C \left(\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left((T-t)^{-1} - (T-t_{i-1})^{-1} + C \right) dt \right)^2 \\
&= C \left(\sum_{i=1}^n \left(\log(T-t_{i-1}) - \log(T-t_i) - \Delta_n (T-t_{i-1})^{-1} + \Delta_n \right) \right)^2 \\
&= C \left\{ \log(n+1) - \sum_{k=2}^{n+1} \frac{1}{k} + n\Delta_n \right\}^2 \\
&\leq C \left\{ \frac{n}{n+1} + n\Delta_n \right\}^2 \\
&\leq C. \tag{3.13}
\end{aligned}$$

When $\frac{1}{2} < \alpha \leq 1$, we have

$$\begin{aligned}
(2\alpha - 1)^2 |U_{2,n}| &\leq \left((T-t)^{-1} - (T-t_{i-1})^{-1} + T^{1-2\alpha} (t-t_{i-1}) (T-t)^{2\alpha-3} \right) \\
&\quad \left((T-s)^{-1} - (T-t_{j-1})^{-1} + T^{1-2\alpha} (t-t_{j-1}) (T-t)^{2\alpha-3} \right),
\end{aligned}$$

which leads to

$$\begin{aligned}
|V_2(n)| &\leq C \left(\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left((T-t)^{-1} - (T-t_{i-1})^{-1} \right. \right. \\
&\quad \left. \left. + T^{1-2\alpha} (t-t_{i-1}) (T-t)^{2\alpha-3} \right) dt \right)^2 \\
&= C \left\{ \log(n+1) - \sum_{k=2}^{n+1} \frac{1}{k} + T^{1-2\alpha} \Delta_n (\Delta_n^{2\alpha-2} - T^{2\alpha-2}) \right\}^2 \\
&\leq C \left\{ \frac{n}{n+1} + T^{2\alpha-2} \Delta_n^{2\alpha-1} - \Delta_n \right\}^2 \\
&\leq C. \tag{3.14}
\end{aligned}$$

For $V_3(n)$, we can write

$$\begin{aligned}
V_3(n) &= 2 \sum_{i < j} \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} \left(\mathbf{E} [Y_{t_{i-1}} (Y_{t_{j-1}} - Y_s)] \mathbf{E} [Y_{t_{j-1}} (Y_{t_{i-1}} - Y_t)] \right. \\
&\quad + \mathbf{E} [Y_{t_{i-1}} (Y_{t_{j-1}} - Y_s)] \mathbf{E} [Y_s (Y_{t_{i-1}} - Y_t)] \\
&\quad + \mathbf{E} [Y_t (Y_{t_{j-1}} - Y_s)] \mathbf{E} [Y_{t_{j-1}} (Y_{t_{i-1}} - Y_t)] \\
&\quad \left. + \mathbf{E} [Y_t (Y_{t_{j-1}} - Y_s)] \mathbf{E} [Y_s (Y_{t_{i-1}} - Y_t)] \right) \\
&=: V_{3,1}(n) + V_{3,2}(n) + V_{3,3}(n) + V_{3,4}(n)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
(2\alpha - 1) \left| \mathbf{E} [Y_{t_{j-1}} (Y_{t_{i-1}} - Y_t)] \right| \\
&= \left| (T-t_{j-1})^{\alpha-1} \left((T-t_{i-1})^{-\alpha} - (T-t)^{-\alpha} \right. \right. \\
&\quad \left. \left. + T^{1-2\alpha} (T-t)^{\alpha-1} - T^{1-2\alpha} (T-t_{i-1})^{\alpha-1} \right) \right|,
\end{aligned}$$

and

$$\begin{aligned}
& (2\alpha - 1) |\mathbf{E} [Y_{t_{i-1}} (Y_{t_{j-1}} - Y_s)]| \\
&= \left| (T - t_{j-1})^{\alpha-1} - (T - s)^{\alpha-1} \right| \left((T - t_{i-1})^{-\alpha} - T^{1-2\alpha} (T - t_{i-1})^{\alpha-1} \right) \\
&\leq \left| (T - t_{j-1})^{\alpha-1} - (T - s)^{\alpha-1} \right| (T - t_{i-1})^{-\alpha}.
\end{aligned}$$

Hence, when $\frac{1}{2} < \alpha < 2$, we have

$$\begin{aligned}
& (2\alpha - 1) |\mathbf{E} [Y_{t_{j-1}} (Y_{t_{i-1}} - Y_t)]| \\
&\leq C (T - t_{j-1})^{\alpha-1} \left((t - t_{i-1}) (T - t)^{-\alpha-1} + (t - t_{i-1}) (T - t)^{\alpha-2} \right),
\end{aligned}$$

and

$$(2\alpha - 1) |\mathbf{E} [Y_{t_{i-1}} (Y_{t_{j-1}} - Y_s)]| \leq C (s - t_{j-1}) (T - t_{i-1})^{-\alpha} (T - s)^{\alpha-2}.$$

When $\alpha \geq 2$, we have

$$\begin{aligned}
& (2\alpha - 1) |\mathbf{E} [Y_{t_{j-1}} (Y_{t_{i-1}} - Y_t)]| \\
&\leq C (T - t_{j-1})^{\alpha-1} \left((t - t_{i-1}) (T - t)^{-\alpha-1} + (t - t_{i-1}) (T - t_{i-1})^{\alpha-2} \right),
\end{aligned}$$

and

$$(2\alpha - 1) |\mathbf{E} [Y_{t_{i-1}} (Y_{t_{j-1}} - Y_s)]| \leq C (s - t_{j-1}) (T - t_{i-1})^{-\alpha} (T - t_{j-1})^{\alpha-2}.$$

As a consequence, when $\alpha \geq 2$, we have

$$\begin{aligned}
& |V_{3,1}(n)| \\
&\leq C \sum_{i < j} \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} (s - t_{j-1}) (T - t_{i-1})^{-\alpha} (T - t_{j-1})^{\alpha-2} (T - t_{j-1})^{\alpha-1} \\
&\quad \left((t - t_{i-1}) (T - t)^{-\alpha-1} + (t - t_{i-1}) (T - t_{i-1})^{\alpha-2} \right) ds dt \\
&\leq C \sum_{i < j} \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} (s - t_{j-1}) (T - t_{i-1})^{-\alpha} (T - t_{j-1})^{\alpha-2} \\
&\quad (T - t_{j-1})^{\alpha-1} (t - t_{i-1}) (T - t)^{-\alpha-1} ds dt \\
&\quad + C \sum_{i < j} \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} (s - t_{j-1}) (T - t_{i-1})^{-\alpha} (T - t_{j-1})^{\alpha-2} \\
&\quad (T - t_{j-1})^{\alpha-1} (t - t_{i-1}) (T - t_{i-1})^{\alpha-2} ds dt \\
&\leq C \sum_{i < j} \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} (s - t_{j-1}) (T - t_{i-1})^{-2} (T - t_{j-1})^{-2} (t - t_{i-1}) ds dt \\
&\quad + C \sum_{i < j} \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} (s - t_{j-1}) (T - t_{i-1})^{-2} T^{2\alpha-3} (t - t_{i-1}) ds dt \\
&\leq C \left(\sum_{k=2}^{n+1} \frac{1}{k^2} \right)^2 + C n \Delta_n^2 \sum_{k=2}^{n+1} \frac{1}{k^2} \\
&\leq C.
\end{aligned}$$

When $1 \leq \alpha \leq 2$, we have

$$\begin{aligned}
& |V_{3,1}(n)| \\
& \leq \sum_{i < j} \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} (s - t_{j-1}) (T - t_{i-1})^{-\alpha} (T - s)^{\alpha-2} (T - t_{j-1})^{\alpha-1} \\
& \quad \left((t - t_{i-1}) (T - t)^{-\alpha-1} + (t - t_{i-1}) (T - t)^{\alpha-2} \right) ds dt \\
& \leq C \sum_{i < j} \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} (s - t_{j-1}) (T - t_{i-1})^{-\alpha} (T - s)^{\alpha-2} \\
& \quad (T - t_{j-1})^{\alpha-1} (t - t_{i-1}) (T - t)^{-\alpha-1} ds dt \\
& \quad + C \sum_{i < j} \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} (s - t_{j-1}) (T - t_{i-1})^{-\alpha} (T - s)^{\alpha-2} \\
& \quad (T - t_{j-1})^{\alpha-1} (t - t_{i-1}) (T - t)^{\alpha-2} ds dt \\
& \leq C \sum_{i < j} \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} (s - t_{j-1}) (T - s)^{-2} (t - t_{i-1}) (T - t)^{-2} ds dt \\
& \quad + C \sum_{i < j} \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} (s - t_{j-1}) (T - s)^{\alpha-2} (t - t_{i-1}) (T - t)^{-2} ds dt \\
& \leq C \left(\sum_{k=1}^n \frac{1}{k} - \log(n+1) \right)^2 + C (\Delta_n^\alpha - T^{\alpha-1} \Delta_n) \left(\sum_{k=1}^n \frac{1}{k} - \log(n+1) \right) \\
& \leq C \left(\frac{n}{n+1} \right)^2 + C (\Delta_n^\alpha - T^{\alpha-1} \Delta_n) \left(\frac{n}{n+1} \right) \leq C'.
\end{aligned}$$

Finally, when $\frac{1}{2} < \alpha \leq 1$, we have

$$\begin{aligned}
& |V_{3,1}(n)| \\
& \leq C \sum_{i < j} \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} (s - t_{j-1}) (T - s)^{2\alpha-3} (t - t_{i-1}) \times \\
& \quad (T - t_{i-1})^{-\alpha} (T - t)^{-\alpha-1} ds dt \\
& \quad + C \sum_{i < j} \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} (s - t_{j-1}) (T - s)^{2\alpha-3} (t - t_{i-1}) (T - t)^{-2} ds dt \\
& \leq C \Delta_n^{-\alpha-1} \sum_{i < j} \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} (s - t_{j-1}) (T - s)^{2\alpha-3} (t - t_{i-1}) (T - t_{i-1})^{-\alpha} ds dt \\
& \quad + C \sum_{i < j} \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} (s - t_{j-1}) (T - s)^{2\alpha-3} (t - t_{i-1}) (T - t)^{-2} ds dt \\
& \leq C (1 - T^{-\alpha} \Delta_n^\alpha - T^{2\alpha-2} \Delta_n^{2-2\alpha} + T^{\alpha-2} \Delta_n^{2-\alpha}) \\
& \quad + C (\Delta_n^{2\alpha-1} - T^{2\alpha-2} \Delta_n) \left(\frac{n}{n+1} \right)
\end{aligned}$$

$$\leq C.$$

Similarly, we can use the same way as $V_{3,1}(n)$, we get that

$$|V_{3,2}(n)| \leq C, |V_{3,3}(n)| \leq C, |V_{3,4}(n)| \leq C.$$

As a consequence,

$$|V_3(n)| \leq C. \quad (3.15)$$

Combining (3.9), (3.10), (3.13), (3.14) and (3.15), we deduce that

$$|b_{2,n}| \leq C. \quad (3.16)$$

Therefore, the facts (3.8) and (3.16) achieve the proof of the desired result. \square

Lemma 3.2. *Suppose that $\alpha > \frac{1}{2}$. Let $0 < \varepsilon < 1$. Then*

$$\mathbb{P}(\lambda_n |B_{T_n}| > \varepsilon) \leq \frac{C}{\varepsilon^2 \log(n+1)}, \quad (3.17)$$

and

$$\mathbb{P}(\lambda_n |B_n - B_{T_n}| > \varepsilon) \leq \frac{C}{\varepsilon^2 \log^2(n+1)}, \quad (3.18)$$

where $\lambda_n = \frac{2\alpha+1}{\log(n+1)}$.

Proof. By using (3.6), we have

$$\begin{aligned} \mathbf{E} |B_{T_n}|^2 &= \mathbf{E} \left| \int_0^{T_n} \frac{X_t}{(T-t)} dW_t \right|^2 \\ &= \mathbf{E} [I_{T_n}] \\ &= \frac{(2\alpha - 1) \log(n+1) + T^{1-2\alpha} \Delta_n^{2\alpha-1} - 1}{(2\alpha - 1)^2}. \end{aligned}$$

On the other hand by Chebyshev inequality we obtain

$$\begin{aligned} \mathbb{P}(\lambda_n |B_{T_n}| > \varepsilon) &\leq \frac{\lambda_n^2 \mathbf{E} |B_{T_n}|^2}{\varepsilon^2} \\ &\leq \frac{C}{\varepsilon^2 \log(n+1)}. \end{aligned}$$

Let us now prove (3.18). By Chebyshev inequality we have

$$\mathbb{P} \left(\lambda_n |B_n - B_{T_n}| > \frac{\varepsilon(1-\varepsilon)}{3} \right) \leq \frac{\lambda_n^2}{\varepsilon^2} \mathbf{E} \left[|B_n - B_{T_n}|^2 \right].$$

Moreover,

$$\begin{aligned} \mathbf{E} \left[|B_n - B_{T_n}|^2 \right] &= \mathbf{E} \left(\left| \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{X_{t_{i-1}}}{(T-t_{i-1})} dW_t - \int_0^{T_n} \frac{X_t}{(T-t)} dW_t \right|^2 \right) \\ &= \mathbf{E} \left(\left| \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left(\frac{X_{t_{i-1}}}{(T-t_{i-1})} - \frac{X_t}{(T-t)} \right) dW_t \right|^2 \right) \end{aligned}$$

$$= \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \mathbf{E} \left[\left(\frac{X_{t_{i-1}}}{(T-t_{i-1})} - \frac{X_t}{(T-t)} \right)^2 \right] dt.$$

On the other hand,

$$\begin{aligned} & (2\alpha - 1) \mathbf{E} \left[\left(\frac{X_{t_{i-1}}}{(T-t_{i-1})} - \frac{X_t}{(T-t)} \right)^2 \right] \\ &= (T-t_{i-1})^{-1} - T^{1-2\alpha} (T-t_{i-1})^{2\alpha-2} + (T-t)^{-1} - T^{1-2\alpha} (T-t)^{2\alpha-2} \\ &\quad - 2(T-t_{i-1})^{-\alpha} (T-t)^{\alpha-1} + 2T^{1-2\alpha} (T-t_{i-1})^{\alpha-1} (T-t)^{\alpha-1} \\ &\leq (T-t_{i-1})^{-1} + (T-t)^{-1} - 2(T-t_{i-1})^{-\alpha} (T-t)^{\alpha-1} \\ &= (T-t_{i-1})^{-1} - (T-t)^{-1} + 2(T-t)^{-1} - 2(T-t_{i-1})^{-\alpha} (T-t)^{\alpha-1} \\ &= (T-t_{i-1})^{-1} - (T-t)^{-1} + 2(T-t)^{\alpha-1} \left((T-t)^{-\alpha} - (T-t_{i-1})^{-\alpha} \right) \\ &\leq (t-t_{i-1})(T-t)^{-2} + 2(T-t)^{\alpha-1} \left(\alpha(t-t_{i-1})(T-t)^{-\alpha-1} \right) \\ &= C(t-t_{i-1})(T-t)^{-2}. \end{aligned}$$

Thus

$$\begin{aligned} \mathbf{E} \left[|B_n - B_{T_n}|^2 \right] &\leq C \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (t-t_{i-1})(T-t)^{-2} dt \\ &\leq C \left(\frac{n}{n+1} \right). \end{aligned}$$

Then the desired result is obtained. \square

3.1. Consistency of the approximate maximum likelihood estimators.

The following result establishes the consistency of the approximate maximum likelihood estimators $\hat{\alpha}_n$ and $\bar{\alpha}_n$.

Theorem 3.3. *Suppose that $\alpha > \frac{1}{2}$. Then, we have*

$$\hat{\alpha}_n \longrightarrow \alpha \text{ and } \bar{\alpha}_n \longrightarrow \alpha,$$

in probability as $n \longrightarrow \infty$.

Proof. By using (1.1), we have

$$\begin{aligned} \alpha - \hat{\alpha}_n &= \frac{\alpha I_n + \sum_{i=1}^n Y_{t_{i-1}} (X_{t_i} - X_{t_{i-1}})}{I_n} \\ &= \frac{\alpha I_n + \sum_{i=1}^n Y_{t_{i-1}} \int_{t_{i-1}}^{t_i} dX_t}{I_n} \\ &= \frac{\alpha I_n + \sum_{i=1}^n Y_{t_{i-1}} \int_{t_{i-1}}^{t_i} (-\alpha Y_t dt + dW_t)}{I_n} \\ &= \frac{\alpha \left(I_n - \sum_{i=1}^n \int_{t_{i-1}}^{t_i} Y_{t_{i-1}} Y_t dt \right) + \sum_{i=1}^n Y_{t_{i-1}} \int_{t_{i-1}}^{t_i} dW_t}{I_n} \\ &= \alpha \frac{A_n}{I_n} + \frac{B_n}{I_n}, \end{aligned} \tag{3.19}$$

and again by (1.1) together with (3.5) we can write

$$\alpha - \bar{\alpha}_n = \frac{2\alpha - 1}{2} \frac{(I_n - I_{T_n})}{I_n} + \frac{B_{T_n}}{I_n}. \quad (3.20)$$

Let $0 < \varepsilon < 1$ and $\lambda_n = \frac{2\alpha-1}{\log(n+1)}$. We have

$$\begin{aligned} \mathbb{P}(|\alpha - \hat{\alpha}_n| > \varepsilon) &= \mathbb{P}\left(\left|\frac{\alpha A_n + B_n}{I_n}\right| > \varepsilon\right) \\ &= \mathbb{P}\left(\left|\frac{\lambda_n(\alpha A_n + B_n)}{\lambda_n I_n}\right|\right) \\ &\leq \mathbb{P}(|\lambda_n(\alpha A_n + B_n)| > \varepsilon(1 - \varepsilon)) + \mathbb{P}(|\lambda_n I_n - 1| > \varepsilon) \\ &:= J_n + K_n. \end{aligned}$$

We begin by studying the term J_n . We have,

$$\begin{aligned} J_n &= \mathbb{P}(|\lambda_n(\alpha A_n + B_n)| > \varepsilon(1 - \varepsilon)) \\ &= \mathbb{P}(|\lambda_n(\alpha A_n + (B_n - B_{T_n}) + B_{T_n})| > \varepsilon(1 - \varepsilon)) \\ &\leq \mathbb{P}\left(\alpha \lambda_n |A_n| > \frac{\varepsilon(1 - \varepsilon)}{3}\right) + \mathbb{P}\left(\lambda_n |B_n - B_{T_n}| > \frac{\varepsilon(1 - \varepsilon)}{3}\right) \\ &\quad + \mathbb{P}\left(\lambda_n |B_{T_n}| > \frac{\varepsilon(1 - \varepsilon)}{3}\right). \end{aligned}$$

Proceeding similar to the estimation of $\mathbf{E} \left[|(I_n - I_{T_n})|^2 \right]$, we deduce that

$$\mathbb{P}\left(\alpha \lambda_n |A_n| > \frac{\varepsilon(1 - \varepsilon)}{3}\right) \leq \frac{C}{\log^2(n+1) (\varepsilon(1 - \varepsilon))^2}. \quad (3.21)$$

Therefore, from Lemma 3.2, we obtain

$$J_n \leq \frac{C}{(\varepsilon(1 - \varepsilon))^2 \log(n+1)}. \quad (3.22)$$

For the term K_n , it follows from Lemma 3.1 that

$$\begin{aligned} K_n &= \mathbb{P}(|\lambda_n(I_n - I_{T_n}) + \lambda_n I_{T_n} - 1| > \varepsilon) \\ &\leq \mathbb{P}\left(|\lambda_n(I_n - I_{T_n})| > \frac{\varepsilon}{2}\right) + \mathbb{P}\left(|\lambda_n I_{T_n} - 1| > \frac{\varepsilon}{2}\right) \\ &\leq \frac{C}{\varepsilon^2 \log(n+1)}. \end{aligned} \quad (3.23)$$

Combining (3.22) and (3.23), we deduce that $\hat{\alpha}_n$ converges in probability to α as $n \rightarrow \infty$. The proof of the consistency of $\bar{\alpha}_n$ is quite similar to the proof above. Thus the desired result is obtained. \square

3.2. Rate of convergence of the approximate maximum likelihood estimators. In order to investigate the Berry-Esseen-type bounds in the Kolmogorov distance for the approximate maximum likelihood estimators $\hat{\alpha}_n$ and $\bar{\alpha}_n$, we will need the following lemmas.

Lemma 3.4 ([10], page 78). *Let f and g be two real-valued random variables with $g \neq 0$ \mathbb{P} -a.s. Then, for any $\epsilon > 0$,*

$$d_{kol} \left(\frac{f}{g}, Z \right) \leq d_{kol}(f, Z) + \mathbb{P}(|g - 1| > \epsilon) + \epsilon,$$

where $Z \sim \mathcal{N}(0, 1)$.

Lemma 3.5 ([1], page 280). *Let f and g be two real-valued random variables. Then, for any $\epsilon > 0$,*

$$d_{kol}(f + g, Z) \leq d_{kol}(f, Z) + \mathbb{P}(|g| > \epsilon) + \frac{\epsilon}{\sqrt{2\pi}},$$

where $Z \sim \mathcal{N}(0, 1)$.

In the next Theorem we give an explicit bounds for the Kolmogorov distance in the central limit theorem for the estimators $\hat{\alpha}_n$ and $\bar{\alpha}_n$.

Theorem 3.6. *Let $0 < \epsilon < 1$ and $\alpha > \frac{1}{2}$. Then there exist a positive constant C , depending only on α and T , such that*

$$\begin{aligned} \sup_{z \in \mathbb{R}} |\mathbb{P}(\beta_n(\alpha - \hat{\alpha}_n) \leq z) - \mathbb{P}(Z \leq z)| & \quad (3.24) \\ & \leq \frac{C}{\sqrt{\log(n+1)}} + \frac{C}{\epsilon^2 \log(n+1)} + 3\epsilon, \end{aligned}$$

and

$$\begin{aligned} \sup_{z \in \mathbb{R}} |\mathbb{P}(\beta_n(\alpha - \bar{\alpha}_n) \leq z) - \mathbb{P}(Z \leq z)| & \quad (3.25) \\ & \leq \frac{C}{\sqrt{\log(n+1)}} + \frac{C}{\epsilon^2 \log(n+1)} + 2\epsilon, \end{aligned}$$

where $\beta_n = \sqrt{\frac{\log(n+1)}{2\alpha-1}}$.

Proof. Using (3.19) and Lemma 3.4, we obtain

$$\begin{aligned} & \sup_{z \in \mathbb{R}} \left| \mathbb{P} \left(\sqrt{\frac{\log(n+1)}{2\alpha-1}} (\alpha - \hat{\alpha}_n) \leq z \right) - \mathbb{P}(Z \leq z) \right| \\ &= \sup_{z \in \mathbb{R}} \left| \mathbb{P} \left(\sqrt{\frac{\log(n+1)}{2\alpha-1}} \left(\frac{\alpha A_n + B_n}{I_n} \right) \leq z \right) - \mathbb{P}(Z \leq z) \right| \\ &= \sup_{z \in \mathbb{R}} \left| \mathbb{P} \left(\sqrt{\frac{2\alpha-1}{\log(n+1)}} (\alpha A_n + B_n) \leq z \right) - \mathbb{P}(Z \leq z) \right| \\ &+ \mathbb{P} \left(\left| \frac{2\alpha-1}{\log(n+1)} I_n - 1 \right| > \epsilon \right) + \epsilon \\ &:= K_1 + K_2 + \epsilon. \end{aligned}$$

On the other hand, from Lemma 3.5, we have

$$K_1 = \sup_{z \in \mathbb{R}} \left| \mathbb{P} \left(\sqrt{\frac{2\alpha-1}{\log(n+1)}} B_n \leq z \right) - \mathbb{P}(Z \leq z) \right|$$

$$\begin{aligned}
& + \mathbb{P} \left(\left| \alpha \sqrt{\frac{2\alpha-1}{\log(n+1)}} A_n \right| > \varepsilon \right) + \varepsilon \\
& := L_1 + L_2 + \varepsilon.
\end{aligned}$$

Again using Lemma 3.5, we obtain

$$\begin{aligned}
L_1 &= \sup_{z \in \mathbb{R}} \left| \mathbb{P} \left(\sqrt{\frac{2\alpha-1}{\log(n+1)}} (B_n - B_{T_n} + B_{T_n}) \leq z \right) - \mathbb{P}(Z \leq z) \right| \\
&= \sup_{z \in \mathbb{R}} \left| \mathbb{P} \left(\sqrt{\frac{2\alpha-1}{\log(n+1)}} B_{T_n} \leq z \right) - \mathbb{P}(Z \leq z) \right| + \\
&\quad \mathbb{P} \left(\sqrt{\frac{2\alpha-1}{\log(n+1)}} |B_n - B_{T_n}| > \varepsilon \right) + \varepsilon.
\end{aligned}$$

From Lemma 3.2, we deduce that

$$\mathbb{P} \left(\sqrt{\frac{2\alpha-1}{\log(n+1)}} |B_n - B_{T_n}| > \varepsilon \right) \leq \frac{C}{\varepsilon^2 \log(n+1)}. \quad (3.26)$$

Moreover, it follows from (1.2) and (2.1) that,

$$\begin{aligned}
& \sqrt{\frac{2\alpha-1}{\log(n+1)}} B_{T_n} \\
&= \sqrt{\frac{2\alpha-1}{\log(n+1)}} \int_0^{T_n} \frac{X_s}{T-s} dW_s \\
&= \sqrt{\frac{2\alpha-1}{\log(n+1)}} \int_0^{T_n} \int_0^s (T-s)^{\alpha-1} (T-r)^{-\alpha} dW_r dW_s \\
&= \frac{1}{2} \sqrt{\frac{2\alpha-1}{\log(n+1)}} \int_0^{T_n} \int_0^{T_n} (T-s \vee r)^{\alpha-1} (T-s \wedge r)^{-\alpha} dW_r dW_s \\
&=: I_2(f_{T_n}),
\end{aligned}$$

where f_{T_n} is a symmetric function defined by

$$f_{T_n}(u, v) = \frac{1}{2} \sqrt{\frac{2\alpha-1}{\log(n+1)}} (T-u \vee v)^{\alpha-1} (T-u \wedge v)^{-\alpha} \mathbf{1}_{[0, T_n]^2}(u, v).$$

Since the function f_{T_n} is symmetric, we have

$$\begin{aligned}
\|f_{T_n}\|_{\mathfrak{H}^{\otimes 2}}^2 &= \frac{2\alpha-1}{4 \log(n+1)} \int_{[0, T_n]^2} (T-x \vee y)^{2\alpha-2} (T-x \wedge y)^{-2\alpha} dx dy \\
&= \frac{2\alpha-1}{2 \log(n+1)} \int_0^{T_n} dy \int_0^y (T-y)^{2\alpha-2} (T-x)^{-2\alpha} dx \\
&= \frac{1}{2 \log(n+1)} \int_0^{T_n} \left((T-y)^{-1} - T^{-2\alpha+1} (T-y)^{2\alpha-2} \right) dy
\end{aligned}$$

$$= \frac{1}{2 \log(n+1)} \left(\log(n+1) + \frac{(\Delta_n/T)^{2\alpha-1}}{(2\alpha-1)} - \frac{1}{(2\alpha-1)} \right).$$

Thus,

$$2\|f_{T_n}\|_{\mathfrak{H}^{\otimes 2}}^2 - 1 = \frac{(\Delta_n/T)^{2\alpha-1} - 1}{(2\alpha-1)\log(n+1)},$$

hence

$$|2\|f_{T_n}\|_{\mathfrak{H}^{\otimes 2}}^2 - 1| \leq \frac{C}{\log(n+1)}.$$

Moreover, from [6], we have

$$\|f_{T_n} \otimes_1 f_{T_n}\|_{\mathfrak{H}^{\otimes 2}}^2 \leq \frac{C}{\log(n+1)}.$$

Consequently, by combining (2.4) and (2.7), we obtain

$$\sup_{z \in \mathbb{R}} \left| \mathbb{P} \left(\sqrt{\frac{2\alpha-1}{\log(n+1)}} B_{T_n} \leq z \right) - \mathbb{P}(Z \leq z) \right| \leq \frac{C}{\sqrt{\log(n+1)}}. \quad (3.27)$$

For the term L_2 , by using (3.21), we deduce that

$$\mathbb{P} \left(\left| \alpha \sqrt{\frac{2\alpha-1}{\log(n+1)}} A_n \right| > \varepsilon \right) \leq \frac{C}{\varepsilon^2 \log(n+1)}. \quad (3.28)$$

Finally, from Lemma 3.1, we have

$$K_2 \leq \frac{C}{\varepsilon^2 \log(n+1)}. \quad (3.29)$$

Combining (3.26), (3.27), (3.28) and (3.29) we deduce (3.24). The proof of (3.25) is quite similar to the proof above. Thus the desired result is obtained. \square

Corollary 3.7. *Let $\frac{1}{6} \leq \beta < \frac{1}{2}$ and $\alpha > \frac{1}{2}$. Then there exist a positive constant C , depending only on α and T , such that*

$$\sup_{z \in \mathbb{R}} \left| \mathbb{P} \left(\sqrt{\frac{\log(n+1)}{2\alpha-1}} (\alpha - \hat{\alpha}_n) \leq z \right) - \mathbb{P}(Z \leq z) \right| \leq C \log^{\beta-\frac{1}{2}}(n+1),$$

and

$$\sup_{z \in \mathbb{R}} \left| \mathbb{P} \left(\sqrt{\frac{\log(n+1)}{2\alpha-1}} (\alpha - \bar{\alpha}_n) \leq z \right) - \mathbb{P}(Z \leq z) \right| \leq C \log^{\beta-\frac{1}{2}}(n+1).$$

Example 3.8. Upon choosing $\beta = \frac{1}{6}$ we obtain

$$\sup_{z \in \mathbb{R}} \left| \mathbb{P} \left(\sqrt{\frac{\log(n+1)}{2\alpha-1}} (\alpha - \hat{\alpha}_n) \leq z \right) - \mathbb{P}(Z \leq z) \right| \leq \frac{C}{\log^{\frac{1}{3}}(n+1)},$$

and

$$\sup_{z \in \mathbb{R}} \left| \mathbb{P} \left(\sqrt{\frac{\log(n+1)}{2\alpha-1}} (\alpha - \bar{\alpha}_n) \leq z \right) - \mathbb{P}(Z \leq z) \right| \leq \frac{C}{\log^{\frac{1}{3}}(n+1)}.$$

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